Geometric methods and the Euler equations of an Ideal fluid

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Abstract

We formulate the Euler equations for Inviscid ideal fluid flow in terms of quaternionic and differential geometry. This presents some interesting ways of investigating solutions to the Euler equations.

Contents

1 Spin geometry on $\mathbb{R}^3$

It is very well known that the Clifford Algebra of $\mathbb{R}^3$ is isomorphic to $\mathbb{H} \oplus \mathbb{H}$ and that the Spin space is just a single copy of $\mathbb{H}$. The spin bundle over $\mathbb{R}^3$ is simply the trivial vector bundle $\underline{\mathbb{H}} \longrightarrow \mathbb{R}^3$. The expression of functions, forms and vectors on $\mathbb{R}^3$ as spinors creates a beautiful and compact notation and unifies the classical operation of div, curl and grad as parts of the same differential operator, the Dirac operator.

1.1 Spinors, vectors and functions

The cross product on $\mathbb{R}^3$ turns $\mathbb{R}^3$ into an non-associative algebra. By adding an extra copy of $\mathbb{R}$ and endowing the result with the algebra structure given by:

$$\begin{array}{rcl} (\mathbb{R} \oplus \mathbb{R}^3) \times (\mathbb{R} \oplus \mathbb{R}^3) & \longrightarrow & (\mathbb{R} \oplus \mathbb{R}^3) \\ ((a, \boldsymbol{u}), (b, \boldsymbol{v})) & \mapsto & (ab - \langle \boldsymbol{u}, \boldsymbol{v} \rangle, a\boldsymbol{v} + b\boldsymbol{u} + \boldsymbol{u} \times \boldsymbol{v}) \end{array}$$

we obtain an associative algebra structure on $\mathbb{R} \oplus \mathbb{R}^3$ The formula may look complicated but a quick calculation to show that

$$(a, \boldsymbol{u})(a, -\boldsymbol{u}) = (a^2 + |\boldsymbol{u}|^2, \boldsymbol{0})$$

indicates that this new algebra is a 4-dimensional division ring and is hence isomorphic to $\mathbb{H}$, the spin space of $\mathbb{R}^3$. Thus we can treat a pair comprising of a scalar quantity and a vector quantity as a spinor. A section of the spin bundle $\underline{\mathbb{H}} \longrightarrow \mathbb{R}^3$ can thus be regarded as a pair comprising of a function and a vector field. This leads us to the following equivalences:

$$\begin{array}{rcccl} \text{scalars} + \text{vectors} & \leftrightarrow & \text{spinors} & \leftrightarrow & \text{quaternions} \\ a + \boldsymbol{u} & \leftrightarrow & (a, \boldsymbol{u}) & \leftrightarrow & a\mathbb{1} + u_1\mathbb{I} + u_2\mathbb{J} + u_3\mathbb{K} \end{array}$$

where in the standard basis $\boldsymbol{u} = (u_1, u_2, u_3)$.

1.2 The Dirac operator

The Dirac operator on the spin bundle $\underline{\mathbb{H}} \longrightarrow \mathbb{R}^3$ is the operator

$$\eth = \mathbb{I}\frac{\partial}{\partial x} + \mathbb{J}\frac{\partial}{\partial y} + \mathbb{K}\frac{\partial}{\partial z}$$

which takes a smooth quaternionic field to another smooth quaternionic field. This is the quaternionic expression of the Dirac operator, and it can easily be checked that as an operator on smooth spinor fields (i.e on scalar fields and vector fields)

$$\eth(f, \boldsymbol{u}) = (-\mathrm{div}\boldsymbol{u}, \mathrm{curl}\boldsymbol{u} + \mathrm{grad}f).$$

Thus all the operators of classical vector analysis can be combined into one operator:

$$\begin{aligned}\eth(f,\mathbf{0}) &= (0,\mathrm{grad} f)\\ \eth(0,\boldsymbol{v}) &= (-\mathrm{div}\boldsymbol{v},\mathrm{curl}\boldsymbol{v}).\end{aligned}$$

Thus we can always recover the classical operators by acting on pure scalars and pure vectors.

Moreover the standard property that the square of the Dirac operator is the (geometers') laplacian can be checked easily.

1.3 The kernel of the Dirac operator

Recall from the standard theory of the Laplace operator that the function

$$\boldsymbol{x} \mapsto 1/|\boldsymbol{x}|$$

forms the kernel of the laplacian on $\mathbb{R}^3$, and from this we may obtain Green's operator G^{Δ} with the property that

$$\Delta \mathrm{G}^{\Delta}\boldsymbol{u} = \boldsymbol{u}$$

for any vector/scalar field. Since

$$\Delta = \eth^2$$

We can immediately see that

$$\mathrm{G}^{\eth}\phi = \eth \mathrm{G}^{\Delta}\phi$$

satisfies

$$\eth \mathrm{G}^{\eth}\phi = \phi$$

on spinors. Thus the solution to

$$\eth\phi = \psi$$

on some region of $\mathbb{R}^3$ is

$$\phi = \mathrm{G}^{\eth}\psi + \eta$$

where η is a spinor field such that $\eth\eta = 0$. The extra function η will be fixed by any boundary conditions.

2 The Euler Equations

According to Gibbon [4] by writing $\boldsymbol{\omega} = \mathrm{curl}\,\boldsymbol{u}$ the Euler Equations for the incompressible Euler Equations can be written

$$\begin{aligned}\mathrm{div}\,\boldsymbol{u} &= 0\\ \frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} &= \boldsymbol{\omega}.\nabla\boldsymbol{u}.\end{aligned}$$

Expanding the material derivative this becomes

$$\frac{\partial\boldsymbol{\omega}}{\partial t} = \boldsymbol{\omega}.\nabla\boldsymbol{u} - \boldsymbol{u}.\nabla\boldsymbol{\omega}.$$

The right-hand side will be familiar to geometers as the Lie bracket of vector fields, indicating that we have the rather familiar Hamiltonian expression

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = [\boldsymbol{\omega}, \boldsymbol{u}] \, .$$

This expression is so very reminiscent of the Euler Equations for rigid body motion. In that case we are considering a curve $\alpha(t)$ in the Lie algebra $\mathfrak{a}$ of an orthogonal group that satisfies

$$\frac{\partial \alpha}{\partial t} = \left[H^{-1}(\alpha), \alpha\right]$$

where $H \in \mathrm{End}\,(\mathfrak{a})$ is the analogue of the inertia tensor. Now the Lie algebra of vector fields is in fact the Lie algebra of the Lie group of diffeomorphisms. Since this Lie group is infinite dimensional we must be sure that we have the definition is correct. Since we require $\mathrm{div}\, \boldsymbol{u} = 0$, and $\mathrm{div}\, \boldsymbol{\omega} = 0$, by construction, we may restrict ourselves to the Lie group of volume preserving diffeomorphisms. The full diffeomorphism group has the unhappy property that the exponential map from its Lie algebra is not a diffeomorphism, so we have to be very careful with our definitions. From our work above, we know that the curl operator is invertible modulo a gradient of a function which can be fixed by applying boundary conditions (e.g 0 at infinity).

2.1 Integrability of the Euler Equations

It is very well known that the Euler equations for a solid body rotating in 3-dimensional space are integrable. By considering eigenvectors of the inertia tensor we can obtain the equations

$$\begin{aligned} \dot{u}_1 &= u_2 u_3 \\ \dot{u}_2 &= u_3 u_1 \\ \dot{u}_3 &= u_1 u_2 \end{aligned}$$

and the algebraic geometry turns these into a linear flow in the jacobian torus of an elliptic curve. As a result the equations can be solved using elliptic functions. We conjecture that it may be possible that the Euler Equations for incompressible flow be integrable.

2.2 The Dirac operator and a reformulation of the Euler equations

We examine the Euler equations in light of the quaternionic geometry that is inherently present within the context of $\mathbb{R}^3$.

Proposition 2.1 *For two imaginary quaternionic fields on $\mathbb{R}^3$, a and b we have*

$$\eth(ab) = \eth(a)b - a\eth b - 2\mathbf{a}.\nabla b$$

where $\mathbf{a}$ is a regarded as a vector field on $\mathbb{R}^3$.

Proof
First we must recall the Clifford algebraic relations satisfied by the quaternions, namely for $q_i, q_j \in \{\mathbb{I}, \mathbb{J}, \mathbb{K}\}$ we have

$$q_i q_j + q_j q_i = -2\delta_{ij} \tag{1}$$

hence for any imaginary quaternion $a = a_1\mathbb{I} + a_2\mathbb{J} + a_3\mathbb{K}$

$$q_i a = -a q_i - 2a_i \tag{2}$$

and for two imaginary quaternions a and b,

$$ba = -ab - 2\langle a, b\rangle. \tag{3}$$

Thus, using the Einstein summation convention

$$\begin{aligned} \eth(ab) &= q_i \partial_i(ab) \\ &= q_i(\partial_i a)b + q_i a \partial_i b \\ &= (\eth a)b - (a q_i + 2a_i)\partial_i b \\ &\quad \text{by (2),} \\ &= (\eth a)b - a\eth b - 2\mathbf{a}.\nabla b \end{aligned}$$

as required. ■

Lemma 2.2 *For two tangent vectors $\boldsymbol{x}, \mathbf{Y} \in \Gamma(\mathrm{T}\mathbb{R}^3)$, denote their expression as imaginary quaternions by $\overrightarrow{X}$ and $\overrightarrow{Y}$ respectively. Then*

$$[\boldsymbol{x}, \mathbf{Y}] = -\frac{1}{2}\Im\eth\left(\left[\overrightarrow{X}, \overrightarrow{Y}\right]_{\mathbb{H}}\right)$$

where the left hand bracket is the Lie bracket of vector fields and the right hand bracket is the Lie bracket of imaginary quaternions, i.e. the bracket on the Lie Algebra $\mathfrak{sp}(1)$

Proof
Examining the bracket of vector fields

$$\begin{aligned} 2[\boldsymbol{x}, \mathbf{Y}] &= 2\boldsymbol{x}.\nabla\mathbf{Y} - 2\mathbf{Y}.\nabla\boldsymbol{x} \\ &= (\eth\overrightarrow{X})\overrightarrow{Y} - \overrightarrow{X}\eth\overrightarrow{Y} - \eth(\overrightarrow{X}\overrightarrow{Y}) \\ &- (\eth\overrightarrow{Y})\overrightarrow{X} + \overrightarrow{Y}\eth\overrightarrow{X} + \eth(\overrightarrow{Y}\overrightarrow{X}) \\ &\quad \text{by Proposition 2.1} \\ &= (\eth\overrightarrow{X})\overrightarrow{Y} - \overrightarrow{X}\eth\overrightarrow{Y} - \eth(\overrightarrow{X}\overrightarrow{Y}) \\ &+ \overrightarrow{X}(\eth\overrightarrow{Y}) + 2\left\langle\overrightarrow{X}, \eth\overrightarrow{Y}\right\rangle \\ &- (\eth\overrightarrow{X})\overrightarrow{Y} - 2\left\langle\eth\overrightarrow{X}, \overrightarrow{Y}\right\rangle \\ &+ \eth(\overrightarrow{Y}\overrightarrow{X}) \\ &= \eth(\overrightarrow{Y}\overrightarrow{X} - \overrightarrow{X}\overrightarrow{Y}) \\ &+ 2\left\langle\overrightarrow{X}, \eth\overrightarrow{Y}\right\rangle - 2\left\langle\eth\overrightarrow{X}, \overrightarrow{Y}\right\rangle \end{aligned}$$

The Left Hand Side regarded as a spinor is purely imaginary, thus by taking the imaginary part of both sides yields the result, since

$$2\left\langle \overrightarrow{X}, \eth\overrightarrow{Y} \right\rangle - 2\left\langle \eth\overrightarrow{X}, \overrightarrow{Y} \right\rangle$$

is a real quantity. ∎

But the structure of $\Im\mathbb{H}$ as a Lie Algebra tells us that for two imaginary quaternions

$$[a,b]_{\mathbb{H}} = 2\mathbf{a}\times\mathbf{b} = 2\Im(ab).$$

Thus we may rewrite the Euler equations in a spinorial form as

$$\frac{\partial\omega}{\partial t} = -\Im\eth\Im(\omega u). \tag{4}$$

Since $\omega = \eth u$ we may rewrite again to get

$$\Re\omega = 0; \tag{5}$$

$$\Im\eth\left(\frac{\partial u}{\partial t} + \Im(\omega u)\right) = 0. \tag{6}$$

The first of these equations is precisely the incompressibility condition. The second may be restated as

$$\frac{\partial u}{\partial t} + \Im(\omega u) = -\nabla\left(p + \frac{|u|^2}{2}\right)$$

which is precisely the original Euler equation for incompressible flow. The question now is: can we encode both of these equations into one separate equation? We first need to make the following observation.

Proposition 2.3 *If $\boldsymbol{u}$ is a solution to the inviscid Navier-Stokes equations*

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p$$

then

$$\frac{\mathrm{D}(\operatorname{div}\boldsymbol{u})}{\mathrm{D}t} = -\Delta p - \partial_i u_j \partial_j u_i$$

Proof

$$\begin{aligned}
\frac{\mathrm{D}(\operatorname{div}\boldsymbol{u})}{\mathrm{D}t} &= \operatorname{div}\frac{\partial\boldsymbol{u}}{\partial t} + \boldsymbol{u}.\nabla\left(\operatorname{div}\boldsymbol{u}\right) \\
&= -\operatorname{div}\left(\nabla p + \boldsymbol{u}.\nabla\boldsymbol{u}\right) + \boldsymbol{u}.\nabla\left(\operatorname{div}\boldsymbol{u}\right) \\
&= -\Delta p - \partial_i\left(u_j\partial_j u_i\right) + u_j\partial_j\partial_i u_i \\
&= -\Delta p - \partial_i u_j \partial_j u_i
\end{aligned}$$

as required. ■

Now in [4] Gibbon states that the incompressibility condition is precisely

$$-\Delta p = \partial_i u_j \partial_j u_i$$

Thus following Gibbon, we have the following equations for incompressible, inviscid flow

$$\begin{aligned} \frac{\mathrm{D}(\operatorname{div}\boldsymbol{u})}{\mathrm{D}t} &= 0 \\ \frac{\mathrm{D}(\operatorname{curl}\boldsymbol{u})}{\mathrm{D}t} &= S \operatorname{curl}\boldsymbol{u} \end{aligned}$$

where S is the symmetric part of the jacobian matrix of $\boldsymbol{u}$.

Now set $\vec{\omega} = \eth\boldsymbol{u}$. Since $\boldsymbol{u}$ is regarded as a purely imaginary quaternion we have

$$\vec{\omega} = \eth(0, \boldsymbol{u}) = (-\operatorname{div}\boldsymbol{u}, \operatorname{curl}\boldsymbol{u})$$

Hence we may write

$$\frac{\mathrm{D}\vec{\omega}}{\mathrm{D}t} = S\vec{\omega}$$

where we extend S to act as 0 on purely real quaternions.

2.3 Symplectic Geometry and the Euler Equations

Let G be a Lie group with a right-invariant metric on its Lie algebra $\mathfrak{g}$. Set $M = G \times \mathfrak{g}$, and define $\theta \in \Omega^1(M)$ by

$$\theta_{(g,v)}(\xi_g, \eta) = \langle R_{g^{-1}}\xi_g, v\rangle$$

where R_g (L_g) is the right (left) translation by g on the tangent bundle of G and $\xi_g \mathrm{T}_g G$. This is the pull back of the canonical form on T^*G and modified by the Riesz representation. The symplectic form on M will therefore be $\mathrm{d}\theta$.

Now allow the group to act on M by

$$g : (h, v) \mapsto (hg^{-1}, v).$$

Now

$$\begin{aligned} (g^*\theta)_{(h,v)}(\xi_h, \eta) &= \theta_{(hg^{-1},v)}(R_{g^{-1}}\xi_h, \eta) \\ &= \left\langle v, R_{(hg^{-1})^{-1}} R_{g^{-1}}\xi_h \right\rangle \\ &= \langle v, R_{gh^{-1}} R_{g^{-1}}\xi_h\rangle \\ &= \langle v, R_{h^{-1}} R_g R_{g^{-1}}\xi_h\rangle \\ &= \langle v, R_{h^{-1}}\xi_h\rangle \\ &= \theta_{(h,v)}(\xi_h, \eta) \end{aligned}$$

Thus the action preserves θ and therefore must be symplectic.

The infinitesimal vector field is given for each $\xi \in \mathfrak{g}$ by

$$X_\xi(g,v) = (-L_g\xi, 0).$$

Now

$$\begin{aligned} X_\xi \lrcorner \mathrm{d}\theta &= \mathcal{L}_{X_\xi}\omega - \mathrm{d}(X_\xi \lrcorner \theta) \\ &= -\mathrm{d}(X_\xi \lrcorner \theta) \end{aligned}$$

since the action is symplectic.

Now

$$(X_\xi \lrcorner \theta)(g,v) = \langle v, -R_{g^{-1}}L_g\xi\rangle = \langle v, \mathrm{Ad}(g)\xi\rangle$$

hence the moment map for this action is

$$\mu(g,v) = -\mathrm{Ad}^*(g^{-1})v.$$

In the case of the Euler equations for the rigid body in $\mathbb{R}^3$ this is simply the statement that the angular momentum in the body lies on in the same coadjoint orbit as the angular momentum in space.

For the case of G being the appropriate Sobolev closure of $\mathcal{D}\mathrm{iff}()\mathrm{vol}^0(\mathbb{R}^3)$ the volume preserving diffeomorphisms of $\mathbb{R}^3$ which are the identity outside a compact set (or which go to the identity sufficiently rapidly on approaching ∞), this says that the vector field

$$\mathrm{Ad}(\phi_t)\omega_{\mathbf{t}}$$

is constant (where ϕ_t is the 1-parameter family of diffeomorphisms generated by u_t). This gives precisely the Euler Equations for an ideal fluid

$$\frac{\partial \boldsymbol{\omega}_t}{\partial t} + [\boldsymbol{u}_t, \boldsymbol{\omega}_t] = 0.$$

2.4 Topology of solutions to the Euler Equations

In [1], Arnol'd states a wonderful classification of particular stationary flows, i.e for those flows for which

$$\boldsymbol{\omega} \times \boldsymbol{u} = \nabla\left(\frac{|\boldsymbol{u}|^2}{2} + p\right)$$

and $\boldsymbol{\omega}$ and $\boldsymbol{u}$ are linearly independent. These are classified as generalised tori (i.e tori which may have one or two generating circles at infinity thus topologically tubes and copies of $\mathbb{R}^2$. We can say more or less the same in the general case. If we set

$$\boldsymbol{v} = (1, \boldsymbol{u})$$

and

$$\boldsymbol{w} = (0, \boldsymbol{\omega})$$

where the first co-ordinate is that of time, then the Euler Equations become a simple equation of vector fields on $\mathbb{R}^4$,

$$[\boldsymbol{v}, \boldsymbol{w}] = 0.$$

If these vector fields are linearly independent over $\mathbb{R}^4$ then by the Frobenius theorem (see for example [7]) they foliate $\mathbb{R}^4$ by integral surfaces N_{q_0} for each $q_0 \in \mathbb{R}^4$. By definition N_{q_0} is the set of points in $\mathbb{R}^4$ that can be described as $\phi^{\boldsymbol{v}}_{s_1} \circ \phi^{\boldsymbol{w}}_{s_2}(\mathbf{x_0})$ where $\phi^{\boldsymbol{v}}_s, \phi^{\boldsymbol{w}}_s$ are the flows associated with $\boldsymbol{v}, \boldsymbol{w}$ respectively and s_1, s_2 lie in the maximal intervals of definition for the appropriate flows. The Frobenius theorem says that provided $\omega \neq 0$, N_{q_0} is a 2 dimensional submanifold (a surface) of $\mathbb{R}^4$.

These surfaces have an action of $\mathbb{R}^2$ and thus they have to be topologically equivalent to generalised tori [1]. The 3-dimensional solutions therefore lie on the projections of these invariant tori to the space co-ordinates. The problems then result when the velocity and the vorticity become linearly dependent. This can obviously only happen when $\boldsymbol{w}$ vanishes, i.e. when the vorticity vanishes.

Now the flow of the vector field $\boldsymbol{v}$ is not compact since the first coordinate is 1, indicating affine linear growth along the t-axis in $\mathbb{R}^4$. Thus we have the following theorem:

Theorem 2.4 *There are only two differential topological possibilities for the integral surface in $\mathbb{R}^4$ namely the tube (i.e diffeomorphic to $\mathbb{R} \times \mathrm{S}^1$) and the sheet (i.e. diffeomorphic to $\mathbb{R}^2$).*

Thus the problem reduces to telling these situations apart.

Remark 2.5 *Given any oriented Riemannian 3-manifold M, we can form the volume preserving vector fields $\boldsymbol{v}$ and $\boldsymbol{w}$ on the manifold $\mathbb{R} \times M$ using the notion of curl given by the metric and the exterior derivative on M. The Euler Equations will take the form*

$$[\boldsymbol{v}, \boldsymbol{w}] = 0.$$

We to make a remark on the nature of $\phi^{\boldsymbol{v}}_t$. Recall that

$$\boldsymbol{v} = (1, \boldsymbol{u}).$$

Thus in solving the differential equation

$$\begin{aligned} \left(\frac{\partial t(s)}{\partial s}, \frac{\partial \boldsymbol{x}(s)}{\partial s} \right) &= (1, \boldsymbol{u}\left(t(s), \boldsymbol{x}(s)\right)) \\ (t(0), \boldsymbol{x}(0)) &= (t_0, \boldsymbol{x}_0) \end{aligned}$$

we see that

$$\phi^{\boldsymbol{v}}_s(t_0, \boldsymbol{x}_0) = (s + t_0, \psi_s\left(t_0, \boldsymbol{x}_0\right))$$

Now since $\phi^{\boldsymbol{v}}_s$ is a 1-parameter family of diffeomorphisms and the restriction of each $\phi^{\boldsymbol{v}}_s$ to the line $\{(t, \boldsymbol{x}_0) | t \in \mathbb{R}\}$ for fixed $\boldsymbol{x}_0 \in \mathbb{R}^3$ is independent of $\boldsymbol{x}_0$, we have the result

Proposition 2.6 *The mappings given by*

$$\psi_{(s,t)}(\boldsymbol{x}) = \psi_s(t, \boldsymbol{x})$$

is a 2-parameter family of diffeomorphisms of $\mathbb{R}^3$ with inverse $\psi_{(-s,t)}$.

Already we can see some results for classical fluid dynamics.

Proposition 2.7 *If $\boldsymbol{w}$ vanishes at $q \in \mathbb{R}^4$, then it vanishes along $\phi_t^{\boldsymbol{v}}(q)$ for all t in the interval of definition of the flow.*

Proof
This is a simple calculation:for any s in the definition of the flow of $\boldsymbol{w}$

$$\begin{aligned} \phi_s^{\boldsymbol{w}}\left(\phi_t^{\boldsymbol{v}}(q)\right) &= \phi_t^{\boldsymbol{v}}\left(\phi_s^{\boldsymbol{w}}(q)\right) \\ &\quad \text{since the flows commute by the Euler equations} \\ &= \phi_t^{\boldsymbol{v}}(q) \\ &\quad \text{since } \boldsymbol{w} \text{ vanishes at } p. \end{aligned}$$

Since the flow of $\boldsymbol{w}$ fixes $\phi_t^{\boldsymbol{v}}(q)$, $\boldsymbol{w}$ must vanish this point also. ■

Corollary 2.8 *If $\boldsymbol{w}$ is non zero at $q \in \mathbb{R}^4$ then it is nonzero on all of N_q.*

Indeed, from the fact that the Euler equations are essentially conservation of Angular momentum, if the vorticity vanishes globally at time 0 then it must vanish at all points of the flow, and we are in a well-known situation of the vortex-free ideal fluid.

We also have the following Corollary.

Corollary 2.9 *Let $B_0 \subset \mathbb{R}^3$ be a region on which the vorticity at time $t = 0$ vanishes. Let B_t be the image of B_0 under the fluid flow at time t, then the vorticity vanishes on all of B_t and the two regions are diffeomorphic and, if compact, they have the same volume. Moreover if ∂B_0 is a 2-dimensional submanifold of $\mathbb{R}^3$ then so is ∂B_t for all t for which the flow is defined.*

Proof
This follows immediately from Propositions 2.7 and 2.6 since $B_t = \psi_{(t,0)}(B_0)$. ■

Remark 2.10 *The issue of the differentiability class of $\boldsymbol{u}$ is of interest to students of fluid-dynamics. For the two dimensional Euler Equations, the vorticity patch problem was solved using analysis. In this case the vorticity had support on certain patches within $\mathbb{R}^3$ but was discontinuous across the boundary. Our results prove the vorticity patch problem for the 3-dimensional Euler Equations certainly for $\mathcal{C}^2$-solutions.*

2.5 Tubes or Sheets?

How can we tell whether the integral surfaces N_q are tubes or sheets? The vector field $\boldsymbol{v}$ certainly does not have a compact flow and is never perpendicular to $\frac{\partial}{\partial t}$. Let $\tilde{\boldsymbol{v}}$ and $\tilde{\boldsymbol{w}}$ be unit vector fields pointing in the directions of $\boldsymbol{v}$ and $\boldsymbol{w}$ respectively. By construction we know that $\tilde{\boldsymbol{w}}$ is always perpendicular to $\frac{\partial}{\partial t}$, and also that $\tilde{\boldsymbol{v}}$ is at worst asymptotically perpendicular to $\frac{\partial}{\partial t}$ (i.e. perpendicular at infinity).

Proposition 2.11 *The topological type of N_q is determined completely by $\boldsymbol{w}$.*

Proof
If $\boldsymbol{w}$ vanishes, then N_q is topologically a line by Proposition 2.7, so assume that $\boldsymbol{w}(q) \neq 0$. Set $\Pi_t = \{(t, \boldsymbol{x}) \in \mathbb{R}^4 | \boldsymbol{x} \in \mathbb{R}^3\}$ and let the maximal interval of the flow $\phi_t^{\boldsymbol{v}}$ through q be I_q. Then the above paragraph states that $\Pi_t \pitchfork N_q$ thus the intersection of N_q with the hypersurface Π_t is a 1-dimensional manifold. We therefore have a surjection

$$\pi : N_q \longrightarrow I_q$$

given by $\pi(q) = t$ such that $q \in \Pi_t \cap N_q$. Since π extends to all of $I_q \times \mathbb{R}^3$ as linear projection, it is a proper submersion onto any closed subinterval of I_q. Thus by the Ehresmann fibration theorem, π is a trivial fibration over every closed subinterval of I_q and thus extends to a fibration over all of I_q. Now by construction $\boldsymbol{w}(t, \boldsymbol{x})$ is perpendicular to $\frac{\partial}{\partial t}(t, \boldsymbol{x})$ thus lies in the tangent space to Π_t which we identify with Π_t. Thus $\boldsymbol{w}$ is a vector field on $\Pi_t \cap N_q$ which forms the fibre over t. The fibre over t is a connected 1-dimensional manifold since N_q is connected and thus can either be topologically a circle or a line (thus providing another proof of Theorem 2.4). Since the 1-dimensional fibre is formed as the integral curve of $\boldsymbol{w}$, a non-zero vector field, the theorem is proved. ■

Corollary 2.12 *The topology of N_q is completely determined by the initial vorticity.*

Proof
Since N_q is a fibration over an interval, its topology is determined by knowing the topology of one fibre. A fibre is determined by an integral curve of $\boldsymbol{\omega}$ at a fixed time. Thus the topology of N_q is determined by knowledge of the integral curve of $\boldsymbol{\omega}$ at $t = 0$. ■

This provides a geometrical proof of the fact that the topology of vortex lines is invariant under fluid flow.

3 Applying Differential Geometry

We have defined the integral surfaces of the fluid in $\mathbb{R}^4$ as those generated by the 4-velocity and the 4-vorticity. Indeed these integral surfaces arises as the orbits of an action of $\mathbb{R}^2$ on $\mathbb{R}^4$, and thus we have a (possibly singular) foliation of $\mathbb{R}^4$ by sheets or cylinders or lines in the case of irrotational flow. The question that we try to answer now is whether any action of $\mathbb{R}^2$ on $\mathbb{R}^4$ arises as an inviscid fluid flow.

3.1 Actions of $\mathbb{R}^2$ on $\mathbb{R}^4$

Recall that the 4-velocity of a fluid was defined as

$$\boldsymbol{v} = (1, \boldsymbol{u})$$

where $\boldsymbol{u}$ is the 3-velocity. For a group action to generate a fluid flow, it is necessary that the orbits should be transverse to the hyperplanes perpendicular to the t-axis. Indeed, since the integral curve of $\boldsymbol{v}$ will have t-component $t \mapsto t+s$, the group action should have one generator which acts as translation in the t-direction. It will not necessarily leave the t-axis invariant.

3.2 When all leaves of the foliation are flat planes

Let us consider the group action of $\mathbb{R}^2$ on $\mathbb{R}^4$ given by

$$(s_1, s_2) \cdot (t, \boldsymbol{x}) = (t + s1, \boldsymbol{x} + s_1\boldsymbol{\alpha} + s_2\boldsymbol{\beta})$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are constant orthogonal unit vectors in $\mathbb{R}^3$, and $\boldsymbol{x}$ as always is (x, y, z). This group action foliates $\mathbb{R}^4$ by planes parallel to the plane generated by

$$\begin{aligned} X_1 &= (1, \boldsymbol{\alpha}) \\ X_2 &= (0, \boldsymbol{\beta}) \end{aligned}$$

We would like to find solutions of the Euler equations such that the integral surfaces described above form the leaves of this foliation. Assume that $\boldsymbol{\beta}$ is non zero, and let $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}$ form a positively oriented orthonormal frame for $\mathbb{R}^3$. Thus the 4-velocity $\boldsymbol{v}$ must have form

$$\boldsymbol{v} = X_1 + fX_2 = (1, \boldsymbol{\alpha} + f\boldsymbol{\beta})$$

since the first coordinate must be 1. Hence the fluid flow lines are straight lines! Thus the 4-vorticity is given by

$$\boldsymbol{w} = (0, \operatorname{curl}(\boldsymbol{\alpha} + f\boldsymbol{\beta})) = (0, \nabla f \times \boldsymbol{\beta}).$$

Where ∇ is the 3-gradient. We shall use $\boldsymbol{\nabla}$ for the 4-gradient. We require $\boldsymbol{v}$ to be divergence free, this means that

$$0 = \operatorname{div}(f\boldsymbol{\beta}) = \langle \nabla f, \boldsymbol{\beta} \rangle$$

Thus we can use the method of characteristics to show that

$$f(t, \boldsymbol{x}) = F(t, \langle \boldsymbol{x}, \boldsymbol{\beta}_1 \rangle, \langle \boldsymbol{x}, \boldsymbol{\beta}_2 \rangle).$$

Set $a_i = \langle \boldsymbol{x}, \boldsymbol{\beta}_i \rangle$ for $i = 1, 2$. Hence

$$\nabla f = \frac{\partial F}{\partial a_1}\boldsymbol{\beta}_1 + \frac{\partial F}{\partial a_2}\boldsymbol{\beta}_2,$$

and

$$\boldsymbol{w} = (0, \frac{\partial F}{\partial a_2}\boldsymbol{\beta}_1 - \frac{\partial F}{\partial a1}\boldsymbol{\beta}_2).$$

Now we require that $\boldsymbol{w}$ lie in the span of X_1 and X_2. Clearly it cannot have any component parallel to X_1 since the first coordinate is 0. But $\boldsymbol{w}$ is also perpendicular to $\boldsymbol{\beta}$ by construction of $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$. It is clear then that $\boldsymbol{w}$ must vanish and that F is a function of time only. Thus the fluid flow in this case depends only on a function of time.

Now, the Euler equations tell us

$$\begin{aligned} -\nabla p &= \mathrm{D}\boldsymbol{v}(\boldsymbol{v}) \\ &= \frac{\partial v}{\partial t} \\ &= \frac{\partial F}{\partial t} X_2 \\ &= \left(0, \frac{\partial F}{\partial t}\boldsymbol{\beta}\right) \end{aligned}$$

Thus the pressure is given by

$$p(t, \boldsymbol{x}) = -\frac{\partial F}{\partial t}\langle \boldsymbol{\beta}, \boldsymbol{x}\rangle.$$

This apparently tells us very little. But looking deeper we see that the flow lines of $\boldsymbol{v}$ are geodesics on the leaves of the foliation.For fixed t, the level sets of p are only submanifolds of $\mathbb{R}^3$ on the provisor that ∇p does not vanish, i.e $\frac{\partial F}{\partial t} \neq 0$, in the case where the level sets are submanifolds, a point of $\mathbb{R}^3$ lies in the level set at time t if it has position vector

$$\lambda_1 \boldsymbol{\beta}_1 + \lambda_2 \boldsymbol{\beta}_2 - \frac{k}{\frac{\partial F}{\partial t}} \boldsymbol{\beta}$$

for constant k. Let

$$M_k = \left\{ \lambda_1 \boldsymbol{\beta}_1 + \lambda_2 \boldsymbol{\beta}_2 - \frac{k}{\frac{\partial F}{\partial t}} \boldsymbol{\beta} \right\}$$

then for each $q \in M_k$, $p(q) = k$ and if k is a regular value for p, then M_k can be shown to be a flat copy of $\mathbb{R}^3$ within $\mathbb{R}^4$, the geodesics of course being straight lines.

3.3 Geodesic curvature of solutions

Recall that given a Riemannian manifold M with Levi-Civita connection D, for a given curve $\gamma : [0, \varepsilon) \longrightarrow M$ we can measure by how much this curve fails to be a geodesic by defining the geodesic curvature k_γ by

$$k_\gamma = |\mathrm{D}_\tau \tau|$$

where τ is the unit length tangent vector field to γ. We know that if k_γ vanishes on an open sub-interval of $[0, \varepsilon)$ then it is necessarily a geodesic on that sub interval.

In our case $M = \mathbb{R}^4$ and D is the Euclidean derivative. So we can compute the geodesic curvature of the flow lines of a incompressible Euler fluid as curves in $\mathbb{R}^4$ by examining

$$k_\gamma = |\mathrm{D}_{\widehat{\boldsymbol{v}}} \widehat{\boldsymbol{v}}| = |\widehat{\boldsymbol{v}}.\boldsymbol{\nabla}\widehat{\boldsymbol{v}}|$$

where $\widehat{\boldsymbol{v}} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|}$ and γ is a flow line in $\mathbb{R}^4$. It is a simple calculation to show that

$$\widehat{\boldsymbol{v}}.\boldsymbol{\nabla}\widehat{\boldsymbol{v}} = \frac{\mathrm{D}_{\boldsymbol{v}}\boldsymbol{v}}{|\boldsymbol{v}|^2} - \left(\widehat{\boldsymbol{v}}.\frac{\mathrm{D}_{\boldsymbol{v}}\boldsymbol{v}}{|\boldsymbol{v}|^2}\right)\widehat{\boldsymbol{v}}$$

where we recall that $\boldsymbol{\nabla}$ is the Euclidean spacetime gradient Thus

$$k_\gamma^2 = \frac{1}{|\boldsymbol{v}|^4}\left(|\mathrm{D}_{\boldsymbol{v}}\boldsymbol{v}|^2 - (\widehat{\boldsymbol{v}}.\mathrm{D}_{\boldsymbol{v}}\boldsymbol{v})^2\right)$$

Thus we have an upper bound

$$k_\gamma \leq \frac{|\mathrm{D}_{\boldsymbol{v}}\boldsymbol{v}|}{|\boldsymbol{v}|^2}$$

with equality when $\mathrm{D}_{\boldsymbol{v}}\boldsymbol{v}$ is perpendicular to $\boldsymbol{v}$. Now the Euler equations tell us that

$$\mathrm{D}_{\boldsymbol{v}}\boldsymbol{v} = -\nabla p$$

so we have

$$k_\gamma \leq \frac{|\nabla p|}{|\boldsymbol{v}|^2}$$

where we recall ∇ is the spatial gradient. This makes sense when we realise that in the absence of pressure, the flow lines are straight as exemplified in this calculation. One should note that $|\boldsymbol{v}|^2 = 1 + |\boldsymbol{u}|^2 \geq 1$, so we can make cruder estimates on the geodesic curvature in

$$k_\gamma \leq |\nabla p|.$$

It would be helpful if we could find a lower bound, and this is also likely to be dependent on the value of $|\nabla p|$. If we could find such a beast then we would be able to find the areas of high turbulence where the geodesic curvature is expected to be high.

3.4 Curvature of Integral surfaces

We recall that if D is the Levi-Civita connection on a Riemannian manifold then for any sub-manifold, the second fundamental form is the projection of D to the normal bundle. For the case of $\mathbb{R}^4$, D is the standard Euclidean derivative and if ν_1 and ν_2 are two orthogonal unit normals to N_q then the second fundamental form $\mathbb{II}$ is a normal-valued symmetric tensor on $\mathbb{R}^4$ given by

$$\mathbb{II}_q(X,Y) = ((\mathrm{D}_X Y).\nu_1(\boldsymbol{x}))\,\nu_1(\boldsymbol{x}) + ((\mathrm{D}_X Y).\nu_2(\boldsymbol{x}))\,\nu_2(\boldsymbol{x})$$

for any X, Y tangent to N_q at $\boldsymbol{x}$. Let us define the projection $\Pi^\nu : \mathrm{T}\mathbb{R}^4 \longrightarrow \mathcal{V}N_q$ given by

$$\Pi^\nu(\mathbf{a}) = \mathbf{a} - (\mathbf{a}.\nu_1)\,\nu_1 - (\mathbf{a}.\nu_2)\,\nu_2.$$

Thus

$$\mathbb{II}(\mathbf{X},\mathbf{Y}) = \Pi^\nu \mathrm{D}_X Y$$

For the Ideal fluid satisfying the Euler Equations, the tangent space to the integral surface is the span of the 4-velocity $\boldsymbol{v}$ and the 4-vorticity $\boldsymbol{w}$, thus the second fundamental form is completely determined by its values on $\boldsymbol{v}$ and $\boldsymbol{w}$. In fact, we have

$$\begin{aligned}
\mathbb{II}(\boldsymbol{v},\boldsymbol{v}) &= \Pi^\nu \mathrm{D}_{\boldsymbol{v}}\boldsymbol{v} = -\Pi^\nu \nabla p \\
\mathbb{II}(\boldsymbol{v},\boldsymbol{w}) &= \Pi^\nu \mathrm{D}_{\boldsymbol{v}}\boldsymbol{w} = \Pi^\nu \mathrm{S}\boldsymbol{w} \\
\mathbb{II}(\boldsymbol{w},\boldsymbol{w}) &= \Pi^\nu \mathrm{D}_{\boldsymbol{w}}\boldsymbol{w}
\end{aligned}$$

where S is the strain matrix of the fluid. This ought to tell us about how the integral surfaces sit inside $\mathbb{R}^4$, and how they bend.

Now, the Gaussian curvature of a surface in $\mathbb{R}^n$ is given by the formula

$$K = \frac{|\mathbb{II}(X,Y)|^2 - \mathbb{II}(X,X).\mathbb{II}(Y,Y)}{|X|^2|Y|^2 - (X.Y)^2}$$

so the gaussian curvature of an integral surface in $\mathbb{R}^4$ will be given by

$$\begin{aligned} K &= \frac{|\mathbb{II}(\boldsymbol{v},\boldsymbol{w})|^2 - \mathbb{II}(\boldsymbol{w},\boldsymbol{w}).\mathbb{II}(\boldsymbol{v},\boldsymbol{v})}{|\boldsymbol{v}|^2|\boldsymbol{w}|^2 - (\boldsymbol{v}.\boldsymbol{w})^2} \\ &= \frac{|\Pi^\nu \mathrm{S}\boldsymbol{w}|^2 + \mathbb{II}(\boldsymbol{w},\boldsymbol{w}).\Pi^\nu\nabla p}{|\boldsymbol{v}|^2|\boldsymbol{w}|^2 - (\boldsymbol{v}.\boldsymbol{w})^2} \\ &= \frac{|\Pi^\nu \mathrm{S}\boldsymbol{w}|^2 + \mathbb{II}(\boldsymbol{w},\boldsymbol{w}).\nabla p}{|\boldsymbol{v}|^2|\boldsymbol{w}|^2 - (\boldsymbol{v}.\boldsymbol{w})^2} \end{aligned}$$

It appears that we get no real information out of these identities. But let us consider a couple of examples

Example 3.1 ***Two dimensional flow***

In this situation we know that $\boldsymbol{v}.\boldsymbol{w} = 0$, $\mathrm{D}_{\boldsymbol{w}}\boldsymbol{w} = 0$ *and* $S\boldsymbol{w} = 0$. *Thus the second fundamental form is determined by*

$$\begin{aligned} \mathbb{II}(\boldsymbol{v},\boldsymbol{v}) &= -\Pi^\nu\nabla p \\ \mathbb{II}(\boldsymbol{v},\boldsymbol{w}) &= 0 \\ \mathbb{II}(\boldsymbol{w},\boldsymbol{w}) &= 0 \end{aligned}$$

So the Gaussian curvature is 0 and we can tell that the surface only bends in the $\boldsymbol{v}$ *directions, the degree of bending controlled by the part of* ∇p *that is normal to this surface.*

We may also see that if the spatial gradient of the pressure lies in the span of $\boldsymbol{v}$ *and* $\boldsymbol{w}$, *then the integral surface is necessarily a flat plane. In which case we return to the previous section.*

Now we have our lower bound for the geodesic curvature of flow lines. It is clear that

$$\mathbb{II}(\widehat{\boldsymbol{v}},\widehat{\boldsymbol{v}}) = -\frac{\Pi^\nu\nabla p}{|\boldsymbol{v}|^2}.$$

If we set

$$\widetilde{\boldsymbol{w}} = \frac{|\boldsymbol{v}|^2\boldsymbol{w} - (\boldsymbol{w}.\boldsymbol{v})\,\boldsymbol{v}}{|\boldsymbol{w}|^2|\boldsymbol{v}|^2 - (\boldsymbol{v}.\boldsymbol{w})^2}$$

then this is a unit vector orthogonal to $\boldsymbol{v}$ which lies in the plan spanned by $\boldsymbol{v}$ and $\boldsymbol{w}$. Thus

$$\Pi^\nu a = a - (a.\widehat{\boldsymbol{v}})\widehat{\boldsymbol{v}} - (a.\widetilde{\boldsymbol{w}})\widetilde{\boldsymbol{w}}$$

Thus one can see that

$$|\Pi^\nu a| \leq |a - (a.\widehat{\boldsymbol{v}})\widehat{\boldsymbol{v}}| \leq |a|.$$

Applying this to $a = \mathrm{D}_{\widehat{\boldsymbol{v}}}\widehat{\boldsymbol{v}}$ we have

$$|\mathbb{II}(\widehat{\boldsymbol{v}},\widehat{\boldsymbol{v}})| \leq k_\gamma.$$

Hence we have

$$\frac{|\Pi^\nu\nabla p|}{|\boldsymbol{v}|^2} \leq k_\gamma \leq \frac{|\nabla p|}{|\boldsymbol{v}|^2} \leq |\nabla p|$$

since $|\boldsymbol{v}|^2 = 1 + |\boldsymbol{u}|^2$ which gives the sought-for lower bound.

3.5 Geometric constraints for the regularity of the Euler Equations

The famous Beale-Kato-Majda condition guarantees the smoothness of solutions to the incompressible Euler equations on an interval $[0,T]$ iff

$$\int_0^T \|\boldsymbol{\omega}_t\|_\infty \mathrm{d}t < \infty.$$

In [3] Peter Constantin, Charles Fefferman and Andrew Majda show that this condition can be weakened under certain other constraints. The first of these constraints is that the vorticity direction vector

$$\boldsymbol{\xi} = \frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|}$$

be locally Lipschitz and if the angle $\phi(\boldsymbol{x},\boldsymbol{y},t))$ is defined by

$$\cos\phi(\boldsymbol{x},\boldsymbol{y},t) = \boldsymbol{\xi}(\boldsymbol{x},t).\boldsymbol{\xi}(\boldsymbol{y},t)$$

then for each $\varepsilon > 0$ there is $R(t) > 0$ such that whenever $|\boldsymbol{x}-\boldsymbol{y}| < \varepsilon$ we have

$$|\sin\phi(\boldsymbol{x},\boldsymbol{y},t)| < \frac{|\boldsymbol{x}-\boldsymbol{y}|}{R(t)}.$$

There is a stronger condition that we can put on the continuity of $\boldsymbol{\xi}$.

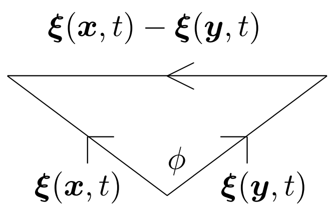
Definition 3.2 *We shall say that a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is uniformly locally Lipschitz if there are $L > 0$ and $K > 0$ such that whenever $|\boldsymbol{x}-\boldsymbol{y}| < L$ we have*

$$|f(\boldsymbol{x}) - f(\boldsymbol{y})| < K|\boldsymbol{x}-\boldsymbol{y}|.$$

Proposition 3.3 *The Constantin-Fefferman-Majda condition is equivalent to $\boldsymbol{\xi}$ being spatially uniformly locally Lipschitz.*

Proof
By spatially uniformly locally Lipschitz we mean that there are $L > 0$ and $K(t) > 0$ such that whenever $|\boldsymbol{x}-\boldsymbol{y}| < L$ we have

$$|\boldsymbol{\xi}(\boldsymbol{x},t) - \boldsymbol{\xi}(\boldsymbol{y},t)| < K(t)|\boldsymbol{x}-\boldsymbol{y}|.$$



By the sine rule

$$\begin{aligned}\frac{\sin\phi(\boldsymbol{x},\boldsymbol{y},t)}{|\boldsymbol{\xi}(\boldsymbol{x},t)-\boldsymbol{\xi}(\boldsymbol{y},t)|} &= \frac{\sin\left(\frac{\pi-\phi(\boldsymbol{x},\boldsymbol{y},t)}{2}\right)}{|\boldsymbol{\xi}(\boldsymbol{y},t)|} \\ &= \cos\frac{\phi(\boldsymbol{x},\boldsymbol{y},t)}{2}.\end{aligned}$$

So we have

$$\sin\phi(\boldsymbol{x},\boldsymbol{y},t) = |\boldsymbol{\xi}(\boldsymbol{x},t)-\boldsymbol{\xi}(\boldsymbol{y},t)|\cos\frac{\phi(\boldsymbol{x},\boldsymbol{y},t)}{2} \tag{7}$$

and

$$2\sin\frac{\phi(\boldsymbol{x},\boldsymbol{y},t)}{2} = |\boldsymbol{\xi}(\boldsymbol{x},t)-\boldsymbol{\xi}(\boldsymbol{y},t)|. \tag{8}$$

Hence, by (7)

$$|\sin\phi(\boldsymbol{x},\boldsymbol{y},t)| \le |\boldsymbol{\xi}(\boldsymbol{x},t)-\boldsymbol{\xi}(\boldsymbol{y},t)| < K(t)|\boldsymbol{x}-\boldsymbol{y}|.$$

So we may take $R(t) = \frac{1}{K(t)}$ for the Constantin-Fefferman-Majda condition. So we have shown that if $\boldsymbol{\xi}$ is spatially uniformly locally Lipschitz, then it satisfies the Constantin-Fefferman-Majda condition.

On the other hand, if there are $L > 0$ and $R(t) > 0$ such that

$$|\sin\phi(\boldsymbol{x},\boldsymbol{y},t)| < \frac{|\boldsymbol{x}-\boldsymbol{y}|}{R(t)}$$

whenever $|\boldsymbol{x}-\boldsymbol{y}| < L$, then by taking a smaller L if necessary, we may assume $|\phi(\boldsymbol{x},\boldsymbol{y},t)| < \frac{2\pi}{3}$. This means that

$$\sin\phi > \sin\frac{\phi}{2}.$$

Thus by (8) we have

$$2\left|\sin\frac{\phi(\boldsymbol{x},\boldsymbol{y},t)}{2}\right| = |\boldsymbol{\xi}(\boldsymbol{x},t)-\boldsymbol{\xi}(\boldsymbol{y},t)|.$$

But our condition above means that

$$|\boldsymbol{\xi}(\boldsymbol{x},t)-\boldsymbol{\xi}(\boldsymbol{y},t)| < 2|\sin\phi(\boldsymbol{x},\boldsymbol{y},t)| < \frac{2|\boldsymbol{x}-\boldsymbol{y}|}{R(t)}.$$

Hence taking $K(t) = \frac{2}{R(t)}$, $\boldsymbol{\xi}$ is spatially uniformly locally Lipschitz. ■

Now Constantin, Fefferman and Majda show that if their condition holds for $\boldsymbol{\xi}$ then

$$\sup_{x\in\mathbb{R}^3}\int_0^T\int_{B_\varepsilon(0)}\frac{|\boldsymbol{\omega}(\boldsymbol{x}+\boldsymbol{y})|}{|\boldsymbol{y}|^2R(t)}\mathrm{d}\boldsymbol{y}\mathrm{d}t < \infty.$$

is a necessary and sufficient condition for the smoothness of solutions for the Euler Equations on the interval $[0,T]$. Thus we have

Corollary 3.4 *If $\boldsymbol{\xi}$ is spatially uniformly locally Lipschitz, then*

$$\sup_{x\in\mathbb{R}^3}\int_0^T\int_{B_\varepsilon(0)}\frac{|\boldsymbol{\omega}(\boldsymbol{x}+\boldsymbol{y})|}{|\boldsymbol{y}|^2R(t)}\mathrm{d}\boldsymbol{y}\mathrm{d}t<\infty.$$

is a necessary and sufficient condition for the smoothness of solutions for the Euler Equations on the interval $[0,T]$.

If further $\boldsymbol{\xi}$ is fully uniformly locally Lipschitz as a vector field on $\mathbb{R}^4$ (that is we demand R to be independent of t) then

$$\int_0^T\|\boldsymbol{\omega}_t\|_{L^p_{unif,loc}}\mathrm{d}t<\infty$$

for $p>3$ is equivalent to the solution being smooth on the interval $[0,T]$.

In fact, Constantin, Fefferman and Majda go further and show that for $\boldsymbol{\xi}$ fully uniformly locally Lipschitz and bounded velocity, we can take $p=1$ in the above equivalence.

4 The Euler equations and exterior differential systems

At present we haven't explored too closely whether the Euler Equations have a quaternionic description, or whether the results in [4] arise from the action of imaginary quaternions on $\mathbb{R}^3$ and extend no further. In order to investigate this, we use the notion of exterior differential systems which involves working on the 1-jet bundle of maps $\mathbb{R}^4\longrightarrow\mathbb{R}^4$.

4.1 General theory

We draw the theory from [5], which uses exterior differential systems (E.D.S) to show that minimal surfaces in $\mathbb{E}^3$ arise from solutions to the Cauchy-Riemann equations.

An E.D.S on a manifold M is a differential ideal $\mathfrak{I}$ in $\Omega^\bullet(M)$ providing a vehicle for generalising PDEs. The motivation for this is

Theorem 4.1 (Frobenius) *Given, $p\in M$, if $\mathfrak{I}\subset\Omega^\bullet(M)$ is a differential ideal (i.e. closed under* d*), then locally there is a unique submanfold Ξ of M containing p whose tangent bundle is maximally annihilated by $\mathfrak{I}$. That is, $\mu(v_1,\ldots,v_p)=0$ for all $v_i\in\Omega^0(\Xi;\mathrm{T}\Xi)$ and $\mu\in\mathfrak{I}$, and if $\Sigma'\subset\Sigma$ is a submanifold of M with dimension strictly larger than Σ, then there is $\mu'\in\mathfrak{I}$, and vector fields $v'_1,\ldots v'_p$ on Σ' with $\mu'(v'_1,\ldots,v'_p)\neq0$ (i.e. Σ is as large as is possible).*

Given a system of 1st order PDEs

$$F^i\left(\boldsymbol{x},\boldsymbol{u},\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{x}}\right)=0\tag{9}$$

with independent variables $\boldsymbol{x}\in\mathbb{R}^n$, dependent variables $\boldsymbol{u}\in\mathbb{R}^l$ and k-functions F^i we may pass to the 1-jet bundle $J:=\mathrm{J}^1(\mathbb{R}^n,\mathbb{R}^l)$ consisting equivalence classes of maps from

$\mathbb{R}^n$ to $\mathbb{R}^l$ under the relation that they agree up to and including the first derivative. As a space,

$$J \cong \mathbb{R}^n \times \mathbb{R}^l \times \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^l)$$

where the components x^i of $\boldsymbol{x}$, u^α of $\boldsymbol{u}$ and the jacobian matrix $\mathrm{D}\boldsymbol{u}_{\boldsymbol{x}}$ become independent variables on J. Thus the functions F^i become a single vector valued function on J

$$\mathbf{F} : J \longrightarrow \mathbb{R}^k$$

and solutions of the PDEs (9) form subspaces Ξ of the space $\Sigma \subset J$ given by $\Sigma = \mathbf{F}^{-1}(0)$. There are two natural differential ideals $\mathcal{I}, \mathcal{J}$ of $\Omega^\bullet(J)$. The ideal $\mathcal{I}$ is locally generated by the forms

$$\theta^\alpha = \mathrm{d}u^\alpha - A_i^\alpha \mathrm{d}x^i \quad \alpha = 1...l,\ i = 1...n$$

and $\mathcal{J}$ by the θ^α and $\mathrm{d}x^i$. This gives us the standard E.D.S on J. Since the ideals $\mathcal{I}$ and $\mathcal{J}$ are locally generated by 1-forms, the E.D.S is known as a Pfaffian differential system, and is called a linear Pfaffian system (L.P.S) if $\mathrm{d}\theta^\alpha \equiv 0$ modulo the ideal $\mathcal{J}$.

A solution to the PDE (9) is therefore an integral submanifold Ξ of $(\Sigma, \iota^*\mathcal{I}, \iota^*\mathcal{J})$ (where $\iota : \Sigma \hookrightarrow J$ is inclusion) on which $\iota^*(\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n)$ vanishes nowhere.

On this manifold Ξ, the coordinates u^α will depend on the x^i and the coordinates A_i^α will become $\frac{\partial u^\alpha}{\partial x^i}$. The condition that $\iota^*(\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n)$ is nowhere zero is the condition that independent variables remain independent variables.

It may be that $\mathcal{I}$ is a differential ideal only on a smaller submanifold Σ' of Σ in which case the solutions to the PDE satisfy a stronger relation, or $\mathcal{I}$ is a differential ideal nowhere meaning that there are no solutions of the PDE.

4.2 The Cartan algorithm for L.P.Ss

We are fortunate that given a linear Pfaffian system $(\mathcal{I}, \mathcal{J})$ on $\Sigma \subset J$ defined by the algebraic relation of the PDE, there is an algorithm that determines the integrability of the system and some data upon it. We follow [5] closely here. Let $\mathcal{I} = \langle \theta^\alpha \rangle$, $\mathcal{J} = \langle \theta^\alpha, \omega^j \rangle$ and $\mathrm{T}^*\Sigma = \langle \theta^\alpha, \omega^j, \pi^\varepsilon \rangle$ be a coframing of $\mathrm{T}^*\Sigma$ with respect to the filtration induced by $\mathcal{I}$ and $\mathcal{J}$. We will fix $p \in \Sigma$ and will now do all our work at that point. Let

$$\begin{aligned} V^* &= \frac{\mathcal{J}_p}{\mathcal{I}_p} \\ W^* &= \mathcal{I}_p \end{aligned}$$

with dual spaces V, W. Let also $v_i \in V$ and $w_\alpha \in W$ be the dual bases with respect to the ω_p^i and θ_p^α respectively. Let

$$\underline{\theta} = \begin{pmatrix} \theta^1 \\ \vdots \\ \theta^l \end{pmatrix} = \theta^\alpha \otimes w_\alpha.$$

Then

$$\mathrm{d}\underline{\theta} \equiv \underline{A} + \underline{T} \bmod \mathcal{I}$$

where $\underline{A}$ and $\underline{T}$ are l-vector of 2-forms given by

$$A^\alpha = A^\alpha_{\varepsilon i}\pi^\varepsilon \wedge \omega^i$$

and

$$T^\alpha = T^\alpha_{ij}\omega^i \wedge \omega^j$$

Set

$$T = T^\alpha_{ij} v^i \wedge v^j \otimes w_\alpha.$$

This T will play a crucial rôle in determining the integrability of the L.P.S. We define the vector space $\mathbb{A} \subset V^* \otimes W$ by

$$\mathbb{A} = \mathbb{A}_p = \operatorname{span}\{\pi_\varepsilon \lrcorner \underline{A}\}.$$

Now let $\delta : V^* \otimes V^* \otimes W \longrightarrow \Lambda^2(V^*) \otimes W$ be exterior multiplication on V^* and set

$$H := \frac{\Lambda^2(V^*) \otimes W}{\delta\left(\mathbb{A} \otimes V^*\right)}.$$

Let $[T] \in H$ denote the projection of T to H. This is called the torsion of the L.P.S.

We now construct some more vector spaces around $\mathbb{A}$. First the prolongation of $\mathbb{A}$ is defined as the space $\mathbb{A}^{(1)} = \delta'(\mathbb{A} \otimes V^*)$ where $\delta' : V^* \otimes V^* \otimes W \longrightarrow \odot^2(V^*) \otimes W$ is the symmetric product. Now we define the spaces

$$\mathbb{A}_{n-j} = \mathbb{A} \cap \big(\operatorname{span}\{v^1, \ldots, v^j\} \otimes W\big)$$

and for the sake of completeness $\mathbb{A}_0 = \mathbb{A}$. Define the characters of the L.P.S to be integers $\{s_1, \ldots, s_n\}$ where

$$\dim \mathbb{A}_{n-j} = \sum_{i=1}^{j} s_i.$$

Proposition 4.2 *[Cartan [2] p120]*

$$\dim \mathbb{A}^{(1)} \leq \sum_{i=0}^{n-1} \dim \mathbb{A}_i$$

We can now state the Cartan-Kähler theorem.

Theorem 4.3 *For an L.P.S and a point $p \in \Sigma$ at which all the numerical invariants are locally constant, if the torsion $[T]$ vanishes and equality holds in Prop 4.2 then there are local solutions to the P.D.E and the solution depends on s_k functions of k variables where k is the integer such that $s_k \neq 0$ and $s_{k'} = 0$ for $k' \geq k$.*

4.3 Quaternions and E.D.Ss

Our principal aim is to investigate the possibility of a quaternionic structure within inviscid fluid dynamics governed by the Euler Equations. First we must investigate the theory of quaternions and E.D.Ss.

First, let us look at the case of complex contact structures. Let M^{2n} be a real manifold with an integrable complex structure $J \in \Omega^0(M; \mathrm{End}(\mathrm{T}M))$, and $L \subset \mathrm{T}M$ a real subbundle of corank 2.

Theorem 4.4 *The following are equivalent:*

1. *L is J-invariant*
2. *Locally there exists a (1,0)-form θ such that each fibre of L is a maximal integral element of $\mathcal{I} = \{\theta, \bar{\theta}\}$.*

The proof is elementary.

Motivated by this and for the purposes of this paper, we make the following definition.

Definition 4.5 *Let $(M^{4n}, \mathbb{I}, \mathbb{J})$ be a hypercomplex manifold and $L \subset \mathrm{T}M$ be a corank 4 subbundle. We call L a quaternionic contact structure if L is invariant under $\mathbb{I}$ and $\mathbb{J}$.*

4.4 The Euler Equations as an E.D.S.

Let us examine the Euler equations on an n-dimensional space. We have

$$\begin{aligned} \boldsymbol{u}_t + \boldsymbol{u}.\nabla \boldsymbol{u} + \nabla p &= 0 \\ \operatorname{div} \boldsymbol{u} &= 0 \end{aligned}$$

For the moment, we forget about the divergence free condition. We will find that we be able to restrict to this space later.

To form our jet space J we see that we have the independent variables $t, x^1, \ldots x^n$ and dependent variables $p, u^1, \ldots, u^n$ and the derivatives $\frac{\partial p}{\partial t}, \frac{\partial p}{\partial x^i}, \frac{\partial u^\alpha}{\partial t}, \frac{\partial u^\alpha}{\partial x^i}$ The coordinates on the jet space J will therefore be

$$t, x^i, p, u^i, r^0, r^i, q^i, A^i_j \quad i, j = 1..n$$

where r^0 corresponds with p_t, $\boldsymbol{r} = (r^1, \ldots, r^n)$ corresponds with ∇p, $\boldsymbol{q} = (q^1, \ldots, q^3)$ corresponds with $\boldsymbol{u}^t$ and $A = (A^j_i)$ corresponds with the jacobian matrix $\left(\frac{\partial u^j}{\partial x^i}\right)$. Thus J is a $(n+2)^2 - 1$-dimensional real vector space.

Within J the Euler equations (minus the divergence condition) become

$$\mathbf{F}(t, \boldsymbol{x}, p, \boldsymbol{u}, r^0, \boldsymbol{r}, \boldsymbol{q}, A) = 0$$

where $\mathbf{F} : J \longrightarrow \mathbb{R}^n$ is given by

$$\mathbf{F}(t, \boldsymbol{x}, p, \boldsymbol{u}, r^0, \boldsymbol{r}, \boldsymbol{q}, A) = \left(\ \boldsymbol{q} + A\boldsymbol{u} + \boldsymbol{r} \ \right).$$

It is not hard to show that $\mathbf{F}$ is a submersion on all of $\Sigma = \mathbf{F}^{-1}(0)$, and thus Σ is a $(n+2)^2 - n$-dimensional submanifold of J.

Now we have a diffeomorphism

$$\begin{array}{rcl} \phi : \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n\times\mathfrak{gl}(n;\mathbb{R}) & \longrightarrow & \Sigma \\ \phi : \big((t,\boldsymbol{x}),(p,\boldsymbol{u}),(r^0,\boldsymbol{r}),A\big) & \mapsto & (t,\boldsymbol{x},p,\boldsymbol{u},r^0,\boldsymbol{r},-A\boldsymbol{u}-\boldsymbol{r},A) \end{array}$$

which permits us to examine the geometry of Σ as a submanifold of J. We thus have an L.D.S on Σ given by the differential ideal $\mathcal{I}$ generated by $\theta^1,\ldots,\theta^n$ and their derivatives where

$$\underline{\theta} = \begin{pmatrix} \theta^0 \\ \theta^1 \\ \vdots \\ \theta^n \end{pmatrix} = \begin{pmatrix} \mathrm{d}p \\ \mathrm{d}\boldsymbol{u} \end{pmatrix} - \begin{pmatrix} r^0 & \boldsymbol{r}^\top \\ -(A\boldsymbol{u}+\boldsymbol{r}) & A \end{pmatrix}\begin{pmatrix} \mathrm{d}t \\ \mathrm{d}\boldsymbol{x} \end{pmatrix}$$

and the differential ideal J generated by $\theta^i, \mathrm{d}t, \mathrm{d}x^j$. So

$$\begin{array}{rcl} \mathrm{d}\underline{\theta} & = & -\begin{pmatrix} \mathrm{d}r^0 & \mathrm{d}\boldsymbol{r}^\top \\ -(\mathrm{d}A\boldsymbol{u}+A\mathrm{d}\boldsymbol{u}+\mathrm{d}\boldsymbol{r}) & \mathrm{d}A \end{pmatrix}\wedge\begin{pmatrix} \mathrm{d}t \\ \mathrm{d}\boldsymbol{x} \end{pmatrix} \\ & \equiv & -\begin{pmatrix} \mathrm{d}r^0\wedge\mathrm{d}t + \mathrm{d}\boldsymbol{r}^\top\wedge\mathrm{d}\boldsymbol{x} \\ -(\mathrm{d}A\boldsymbol{u}+\mathrm{d}\boldsymbol{r})\wedge\mathrm{d}t + \mathrm{d}A\wedge\mathrm{d}\boldsymbol{x} \end{pmatrix} - \begin{pmatrix} 0 \\ A^2\mathrm{d}\boldsymbol{x}\wedge\mathrm{d}t \end{pmatrix} \quad \mathrm{mod}\ \mathcal{I}. \end{array}$$

Thus

$$T = A^i_j A^j_k v^0 \wedge v^k \otimes w_i$$

where w_0, w_i are dual to the θ^0, θ^i,$v^0 = \mathrm{d}t$ and $v^i = \mathrm{d}x^i$. Now the space $\mathbb{A}$ is spanned by the vectors

$$\begin{array}{rl} v^0\otimes w_0, & \\ v^i\otimes w_0 - v^0\otimes w_i, & i = 1\ldots n; \\ (v^i - u^i v^0)\otimes w_j, & i,j = 1\ldots n. \end{array}$$

Hence $\dim\mathbb{A} = n^2+n+1$.

It is an elementary (though tedious) exercise to show that $\delta(\mathbb{A}\otimes V^*)$ is all of $\Lambda^2(V^*)\otimes W$ (both are $\frac{1}{2}n(n+1)^2$-dimensional) and thus both H and $[T]$ are trivial. Since

$$(A)\otimes V^* = ((A)\otimes V^*)\cap\Lambda^2(V^*)\otimes W \oplus ((A)\otimes V^*)\cap\odot^2(V^*)\otimes W$$

we know that $\dim\mathbb{A}^{(1)} = (n+1)(n^2+n+1) - \frac{n}{2}(n+1)^2 = \frac{1}{2}(n^3+2n^2+3n+2)$. Now we need to calculate the dimensions of the $\mathbb{A}_i$. First

$$\mathbb{A}_n = \mathbb{A}\cap\mathrm{span}\{v^0\}\otimes W = \mathrm{span}\{v^0\otimes w_0\},$$

so $\dim\mathbb{A}_n = 1$. Now we notice that for $j > 1$

$$\begin{array}{rcl} \mathbb{A}_{n-j} & = & \mathbb{A}\cap\mathrm{span}\{v^0,\ldots,v^j\}\otimes W \\ & = & \mathbb{A}_{n+1-j}\oplus\mathrm{span}\{v^j\otimes w_0 - v^0\otimes w_j, (v^j - u^j v^0)\otimes w_k\}, \end{array}$$

So $\dim \mathbb{A}_{n-j} = \dim \mathbb{A}_{n+1-j} + n + 1$ for $j = 1 \dots n$ and hence

$$\dim \mathbb{A}_{n-j} = 1 + (n+1)j.$$

Thus with $\mathbb{A}_0 = \mathbb{A}$ we have

$$\sum_{i=0}^{n} \dim \mathbb{A}_{n-i} \quad = \quad \frac{1}{2}(n^3 + 2n^2 + 3n + 2)$$

Since

$$\dim \mathbb{A}^{(1)} = \sum_{i=0}^{n} \dim \mathbb{A}_{n-i}$$

we know that this L.P.S is integrable for any n and that (predictably) the Euler system can be completely described by $n+1$ functions of $n+1$ variables.

Now the orthogonal complement of $\mathbb{A}$ in $V^* \otimes W$ is

$$\text{Span}\left\{v^i \otimes w_0 + (v^0 + u^j v^j) \otimes w_i\right\}$$

which yields the symbol matrix for $\xi^0, \underline{\xi} = (\xi^1, \dots \xi^n) \in V^*$

$$\sigma_p(\xi^0, \underline{\xi}) = \begin{pmatrix} 0 & \underline{\xi}^\top \\ \underline{\xi} & (\xi^0 + \boldsymbol{u}^\top \underline{\xi})\mathbb{1}_n \end{pmatrix}.$$

This matrix has determinant $-(\xi^0 + \boldsymbol{u}^\top \underline{\xi})^{n-2}(\xi^i \xi^i)$ so the characteristic variety is the union of a nonsingular conic $(\xi^1)^2 + \dots + (\xi^n)^2 = 0$ and a hyperplane $(\xi^0 + \boldsymbol{u}^\top \underline{\xi})$ in $\mathbb{P}V^*_{\mathbb{C}}$. The hyperplane occurs as the locus of a (very!) singular polynomial of degree $n-2$. The the real characteristic variety is just the hyperplane which describes a certain slice of codimension 1 through any integral manifold.

4.5 Integral elements of the Euler L.P.S

We are obliged to consider the integral manifolds of the Euler L.P.S. The tangent spaces to any integral manifold occur as integral elements of the E.D.S, i.e. vector subspaces $E \subset \mathrm{T}_p J$ such that

$$\phi_p(v_1, \dots, v_k) = 0 \text{ for each } \phi \in \mathfrak{I} \text{ and } v_1, \dots v_k \in E.$$

Our search for integral submanifolds is restricted to those for which the restriction of $\mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$ vanishes nowhere. This singles t, x, y, z out as independent variables.

A generic integral element will be cut out by

$$\begin{aligned} \mathrm{d}r^0 &= \rho_0 \mathrm{d}t + \alpha^i \mathrm{d}x^i \\ \mathrm{d}r^i &= \beta^i \mathrm{d}t + R^i_j \mathrm{d}x^j \\ \mathrm{d}A^j_i &= a^j_{0i} \mathrm{d}t + a^{\ j}_{ki} \mathrm{d}x^k \end{aligned}$$

into $\mathrm{d}\underline{\theta}$ and solving for these linear variables, we find that at a generic point, any $(n+1)$-dimensional integral element is parameterised by

$$(\rho_0, \underline{\rho}, R, \underline{a}) \in \mathbb{R} \oplus \mathbb{R}^n \oplus \odot^2\mathbb{R}^n \oplus \left(\odot^2 (\mathbb{R}^n)^* \otimes \mathbb{R}^n\right) \cong (\mathbb{R} \oplus \mathbb{R}^n) \otimes (\mathbb{R} \oplus \odot^2\mathbb{R}^n)$$

The parameterisation of integral elements is given by

$$\begin{aligned}
\mathrm{d}p &= r^0\mathrm{d}t + \boldsymbol{r}^\top \mathrm{d}\boldsymbol{x}\\
\mathrm{d}\boldsymbol{u} &= -(A\boldsymbol{u}+\boldsymbol{r})\mathrm{d}t + A\mathrm{d}\boldsymbol{x}\\
\mathrm{d}r^0 &= \rho_0\mathrm{d}t + \underline{\rho}^\top\mathrm{d}\boldsymbol{x}\\
\mathrm{d}\boldsymbol{r} &= \underline{\rho}\mathrm{d}t + (R-\odot(A^2))\mathrm{d}\boldsymbol{x}\\
\mathrm{d}A &= -\left(\wedge(A^2)+R+\underline{a}(\boldsymbol{u})\right)\mathrm{d}t + \underline{a}(\mathrm{d}\boldsymbol{x})
\end{aligned}$$

where $\odot(M) = \frac{1}{2}(M+M^\top)$. These are the linear equations which cut out the integral elements at a generic point.

We can realise the condition that $\operatorname{div}\boldsymbol{u} = 0$ (i.e $\operatorname{tr} A = 0$) by requiring that $\operatorname{tr} R = \operatorname{tr}\underline{a}(v) = 0$ for any vector v. Implicit in this is the condition that on an integral manifold (i.e. a solution of Euler's Fluid equations)

$$\operatorname{tr}(\mathrm{d}\boldsymbol{r}\odot\mathrm{d}\boldsymbol{x}) = -\operatorname{tr}\odot(A^2) = -\operatorname{tr}A^2$$

integrates to

$$\Delta p = -\operatorname{tr}(\mathrm{D}\boldsymbol{u})^2.$$

This yields a $\frac{n}{2}(n+1)^2$ dimensional vector space of $(n+1)$-dimensional integral elements at a generic point. Obviously at certain points of Σ, the dimension of the space of $(n+1)$-dimensional integral elements increases.

By splitting A into its trace-free symmetric and skew-symmetric parts

$$A = S + \Omega$$

we find that $(n+1)$-dimensional integral elements are cut out by the equations

$$\begin{aligned}
\mathrm{d}p &= r^0\mathrm{d}t + \boldsymbol{r}^\top \mathrm{d}\boldsymbol{x}\\
\mathrm{d}\boldsymbol{u} &= -(A\boldsymbol{u}+\boldsymbol{r})\mathrm{d}t + A\mathrm{d}\boldsymbol{x}\\
\mathrm{d}r^0 &= \rho_0\mathrm{d}t + \underline{\rho}^\top\mathrm{d}\boldsymbol{x}\\
\mathrm{d}\boldsymbol{r} &= \underline{\rho}\mathrm{d}t + (R-S^2-\Omega^2)\mathrm{d}\boldsymbol{x}\\
\mathrm{d}S &= -\left(R+\odot(\underline{a}(\boldsymbol{u}))\right)\mathrm{d}t + \odot(\underline{a}(\mathrm{d}\boldsymbol{x}))\\
\mathrm{d}\Omega &= -(S\Omega+\Omega S+\wedge(\underline{a}(\boldsymbol{u})))\mathrm{d}t + \wedge(\underline{a}(\mathrm{d}\boldsymbol{x})).
\end{aligned}$$

Since an integral element is cut out by $(n+1)^2+1$ quantities depending on $n+1$ independent variables, we expect that there must be n^2+n+1 "constants of motion", i.e. n^2+n+1 equations of the form

$$K\mathrm{d}p + L_i\mathrm{d}u^i + M\mathrm{d}r^0 + N_i\mathrm{d}r^i + P_i^j\mathrm{d}A_j^i = 0,$$

where K, L_i, M, N_i, P_i^j are rational functions of $p, \boldsymbol{u}, r^0, \boldsymbol{r}, A, \rho^0, \underline{\rho}, R$ and $\underline{a}$. Any integral element is thus spanned by the vector fields

$$\begin{aligned}
\zeta_0 &= \frac{\partial}{\partial t} + r^0\frac{\partial}{\partial p} - (A\boldsymbol{u}+\boldsymbol{r})^i\frac{\partial}{\partial u^i} + \rho^0\frac{\partial}{\partial r^0} + \rho^i\frac{\partial}{\partial r^i} - (S\Omega+\Omega S+\underline{a}(\boldsymbol{u}))^i_j\frac{\partial}{\partial A^i_j}\\
\zeta_i &= \frac{\partial}{\partial x^i} + r^i\frac{\partial}{\partial p} + A_i^j\frac{\partial}{\partial u^j} + \rho^i\frac{\partial}{\partial r^0} + (R-S^2-\Omega^2)^j_i\frac{\partial}{\partial r^j} + a^j_{ik}\frac{\partial}{\partial A^j_k}
\end{aligned}$$

5 Further Spin Geometric Aspects of Incompressible Inviscid Hydrodynamics

The Euler-Equations can be expressed as the Newtonian limit of Relativistic Hydrodynamics. Since this is naturally 4-dimensional we hope to recover some essential quaternionic features of fluid dynamical theory. We recall that under the Lorentz metric on 4 dimensional Minkowski space

$$\eta = (\mathrm{d}x^0)^2 - (\mathrm{d}x^1)^2 - (\mathrm{d}x^2)^2 - (\mathrm{d}x^3)^2,$$

the Relativistic Hydrodynamics equation is given by

$$\operatorname{div} T = 0$$

where

$$T = (P + \rho)\boldsymbol{u} \otimes \boldsymbol{u}^\top - P\mathbb{1}$$

is the symmetric energy-momentum tensor.

5.1 Preliminaries

Of course, there is a great deal of difference between the spin geometries of Euclidean and Lorentzian spaces. However, there is a unification if we are prepared to work on a 4-dimensional complex vector space with a complex-bilinear form. In this case the idea of signature is irrelevant, so we assume that this form is given by

$$g = \mathrm{d}z^1 \otimes \mathrm{d}z^1 + \mathrm{d}z^2 \otimes \mathrm{d}z^2 + \mathrm{d}z^3 \otimes \mathrm{d}z^3 + \mathrm{d}z^4 \otimes \mathrm{d}z^4.$$

We can easily recover the Euclidean and Lorentzian geometries by restricting to 4 dimensional real slices. The symmetry group of $(\mathbb{C}^4, g)$ is $\mathrm{SO}(4;\mathbb{C})$ which has a double cover

$$\mathrm{SL}(2;\mathbb{C}) \times \mathrm{SL}(2;\mathbb{C}) \longrightarrow \mathrm{SO}(4;\mathbb{C}).$$

By using the holomorphic volume form and g we can recover the Hodge $*$ operator and the notion of self-duality on holomorphic 1-forms.

Now, we can obtain our spin geometry from

$$\mathbb{C}^4 \cong S^+ \otimes S^-$$

In which case

$$\begin{aligned}
\mathbb{C}^4 \otimes \mathbb{C}^4 &\cong (S^+ \otimes S^-) \otimes (S^+ \otimes S^-) \\
&\cong (S^+ \otimes S^+) \otimes (S^- \otimes S^-) \\
&\cong (\wedge^{2,0}S^+ \oplus \odot^{2,0}S^+) \otimes (\wedge^{2,0}S^- \oplus \odot^{2,0}S^-) \\
&\cong (\mathbb{C} \oplus \odot^{2,0}S^+) \otimes (\mathbb{C} \oplus \odot^{2,0}S^-) \\
&\cong \mathbb{C} \oplus \odot^{2,0}S^+ \oplus \odot^{2,0}S^- \oplus \left(\odot^{2,0}S^+ \otimes \odot^{2,0}S^-\right)
\end{aligned}$$

Now, we note that

$$\Lambda^{2,0}(\mathbb{C}^4) \cong \odot^{2,0} S^+ \oplus \odot^{2,0} S^-,$$

hence

$$\odot^{2,0}(\mathbb{C}^4) \cong \mathbb{C} \oplus \left(\odot^{2,0} S^+ \otimes \odot^{2,0} S^-\right).$$

We may also conclude that

$$\begin{aligned} \odot^{2,0}(\mathbb{C}^4) &\cong \mathbb{C} \oplus \Lambda_+^{2,0} \otimes \Lambda_-^{2,0} \\ &\cong \mathbb{C} \oplus \left(\Im\mathbb{H}^+ \otimes \Im\mathbb{H}^-\right) \otimes_{\mathbb{R}} \mathbb{C} \end{aligned}$$

where $\Lambda_\pm^{2,0}$ are the spaces of (anti)self-dual holomorphic 2-forms on $\mathbb{C}^4$ and $\Im\mathbb{H}^\pm$ the space of imaginary quaternions which come from regarding (anti) self-dual 2-forms as complex linear endomorphisms on $\mathbb{C}^4$. Thus a traceless symmetric (both wrt g) holomorphic 2-tensor on $\mathbb{C}^4$ can be expressed as the tensor product of a self-dual 2-form and an anti self-dual 2-form.If we work in the basis $\partial_i = \frac{\partial}{\partial z^i}$, then $\mathbb{I}^-, \mathbb{J}^-$ and $\mathbb{K}^-$ have matrices

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

and $\mathbb{I}^+, \mathbb{J}^+$, and $\mathbb{K}^+$ have matrices

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The explicit isomorphism

$$(\Im\mathbb{H}^+ \otimes \Im\mathbb{H}^-) \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow \odot_0^{2,0}\mathbb{C}^4$$

is given by composition

$$A \otimes B \mapsto AB.$$

The inverse is given by

$$T \mapsto -\frac{1}{2}\left((T\mathbb{I}^+ + \mathbb{I}^+T) \otimes \mathbb{I}^- + (T\mathbb{J}^+ + \mathbb{J}^+T) \otimes \mathbb{J}^- + (T\mathbb{K}^+ + \mathbb{K}^+T) \otimes \mathbb{K}^-\right).$$

We note that

$$\frac{1}{2}\left(\mathbb{I}^{+}+\mathbb{I}^{-}\right) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix};$$

$$\frac{1}{2}\left(\mathbb{J}^{+}+\mathbb{J}^{-}\right) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix};$$

$$\frac{1}{2}\left(\mathbb{K}^{+}+\mathbb{K}^{-}\right) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This result is significant due to the fact that these matrices define an embedding of $\Lambda^{2,0}(\mathbb{C}^3)$ in $\Lambda^{2,0}(\mathbb{C}^4)$. However, we know that the Lie algebra $\Lambda^{2,0}(\mathbb{C}^3)$ with twice the usual Lie bracket, is isomorphic with $\mathbb{C}^3$ with the (complexified) cross product via

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}.$$

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