Numerical Continuation of Bifurcations

An Introduction

Part II: Connecting Orbits

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Computation of connecting orbits

Let x_- and x_+ be equilibria and let $\Gamma = \{\gamma(t) : t \in \mathbb{R}\}$ be a connecting orbit from x_- to x_+. Then Γ is characterized as follows:

$$\dot{\gamma} = f(\gamma, \lambda)$$
$$f(x_-, \lambda) = 0$$
$$f(x_+, \lambda) = 0$$
$$\lim_{t\to-\infty} \gamma(t) = x_-$$
$$\lim_{t\to\infty} \gamma(t) = x_+$$

Computation of connecting orbits II

For numerical computations the equations have to be transferred to a finite time-interval, say $[-T, T]$.

For simplicity: restrict to *homoclinic orbits*, ($x_- = x_+ = x^*$) and assume that x^* is a *hyperbolic equilibrium*.

Let $A := D_x f(x^*, \lambda)$ have n_u positive and n_s negative eigenvalues, and let E^u and E^s denote the unstable and stable eigenspace.

(Un)Stable Manifold Theorem: Locally near x^* there exist an n_u-dimensional unstable manifold W^u, tangent to E^u and an n_s-dimensional stable manifold W^s tangent to E^s.

Computation of connecting orbits III

The orbit Γ has to approach x^* along $W^{s(u)}$ for $t \to \pm\infty$. We use $E^{s(u)}$ as linear approximations to $W^{s(u)}$.

Thus, let $L_{s(u)}$ be projections on $E^{s(u)}$. Then we demand

$$L_s(\gamma(-T) - x^*) = 0$$
$$L_u(\gamma(T) - x^*) = 0$$

Finally, we need the standard phase condition (also truncated from $\mathbb{R}$ to $[-T, T]$)

$$\int_{-T}^{T} (\gamma(t) - x_0(t))^T \dot{\gamma}(t) dt = 0,$$

where x_0 is a reference solution (initial guess).

Computation of connecting orbits IV

The boundary value problem for the computation of a homoclinic orbit thus reads

$$\begin{aligned} \dot\gamma(t) &= f(\gamma(t),\lambda), \quad \text{for } t \in [-T,T] \\ L_s(\gamma(-T) - x^*) &= 0 \\ L_u(\gamma(T) - x^*) &= 0 \end{aligned}$$

with phase condition

$$\int_{-T}^{T} (\gamma(t) - x_0(t))^T \dot\gamma(t) dt = 0,$$

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