EE2 Mathematics

The role of grad, div and curl in vector calculus

The gradient operator ∇ is defined as

$$\nabla = \hat{\boldsymbol{i}}\,\frac{\partial}{\partial x} + \hat{\boldsymbol{j}}\,\frac{\partial}{\partial y} + \hat{\boldsymbol{k}}\,\frac{\partial}{\partial z}.$$

Using this operator on a scalar field $\phi = \phi(x, y, z)$ gives the *gradient* of ϕ

$$\operatorname{grad}\phi = \nabla\phi = \hat{\boldsymbol{i}}\,\frac{\partial \phi}{\partial x} + \hat{\boldsymbol{j}}\,\frac{\partial \phi}{\partial y} + \hat{\boldsymbol{k}}\,\frac{\partial \phi}{\partial z}.$$

Note that $\nabla\phi$ is a vector. Since ∇ is a *vector* operator it can act upon a vector field

$$\boldsymbol{A}(x, y, z) = \hat{\boldsymbol{i}}\,A_1(x, y, z) + \hat{\boldsymbol{j}}\,A_2(x, y, z) + \hat{\boldsymbol{k}}\,A_3(x, y, z)$$

through the dot product. This defines the *divergence* of a vector (which is itself a scalar)

$$\operatorname{div}\boldsymbol{A} = \nabla \cdot \boldsymbol{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

which physically expresses how the vector $\boldsymbol{A}$ changes spatially through the six faces of a 3D-box[1]. Note, however, that $\nabla \cdot \boldsymbol{A} \neq \boldsymbol{A} \cdot \nabla$: the RHS is a differential operator while the LHS is a scalar function.

The *curl* of a vector (which is itself a vector) is defined by

$$\operatorname{curl}\boldsymbol{A} = \nabla \times \boldsymbol{A} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}.$$

This expresses how much swirl is in a vector field; if $\boldsymbol{r}$ is the line from the origin to the point (x, y, z) then $\operatorname{curl}\boldsymbol{r} = 0$.

There are various identities that are useful:

1. The gradient of the product of two scalars ϕ and ψ
$$\nabla(\phi\psi) = \psi\nabla\phi + \phi\nabla\psi$$
2. The divergence of the product of a scalar ϕ with a vector $\boldsymbol{A}$
$$\operatorname{div}(\phi\boldsymbol{A}) = \phi\operatorname{div}\boldsymbol{A} + (\nabla\phi)\cdot\boldsymbol{A}$$
3. The curl of the product of a scalar ϕ with a vector $\boldsymbol{A}$
$$\operatorname{curl}(\phi\boldsymbol{A}) = \phi\operatorname{curl}\boldsymbol{A} + (\nabla\phi)\times\boldsymbol{A}$$
4. The curl of the gradient of any scalar ϕ
$$\operatorname{curl}(\nabla\phi) = \nabla\times\nabla\phi = 0.$$
5. The divergence of the curl of any vector $\boldsymbol{A}$
$$\operatorname{div}(\operatorname{curl}\boldsymbol{A}) = \nabla\cdot(\nabla\times\boldsymbol{A}) = 0$$
The cyclic rule for the scalar triple product shows that this is zero for all vectors $\boldsymbol{A}$ because two vectors in the triple (∇) are the same.

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[1] One can think of a vector field that has zero divergence as 'incompressible'.