Chapter IX

Measures on Fractals

1 Introduction to Invariant Measures on Fractals

In this section we give an intuitive introduction to measures. We focus on measures that arise from iterated function systems in $\mathbb{R}^2$.

In Chapter III, section 8 we introduced the Random Iteration Algorithm. This algorithm is a means for computing the attractor of a hyperbolic IFS in $\mathbb{R}^2$. In order to run the algorithm one needs a set of probabilities, in addition to the IFS.

Definition 1.1 *An iterated function system* with probabilities *consists of an IFS*

$$\{\mathbf{X}; w_1, w_2, \ldots, w_N\}$$

together with an ordered set of numbers $\{p_1, p_2, \ldots, p_N\}$, *such that*

$$p_1 + p_2 + p_3 + \cdots + p_N = 1 \text{ and } p_i > 0 \qquad \text{for } i = 1, 2, \ldots, N.$$

The probability p_i is associated with the transformation w_i. The nomenclature "IFS with probabilities" is used for "iterated function system with probabilites." The full notation for such an IFS is

$$\{\mathbf{X}; w_1, w_2, \ldots, w_N; p_1, p_2, \ldots, p_N\}.$$

Explicit reference to the probabilities may be suppressed.

An example of an IFS with probabilities is

$$\{\mathbb{C}; w_1(z), w_2(z), w_3(z), w_4(z); 0.1, 0.2, 0.3, 0.4\},$$

where

$$w_1(z) = 0.5z,\ w_2(z) = 0.5z + 0.5,$$
$$w_3(z) = 0.5z + (0.5)i,\ w_4(z) = 0.5z + 0.5 + (0.5)i.$$

It can be represented by the IFS code in Table IX.1. The attractor is the filled square ■, with corners at $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$.

Here is how the Random Iteration Algorithm proceeds in the present case. An initial point, $z_0 \in \mathbb{C}$, is chosen. One of the transformations is selected "at random"

Table IX.1. IFS code for a measure on ■.

w	a	b	c	d	e	f	p
1	0.5	0	0	0.5	1	1	0.1
2	0.5	0	0	0.5	50	1	0.2
3	0.5	0	0	0.5	1	50	0.3
4	0.5	0	0	0.5	50	50	0.4

from the set $\{w_1, w_2, w_3, w_4\}$. The probability that w_i is selected is p_i, for $i = 1, 2, 3, 4$. The selected transformation is applied to z_0 to produce a new point $z_1 \in \mathbb{C}$. Again a transformation is selected, in the same manner, independently of the previous choice, and applied to z_1 to produce a new point z_2. The process is repeated a number of times, resulting in a finite sequence of points $\{z_n : n = 1, 2, \ldots, numits\}$, where *numits* is a positive integer. For simplicity, we assume that $z_0 \in$ ■. Then, since $w_i(■) \subset$ ■, for $i = 1, 2, 3, 4$, the "orbit" $\{z_n : n = 1, 2, \ldots, numits\}$ lies in ■.

Consider what happens when we apply the algorithm to the IFS code in Table IX.1. If the number of iterations is sufficiently large, a picture of ■ will be the result. That is, every pixel corresponding to ■ is visited by the "orbit" $\{z_n : n = 1, 2, \ldots, numits\}$. The rate at which a picture of ■ is produced depends on the probabilities. If $numits = 10,000$, then we expect that because the images of ■ are just-touching,

$$\text{the number of computed points in } w_1(■) \approx 1000,$$

$$\text{the number of computed points in } w_2(■) \approx 1000,$$

$$\text{the number of computed points in } w_3(■) \approx 1000,$$

$$\text{the number of computed points in } w_4(■) \approx 1000.$$

These estimates are supported by Figure IX.247, which shows the result of running a modified version of Program 2 in Chapter III, with the IFS code in Table IX.1, and $numits = 100,000$.

In Figure IX.248 we show the result of running a modified version of Program 2 in Chapter III, for the IFS code in Table IX.1, with various choices for the probabilities. In each case we have halted the program after a relatively small number of iterations, to stop the image from becoming "saturated." The results are diverse textures. In each case the attractor of the IFS is the same set, ■. However, the points produced by the Random Iteration Algorithm "rain down" on ■ with different frequencies at different places. Places where the "rainfall" is highest appear "darker" or "more dense" than those places where the "rainfall" is lower. In the end all places on the attractor get wet.

The pictures in Figure IX.248 (a)–(c) suggest a wonderful idea. They suggest that associated with an IFS with probabilities there is a unique "density" on the

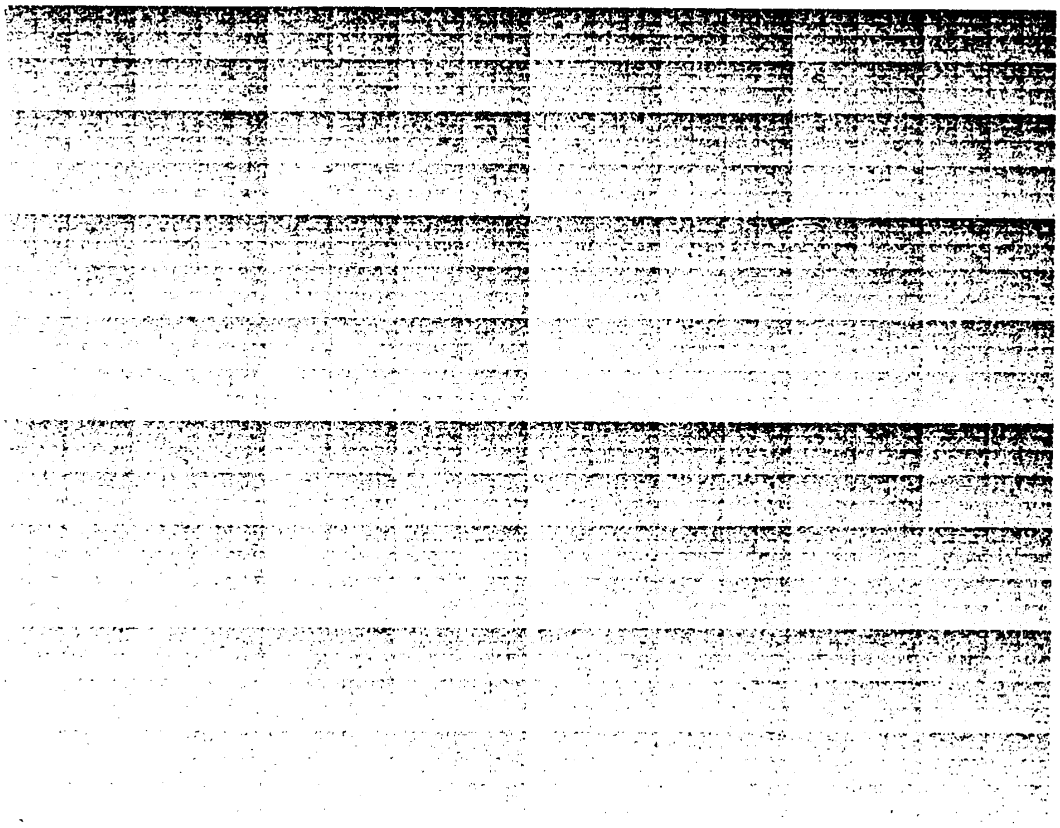


Figure IX.247. The Random Iteration Algorithm, Program 1 in Chapter III, is applied to the IFS code in Table IX.1, with numits = 100,000. Verify that the number of points that lie in $w_i(\blacksquare)$ is approximately $(numits)p_i$, for $i = 1, 2, 3, 4$.

attractor of the IFS. The Random Iteration Algorithm gives one a glimpse of this "density," but one loses sight of it as the number of iterations is increased. This is true, and much more as well! As we will see, the "density" is so beautiful that we need a new mathematical concept to describe it. The concept is that of a *measure.* Measures can be used to describe intricate distributions of "mass" on metric spaces. They are introduced formally further on in this chapter. The present section provides an intuitive understanding of what measures are and of how an interesting class of measures arises from IFS's with probabilities.

As a second example, consider the IFS with probabilities

$$\{\mathbb{C};\ w_1(z) = 0.5z + 24 + 24i,\ w_2(z) = 0.5z + 24i,\ w_3(z) = 0.5z;\ 0.25, 0.25, 0.5\}.$$

The attractor is a Sierpinski triangle $\blacktriangle$. The probability associated with w_3 is twice that associated with either w_1 or w_2. In Figure IX.249 we show the result of applying the Random Iteration Algorithm, with these probabilities, to compute 10,000 points belonging to $\blacktriangle$. There appear to be different "densities" at different places on $\blacktriangle$. For example, $w_3(\blacktriangle)$ appears to have more "mass" than either $w_1(\blacktriangle)$ or $w_2(\blacktriangle)$.

In Figure IX.250 we show the result of applying the Random Iteration Algorithm to another IFS with probabilities, for three different sets of probabilities. The IFS is $\{\mathbb{R}^2;\ w_1, w_2, w_3, w_4\}$, where w_i is an affine transformation for $i = 1, 2, 3, 4$. The attractor of the IFS is a leaf-like subset of $\mathbb{R}^2$. In each case we see a different pattern

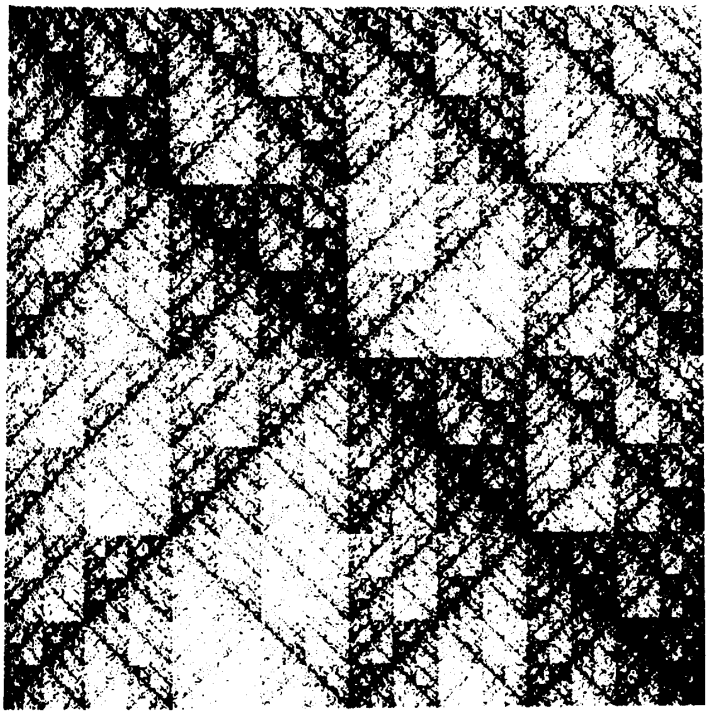


Figure IX.248. The Random Iteration Algorithm is applied to the IFS code in Table IX.1, but with various different sets of probabilities. The result is that points rain down on the attractor of the IFS at different rates at different places. What we are seeing are the faint traces of wonderful mathematical entities called *measures*. These are the true fractals. Their supports, the attractors of IFS, are merely sets upon which the measures live.

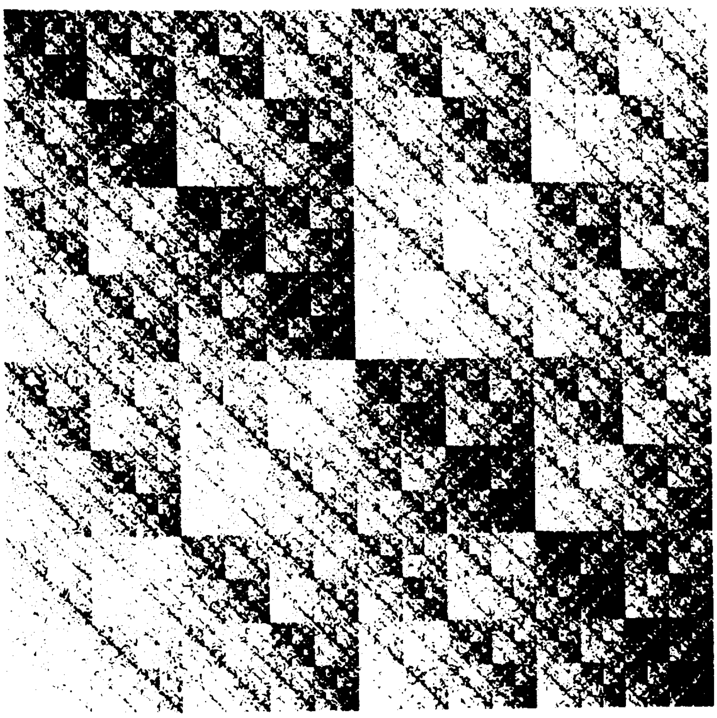
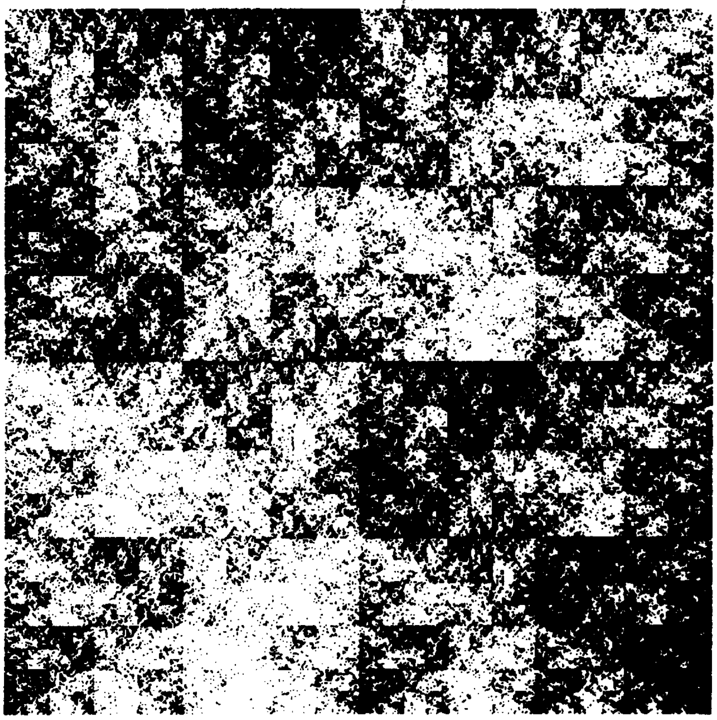


Figure IX.248. (b)

Figure IX.248. (c)



of "mass" on the attractor of the IFS. It appears that each "density" is itself a fractal object.

Examples & Exercises

1.1. Carry out the following numerical experiment. Apply the Random Iteration Algorithm to the IFS code in Table IX.1, for *numits* $= 1000, 2000, 3000, \ldots$. In each case record the number, $\mathcal{N}$, of computed points that land in $B = \{(x, y) \in \mathbb{R}^2 : (x-1)^2 + (y-1)^2 \leq 1\}$, and make a table of your results. Verify that the ratio $\mathcal{N}/\mathit{numits}$ appears to approach a constant.

1.2. Repeat the computergraphical experiment that produced Figure IX.247. Verify that you obtain "similar-looking" output to that shown in Figure IX.247, even though you (probably) use a different random-number sequence.

1.3. The Random Iteration Algorithm is used to compute 100,000 points belonging to ■, using the IFS code in Table IX.1. How many of these points, do you expect, would belong to $w_1 \circ w_3(■)$? Why?

Let $(\mathbf{X}, d)$ be a complete metric space. Let $\{\mathbf{X}; w_1, \ldots, w_N; p_1, \ldots, p_N\}$ be an IFS with probabilities. Let A denote the attractor of the IFS. Then there exists a thing called the *invariant measure* of the IFS, which we denote here by μ. μ assigns "mass" to many subsets of $\mathbf{X}$. For example, $\mu(A) = 1$ and $\mu(\emptyset) = 0$. That is, the "mass" of the attractor is one unit, and the "mass" of the empty set is zero. Also $\mu(\mathbf{X}) = 1$, which says that the whole space has the same "mass" as the attractor of the IFS; the "mass" is located on the attractor.

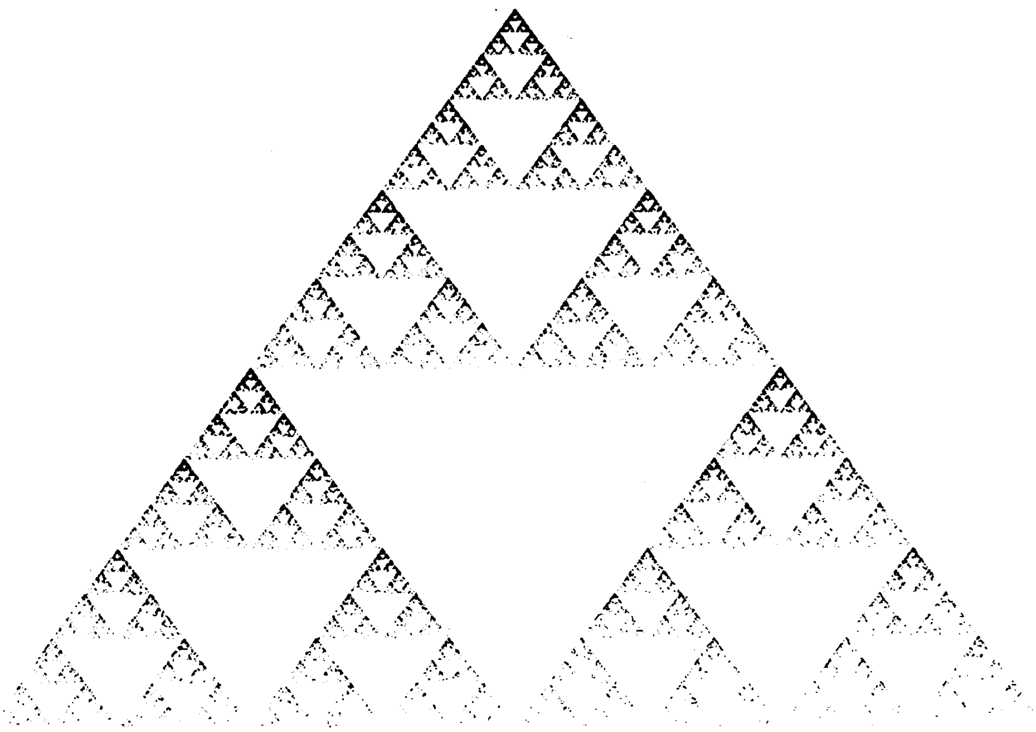


Figure IX.249. The Random Iteration Algorithm is used to compute an image of the Sierpinski triangle $\blacktriangle$. The probability associated with w_3 is twice that associated with w_1 or w_2. One thousand points have been computed. The result is that $w_3(\blacktriangle)$ appears denser than $w_1(\blacktriangle)$ or $w_2(\blacktriangle)$. This appearance is lost when the number of iterations is increased. We are led to the idea of a "mass" or measure that is supported on the fractal.

Not all subsets of $\mathbf{X}$ have a "mass" assigned to them. The subsets of $\mathbf{X}$ that do have a "mass" are called the *Borel subsets* of $\mathbf{X}$, denoted by $\mathcal{B}(\mathbf{X})$. The Borel subsets of $\mathbf{X}$ include the compact nonempty subsets of $\mathbf{X}$, so that $\mathcal{H}(\mathbf{X}) \subset \mathcal{B}(\mathbf{X})$. Also, if $\mathcal{O}$ is an open subset of $\mathbf{X}$, then $\mathcal{O} \in \mathcal{B}(\mathbf{X})$. So there are plenty of sets that have "mass." Let B denote a closed ball in $\mathbf{X}$. Here is how to calculate the "mass" of the ball, $\mu(B)$. Apply the Random Iteration Algorithm to the IFS with probabilities, to produce a sequence of points $\{z_n\}_{n=0}^{\infty}$. Let

$$\mathcal{N}(B, n) = \text{number of points in } \{z_0, z_1, z_2, z_3, \ldots, z_n\} \cap B, \quad \text{for } n = 0, 1, 2, \ldots.$$

Then, almost always,

$$\mu(B) = \lim_{n\to\infty} \left\{ \frac{\mathcal{N}(B, n)}{(n+1)} \right\}.$$

That is, the "mass" of the ball B is the proportion of points, produced by the Random Iteration Algorithm, which land in B. (To be precise we also have to require that the "mass" of the boundary of B is zero; see Corollary 7.1.)

By now you should be bursting with questions. How do we know that this formula "almost always" gives the same answer? What are Borel sets? Why don't all sets have "mass"? Welcome to measure theory!

As an example, we evaluate the measure of some subsets of $\mathbb{C}$, for the IFS with probabilities

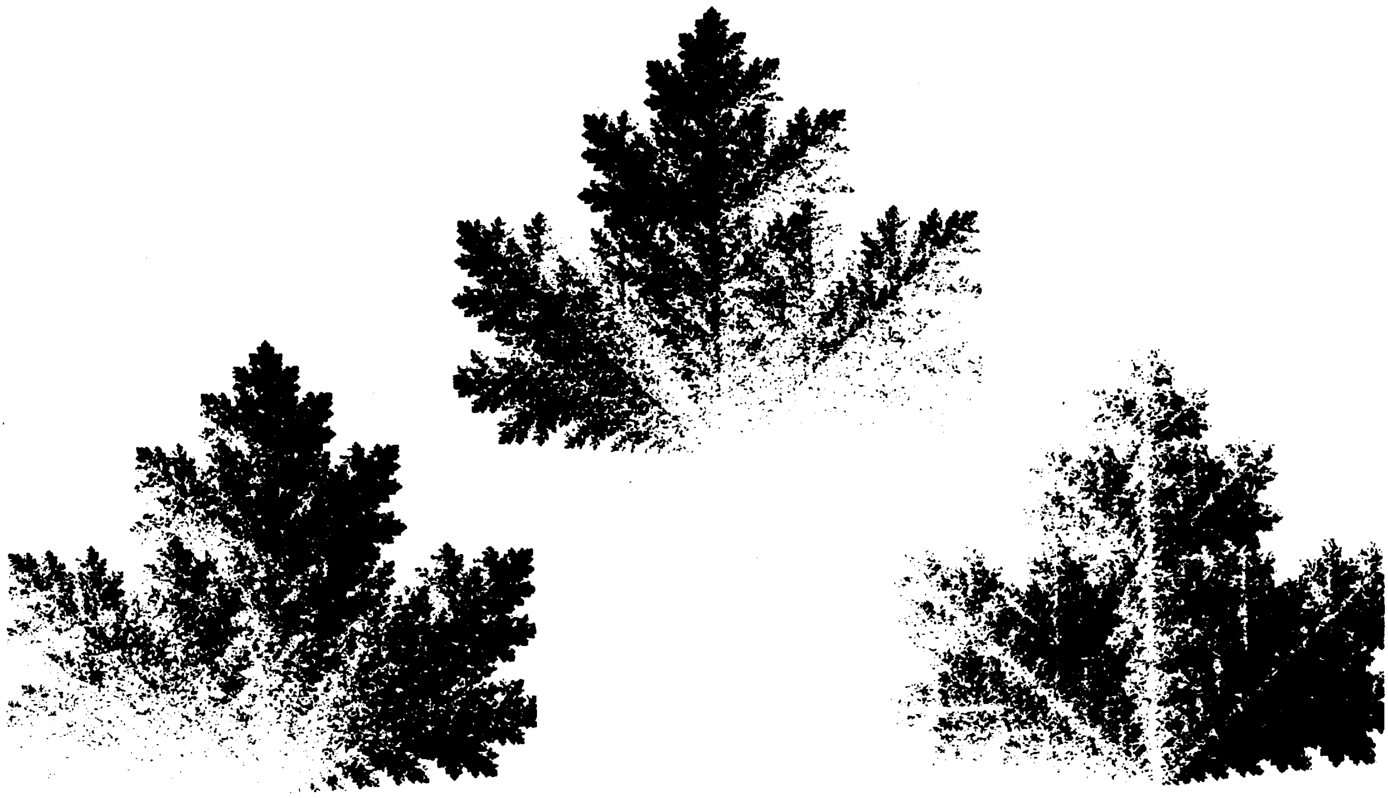


Figure IX.250. The Random Iteration Algorithm is used to compute an image of a leaf. Different sets of probabilities lead to different distributions of "mass" on the leaf.

$$\{\mathbb{C};\ w_1(z) = 0.5z,\ w_2(z) = 0.5z + (0.5)i,\ w_3(z) = 0.5z + 0.5;\ 0.33, 0.33, 0.34\}.$$

The attractor is a Sierpinski triangle $\blacktriangle$ with vertices at 0, i, and 1. We compute the measures of the following sets:

$$B_1 = \{z \in \mathbb{C} : |z| \leq 0.5\}$$
$$B_2 = \{z \in \mathbb{C} : |z - (0.5 + 0.5i)| \leq 0.2\}$$
$$B_3 = \{z \in \mathbb{C} : |z - (0.5 + 0.5i)| \leq 0.5\}$$
$$B_4 = \{z \in \mathbb{C} : |z - (2 + i)| \leq \sqrt{2}\}.$$

The results are presented in Table IX.2.

Figure IX.251 illustrates the ideas introduced here.

Examples & Exercises

1.4. Explain why $\mu(B_4) \approx 0$ in Table IX.2.

1.5. What value, approximately, would have been obtained for $\mu(B_1)$ in Table IX.2, if the probabilities on the three maps had been $p_1 = 0.275$, $p_2 = 0.125$, and $p_3 = 0.5$?

Table IX.2. The measures of some subsets of $\mathbb{A}$ are computed by random iteration.

n	$\mathcal{N}(B_1, n)/n$	$\mathcal{N}(B_2, n)/n$	$\mathcal{N}(B_3, n)/n$	$\mathcal{N}(B_4, n)/n$
5,000	0.3313	0.1036	0.6385	0.0004
10,000	0.3314	0.1050	0.6500	0.0002
15,000	0.3323	0.1041	0.6512	0.0001
20,000	0.3330	0.1030	0.6525	0.0000
50,000	0.3326	0.1041	0.6527	0.0000
100,000	0.3325	0.1054	0.6497	0.0000

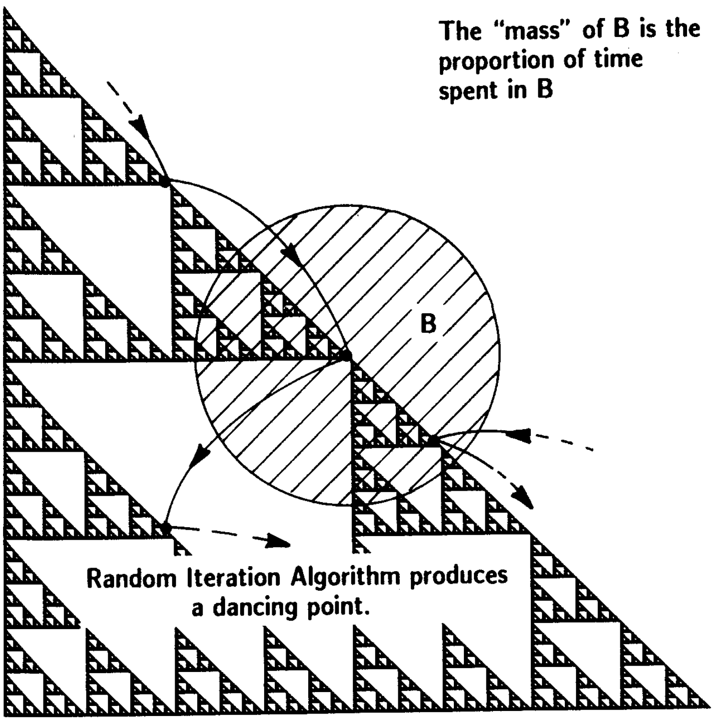


Figure IX.251. Diagram of the Random Iteration Algorithm running, and a dancing point coming and going from the ball B. The "mass" or measure of the ball is $\mu(B)$. It is equal to the proportion of points that land in B.

1.6. Why, do you think, is the phrase "almost always" written in connection with the formula for $\mu(B)$, given above?

2 Fields and Sigma-Fields

Definition 2.1 *Let* $\mathbf{X}$ *be a space. Let* $\mathcal{F}$ *denote a nonempty class of subsets of a space* $\mathbf{X}$, *such that*

(1) $A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$;
(2) $A \in \mathcal{F} \Rightarrow \mathbf{X} \setminus A \in \mathcal{F}$.

Then $\mathcal{F}$ is called a field.

(In exercise 2.12 you will be asked to prove that $\mathbf{X} \in \mathcal{F}$.)

Theorem 2.1 *Let $\mathbf{X}$ be a space. Let $\mathcal{G}$ be a nonempty set of subsets of $\mathbf{X}$. Let $\mathcal{F}$ be the set of subsets of $\mathbf{X}$ which can be built up from finitely many sets in $\mathcal{G}$ using the operations of union, intersection, and complementation with respect to $\mathbf{X}$. Then $\mathcal{F}$ is a field.*

Proof Elements of $\mathcal{F}$ consist of sets such as

$$\mathbf{X} \setminus (((\mathbf{X} \setminus (G_1 \cup G_2)) \cap G_3) \cup (G_5 \cap G_6)),$$

where G_1, G_2, G_3, $G_3, \ldots$ denote elements of $\mathcal{G}$. That is, $\mathcal{F}$ is made of all those sets that can be expressed using a finite chain of parentheses, $\setminus$, $\cup$, $\cap$, elements of $\mathcal{G}$, and $\mathbf{X}$. (In fact, using de Morgan's laws one can prove that it is not necessary to use the intersection operation.) If we form the union of any two such expressions we obtain another one. Similarly, if we form the complement of such an expression with respect to $\mathbf{X}$, we obtain another such expression. So conditions (i) and (ii) in Definition 2.1 are satisfied. This completes the proof.

Definition 2.2 *The field referred to in Theorem 2.1 is called the field* generated *by $\mathcal{G}$.*

Examples & Exercises

2.1. Let $\mathbf{X}$ be a space and let $A \subset \mathbf{X}$. Then $\mathcal{F} = \{\mathbf{X}, A, \mathbf{X} \setminus A, \emptyset\}$ is a field.

2.2. Let $\mathbf{X}$ be the set of all leaves on a certain tree and let $\mathcal{F}$ be the set of all subsets of $\mathbf{X}$. Then $\mathcal{F}$ is a field. Let A denote the set of all the leaves on the lowest branch of the tree. Then $A \in \mathcal{F}$. Prove that $\mathcal{F}$ is generated by the leaves.

2.3. Let $\mathbf{X} = [0, 1] \subset \mathbb{R}$. Let $\mathcal{G}$ denote the set of all subintervals (open, closed, half-open) of $[0, 1]$. Let $\mathcal{F}$ denote the field generated by $\mathcal{G}$. Examples of members of $\mathcal{F}$ are $[0.5, 0.6) \cup (0.7, 0.81)$; $[0, 1]$; $[1, 1]$; and $(\frac{1}{2}, 1) \cup (\frac{1}{4}, \frac{1}{3}) \cup \cdots \cup (\frac{1}{100}, \frac{1}{99})$. Show that

$$\cup_{n=1}^{\infty} \left(\frac{1}{(n+1)}, \frac{1}{n}\right) = \left(\frac{1}{2}, 1\right) \cup \left(\frac{1}{3}, \frac{1}{2}\right) \cup \left(\frac{1}{4}, \frac{1}{3}\right) \cup \cdots$$

is a subset of $\mathbf{X}$ but it is not a member of $\mathcal{F}$.

2.4. Let $\mathbf{X} = \blacksquare \subset \mathbb{R}^2$. Let $\mathcal{G}$ denote the set of closed rectangles contained in $\mathbf{X}$, whose sides are parallel to the coordinate axes and whose corners have rational coordinates. Let $\mathcal{F}$ denote the field generated by $\mathcal{G}$. An example of an element of $\mathcal{F}$ is

$$((\blacksquare \setminus ((\blacksquare \setminus R_1) \cup R_2) \cap R_3) \cup (R_4 \cap (\blacksquare \setminus R_5)),$$

where R_1, R_2, R_3, R_4, and R_5 are rectangles in $\mathcal{G}$. Let $S \in \mathcal{F}$. Prove that the area of S is a rational number. Deduce that $\mathcal{F}$ does not contain the ball $B(O, 1) = \{(x, y) \in \blacksquare : x^2 + y^2 \leq 1\}$.

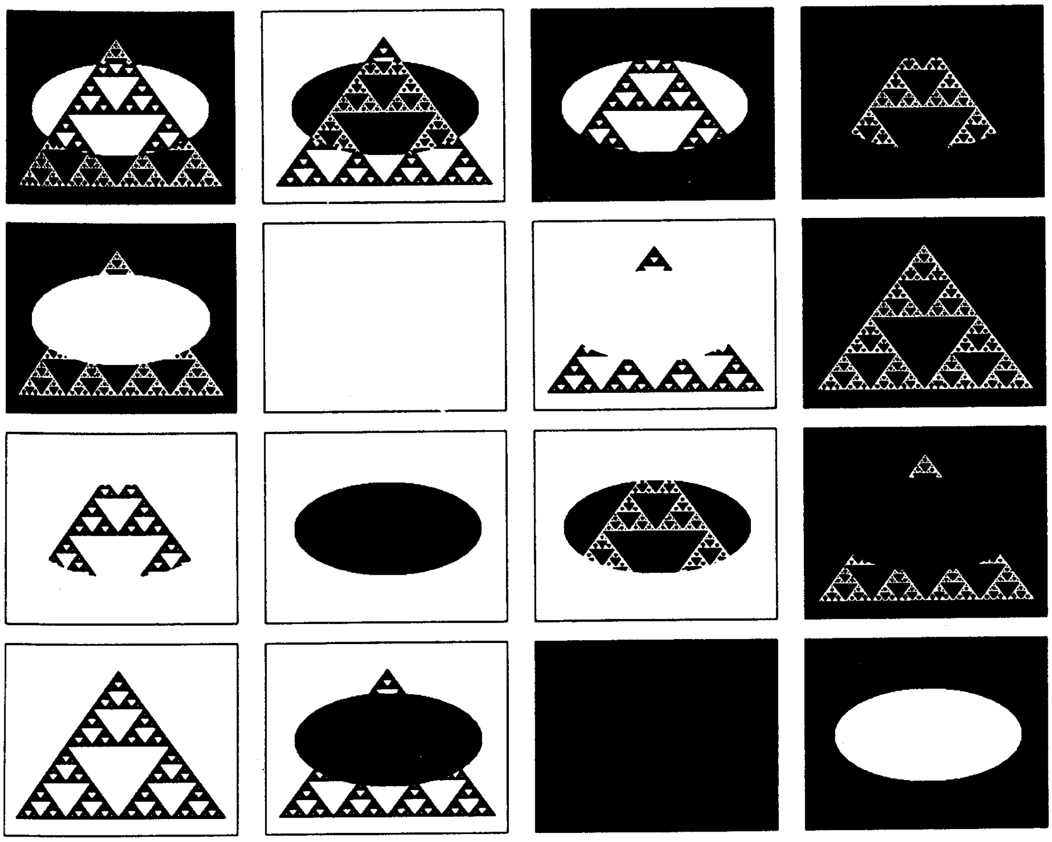


Figure IX.252. A field whose elements are sets of pixels. Can you find two elements of the field which generate the field?

2.5. Let **X** denote the set of pixels corresponding to a certain computer graphics display device. The set of all monochrome images that can be produced on this device forms a field. Figure IX.252 shows an example of a small field of subsets of **X**. It is generated by the pair of images, G_1 and G_2, in the middle of the second row, together with the set **X**. **X** is represented by the black rectangle. The empty set is represented by a blank screen. Find formulas for all of the images in Figure IX.252, in terms of G_1, G_2, and **X**.

2.6. Let Σ denote the code space on two symbols 1 and 2. Let $n \in \{1, 2, 3, \ldots\}$ and $e_i \in \{1, 2\}$ for $i = 1, 2, \ldots, n$. Let

$$C(e_1, e_2, \ldots, e_n) = \{x \in \Sigma : x_i = e_i \text{ for } i = 1, 2, \ldots, n\}.$$

Any subset of Σ that can be written in this form is called a *cylinder subset* of Σ. Let $\mathcal{F}$ denote the field generated by the cylinder subsets of Σ. Find a subset of Σ that is not in $\mathcal{F}$.

2.7. Let **X** be a space. Let $\mathcal{F}$ denote the set of all subsets of **X**. The customary notation for this field is $\mathcal{F} = 2^{\mathbf{X}}$. Show that $\mathcal{F}$ is a field.

Definition 2.3 *Let $\mathcal{F}$ be a field such that*

$$A_i \in \mathcal{F} \text{ for } i \in l, 2, 3, \ldots \Rightarrow \cup_{i=1}^{\infty} A_i \in \mathcal{F}.$$

*Then $\mathcal{F}$ is called a σ-*field *(sigma-field).*

Given any field, there always is a minimal, or smallest, σ-field which contains it.

Theorem 2.2 *Let $\mathbf{X}$ be a space and let $\mathcal{G}$ be a set of subsets of $\mathbf{X}$. Let $\{\mathcal{F}_\alpha : \alpha \in I\}$ denote the set of all σ-fields on $\mathbf{X}$ which contain $\mathcal{G}$. Then $\mathcal{F} = \cap_\alpha \mathcal{F}_\alpha$ is a σ-field.*

Proof Note that there is at least one σ-field that contains $\mathcal{G}$, namely $2^{\mathbf{X}}$, the field consisting of all subsets of $\mathbf{X}$. We have to show that $\cap_\alpha \mathcal{F}_\alpha$ is a σ-field if each $\mathcal{F}_\alpha$ is a σ-field that contains $\mathcal{G}$. Suppose that $A_i \in \cap_\alpha \mathcal{F}_\alpha$; then, for each α, A_i is an element of the σ -field $\mathcal{F}_\alpha$ and so $\cup_{i=1}^{\infty} A_i \in \mathcal{F}_\alpha$. Suppose $A \in \cap_\alpha \mathcal{F}_\alpha$; then, for each α, $A \in \mathcal{F}_\alpha$ and so $\mathbf{X} \setminus A \in \mathcal{F}_\alpha$. Hence $\mathbf{X} \setminus A \in \cap_\alpha \mathcal{F}_\alpha$. This completes the proof.

Definition 2.4 *Let $\mathcal{G}$ be a set of subsets of a space $\mathbf{X}$. The minimal σ-field which contains $\mathcal{G}$, defined in Theorem 2.2, is called the σ-field* generated *by $\mathcal{G}$.*

Definition 2.5 *Let $(\mathbf{X}, d)$ be a metric space. Let $\mathcal{B}$ denote the σ-field generated by the open subsets of $\mathbf{X}$. $\mathcal{B}$ is called the* Borel field *associated with the metric space. An element of $\mathcal{B}$ is called a* Borel subset *of $\mathbf{X}$.*

The following theorem gives the flavor of ways in which the Borel field can be generated.

Theorem 2.3 *Let $(\mathbf{X}, d)$ be a compact metric space. Then the associated Borel field $\mathcal{B}$ is generated by a countable set of balls.*

Proof We prove a more general result first. Let $\mathcal{G} = \{b_n \subset \mathbf{X} : n = 1, 2, 3, \ldots; b_n \text{ open }\}$ be a *countable base* for the open subsets of $\mathbf{X}$. That is, every open set in $\mathbf{X}$ can be written as a union of sets in $\mathcal{G}$. Then $\mathcal{B}$ is generated by $\mathcal{G}$. To see this, let $\tilde{\mathcal{B}}$ denote the σ-field generated by $\mathcal{G}$. Then $\tilde{\mathcal{B}} \subset \mathcal{B}$ because $\mathcal{G}$ is contained in the set of open subsets of $\mathbf{X}$. On the other hand, $\mathcal{B} \subset \tilde{\mathcal{B}}$ because $\tilde{\mathcal{B}}$ contains all the generators of $\mathcal{B}$. Hence $\mathcal{B} = \tilde{\mathcal{B}}$.

It remains to construct a countable base for the open subsets of $\mathbf{X}$ using balls. For $R > 0$ let

$$B(x, R) = \{y \in \mathbf{X} : d(x, y) < R\}.$$

Let n be a positive integer. Then $\mathbf{X} = \cup_{x \in \mathbf{X}} B(x, \frac{1}{n})$. Hence $\{B(x, \frac{1}{n}) : x \in \mathbf{X}\}$ is an open covering of $\mathbf{X}$. Since $\mathbf{X}$ is compact it contains a finite subcovering $\{B(x_m^{(n)}, \frac{1}{n}) : m = 1, 2, \ldots, M(n)\}$ for some integer $M(n)$. We claim that

$$\mathcal{D} = \{B(x_m^{(n)}, \frac{1}{n}) : m = 1, 2, \ldots, M(n); n = 1, 2, 3, \ldots\}$$

is a countable base for the open subsets of $\mathbf{X}$. Let $\mathcal{O}$ be an open subset of $\mathbf{X}$, and let $x \in \mathcal{O}$. Then there is an open ball, of radius $R > 0$, such that $B(x, R) \subset \mathcal{O}$. Let n be large enough that $\frac{1}{n} < \frac{R}{2}$. Then there is $m \in \{1, 2, \ldots, M(n)\}$ so that x is in the ball $B(x_m^{(n)}, \frac{1}{n})$, and this ball is contained in $\mathcal{O}$. Each x in $\mathcal{O}$ is contained in such a ball, belonging to $\mathcal{D}$. Hence $\mathcal{D}$ is indeed a countable base for the open subsets of $\mathbf{X}$. This completes the proof.

Examples & Exercises

2.8. Let $\mathcal{B}$ denote the σ-field generated by the field in exercise 2.4. Then $\mathcal{B}$ contains the ball $B(O, 1)$. Similarly it contains all balls in $\blacksquare \subset \mathbb{R}^2$. Show that $\mathcal{B}$ is the Borel field associated with ($\blacksquare$, Manhattan).

2.9. Let Σ denote the code space on the two symbols $\{0, 1\}$. Show that the Borel field associated with (Σ, code space metric) is generated by the cylinder subsets of Σ, defined in exercise 2.5.

2.10. Let $\blacktriangle \subset \mathbb{R}^2$ denote a Sierpinski triangle. Let $\mathcal{G}$ denote the set of connected components of $\mathbb{R}^2 \setminus \blacktriangle$. Let $\mathcal{F}$ denote the σ-field generated by $\mathcal{G}$. Show that $\mathcal{F}$ is contained in, but not equal to, the Borel field associated with ($\mathbb{R}^2$,Euclidean).

2.11. Let $\mathbf{X}$ be a space and let $\mathcal{G}$ be a set of subsets of $\mathbf{X}$. Let $\mathcal{F}_1$ be the field generated by $\mathcal{G}$, let $\mathcal{F}_2$ be the σ-field generated by $\mathcal{G}$, and let $\mathcal{F}_3$ be the σ-field generated by $\mathcal{F}_1$. Prove that $\mathcal{F}_3 = \mathcal{F}_2$.

2.12. Let $\mathcal{F}$ be a field of subsets of a space $\mathbf{X}$. Prove that $\mathbf{X} \in \mathcal{F}$.

3 Measures

A measure is defined on a field. Each member of the field is assigned a nonnegative real number, which tells us its "mass."

Definition 3.1 *A measure μ, on a field $\mathcal{F}$, is a real nonnegative function $\mu : \mathcal{F} \to [0, \infty) \subset \mathbb{R}$, such that whenever $A_i \in \mathcal{F}$ for $i = 1, 2, 3, \ldots$, with $A_i \cap A_j = \emptyset$ for $i \neq j$ and $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$, we have*

$$\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i).$$

(In other texts a measure as defined here is usually referred to as a finite measure.)

Definition 3.2 *Let $(\mathbf{X}, d)$ be a metric space. Let $\mathcal{B}$ denote the Borel subsets of $\mathbf{X}$. Let μ be a measure on $\mathcal{B}$. Then μ is called a* Borel measure.

Some basic properties of measures are summarized below.

Theorem 3.1 *Let $\mathcal{F}$ be a field and let $\mu : \mathcal{F} \to \mathbb{R}$ be a measure. Then*

(1) If $B \supset A$, then $\mu(B) = \mu(B \setminus A) + \mu(B)$, for $A, B \in \mathcal{F}$;
(2) If $B \supset A$, then $\mu(B) \geq \mu(A)$;
(3) $\mu(\emptyset) = 0$;
(4) If $A_i \in \mathcal{F}$ for $i = 1, 2, 3, \ldots$, and $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$, then $\mu\left(\cup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i)$;
(5) If $\{A_i \in \mathcal{F}\}$ obeys $A_1 \subset A_2 \subset A_3 \subset \ldots$, and if $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$, then $\mu(A_i) \to \mu(\cup_{i=1}^{\infty} A_i)$.

(6) If $\{A_i \in \mathcal{F}\}$ obeys $A_1 \supset A_2 \supset A_3 \supset \ldots$, and if $\cap_{i=1}^{\infty} A_i \in \mathcal{F}$, then $\mu(A_i) \rightarrow \mu(\cap_{i=1}^{\infty} A_i)$.

Proof [Rudin 1966] Theorem 1.19, p. 17. These are fun to prove for yourself!

We are concerned with measures on compact subsets of metric spaces such as ($\mathbb{R}^2$, Euclidean). The natural underlying σ-field is the Borel field, generated by the open subsets of the metric space. The following theorem allows us to work with the restriction of the measure to any field that generates the σ-field.

Theorem 3.2 *[Caratheodory] Let μ denote a measure on a field $\mathcal{F}$. Let $\hat{\mathcal{F}}$ denote the σ-field generated by $\mathcal{F}$. Then there exists a unique measure $\hat{\mu}$ on $\hat{\mathcal{F}}$ such that $\mu(A) = \hat{\mu}(A)$ for all $A \in \mathcal{F}$.*

Sketch of proof The proof can be found in most books on measure theory; see [Eisen 1969] Theorem 5, p. 180, Chapter 6, for example. First μ is used to define an "outer measure" μ^0 on the set of subsets of $\mathbf{X}$. μ^0 is defined by

$$\mu^0(A) = \inf\left\{\sum_{n=1}^{\infty} \mu(A_n) : A \subset \cup_{n=1}^{\infty} A_n,\ A_n \in \mathcal{F} \forall n \in \mathcal{Z}^+\right\}.$$

μ^0 is not usually a measure. However, one can show that the class $\mathcal{F}^0$ of subsets A of $\mathbf{X}$ such that—this was Caratheodory's smart idea—

$$\mu^0(E) = \mu^0(A \cap E) + \mu^0((\mathbf{X} \setminus A) \cap E) \qquad \text{for all } E \in 2^{\mathbf{X}}$$

is a σ-field that contains $\mathcal{F}$. One can also show that μ^0 is a measure on $\mathcal{F}^0$. Note that $\mathcal{F}^0 \supset \hat{\mathcal{F}}$. $\hat{\mu}$ is defined by restricting μ^0 to $\hat{\mathcal{F}}$. Finally one shows that this extension of μ to $\hat{\mathcal{F}}$ is unique. This completes the sketch.

In the above sketch we have discovered how to evaluate the extended measure $\hat{\mu}$ in terms of its values on the original field.

Theorem 3.3 *Let a measure μ on a field $\mathcal{F}$ be extended to a measure $\hat{\mu}$ on the minimal σ-field $\hat{\mathcal{F}}$ that contains $\mathcal{F}$. Then, for all $A \in \hat{\mathcal{F}}$,*

$$\hat{\mu}(A) = \inf\left\{\sum_{n=1}^{\infty} \mu(A_n) : A \subset \cup_{n=1}^{\infty} A_n,\ A_n \in \mathcal{F} \forall n = 1, 2, \ldots\right\}.$$

Examples & Exercises

3.1. Consider the field $\mathcal{F} = \{\mathbf{X}, A, \mathbf{X} \setminus A, \emptyset\}$, where $A \neq \mathbf{X}$ and $A \neq \emptyset$. A measure $\mu : \mathcal{F} \rightarrow \mathbb{R}$ is defined by $\mu(\mathbf{X}) = 7.2$, $\mu(A) = 3.5$, $\mu(\mathbf{X} \setminus A) = 3.7$, and $\mu(\emptyset) = 0$. $\mathcal{F}$ is also a σ-field. The extension of the measure promised by Caratheodory's theorem is just the measure itself.

3.2. Let $\mathcal{F}$ be the field made of sets of leaves on a certain tree, at a certain instant in time, and let $\mu(A)$ be the number of aphids on all the leaves in $A \in \mathcal{F}$. Then μ is a measure on a finite σ-field.

3.3. Let $\mathbf{X} = [0, 1] \subset \mathbb{R}$. Let $\mathcal{F}$ be the field generated by the set of subintervals of $[0, 1]$. Let $a, b \in [0, 1]$ and define $\mu((a, b)) = \mu([a, b]) = b - a$, for $a \leq b$; and more generally let

$$\mu(\text{element of } \mathcal{F}) = \text{ sum of lengths of subintervals which comprise the element.}$$

Show that μ is a measure on $\mathcal{F}$. The σ-field $\hat{\mathcal{F}}$ generated by $\mathcal{F}$ is the Borel field for ([0, 1],Euclidean). Show that $S = \{x \in [0, 1] : x \text{ is a rational number }\}$ belongs to $\hat{\mathcal{F}}$ but not to $\mathcal{F}$. Evaluate $\hat{\mu}(S)$, where $\hat{\mu}$ is the extension of μ to $\hat{\mathcal{F}}$.

3.4. Let $\mathbf{X} = \Sigma$, the code space on the two symbols 1 and 2. Let $\mathcal{F}$ denote the field generated by the cylinder subsets of Σ, as defined in exercise 2.5. Let $0 \leq p_1 \leq 1$ and $p_2 = 1 - p_1$. Define

$$\mu(C(e_1, e_2, \ldots, e_n)) = p_{e_1} p_{e_2} \cdots p_{e_n},$$

for each cylinder subset $C(e_1, e_2, \ldots, e_n)$ of Σ. Show how μ can be defined on the other elements of $\mathcal{F}$ in such a way as to provide a measure on $\mathcal{F}$. Evaluate

$$\mu(\{x \in \Sigma : x_7 = 1\}) \text{ and } \mu(\Sigma).$$

Extend $\mathcal{F}$ to the field $\hat{\mathcal{F}}$ generated by $\mathcal{F}$, and correspondingly extend μ to $\hat{\mu}$. Show that

$$S = \{x \in \Sigma : x_{\text{odd}} = 1\} \in \hat{\mathcal{F}}$$

and evaluate $\hat{\mu}(S)$.

3.5. This example takes place in the metric space ([0, 1],Euclidean). Consider the IFS with probabilities

$$\{[0, 1]; w_1(x) = \frac{1}{3}x, w_2(x) = \frac{1}{3}x + \frac{2}{3}; p_1, p_2\}.$$

Let $\mathcal{F}$ denote the field generated by the set of intervals that can be expressed in the form

$$w_{e_1} \circ w_{e_2} \circ \ldots \circ w_{e_n}([0, 1]),$$

where $n \in \{1, 2, \ldots\}$ and $e_i \in \{1, 2\}$ for each $i = 1, 2, \ldots, n$. Let $0 \leq p_1 \leq 1$ and $p_2 = 1 - p_1$. Show that one can define a measure on $\mathcal{F}$ so that, for every such interval,

$$\mu(w_{e_1} \circ w_{e_2} \circ \cdots \circ w_{e_n}([0, 1])) = p_{e_1} p_{e_2} \cdots p_{e_n}.$$

Let A denote the attractor of the IFS. Evaluate $\mu(A)$, $\mu(\mathbf{X} \setminus A)$, and $\mu([\frac{1}{3}, \frac{2}{3}])$.

3.6. What happens in exercise 3.5 if the IFS is replaced by

$$\{[0, 1]; w_1(x) = \frac{1}{2}x, w_2(x) = \frac{1}{2}x + \frac{1}{2}; p_1, p_2\}?$$

For what value of p_1 is the extension of the measure to the σ-field generated by $\mathcal{F}$ the same as the Borel measure defined in exercise 3.3?

Definition 3.3 *Let* $(\mathbf{X}, d)$ *be a metric space, and let* μ *be a Borel measure. Then the* support *of* μ *is the set of points* $x \in \mathbf{X}$ *such that* $\mu(B(x, \epsilon)) > 0$ *for all* $\epsilon > 0$, *where* $B(x, \epsilon) = \{y \in \mathbf{X} : d(y, x) < \epsilon\}$.

The support of a measure is the set on which the measure lives. The following is an easy exercise.

Theorem 3.4 *Let* $(\mathbf{X}, d)$ *be a metric space, and let* μ *be a Borel measure. Then the support of* μ *is closed.*

Examples & Exercises

3.7. Let $(\mathbf{X}, d)$ be a compact metric space and let μ be a Borel measure on $\mathbf{X}$ such that $\mu(\mathbf{X}) \neq 0$. Show that the support of μ belongs to $\mathcal{H}(\mathbf{X})$, the space of nonempty compact subsets of $\mathbf{X}$.

3.8. Prove the following. "Let μ be a measure on a σ-field $\mathcal{F}$, and let $\overline{\mathcal{F}}$ be the class of all sets of the form $A \cup B$ where $A \in \mathcal{F}$ and B is a subset of a set of measure zero. Then $\overline{\mathcal{F}}$ is a σ-field and the function $\overline{\mu} : \overline{\mathcal{F}} \to \mathbb{R}$ defined by $\overline{\mu}(A \cup B) = \overline{\mu}(A)$ is a measure." The measure $\overline{\mu}$ referred to here is called the *completion* of μ. The completion of the measure in exercise 3.3 is called the *Lebesgue* measure on $[0, 1]$.

4 Integration

In the next section we will introduce a remarkable compact metric space. It is a space whose points are measures! In order to define the metric on this space we need to be able to integrate continuous real-valued functions with respect to measures. Can one integrate a continuous function defined on a fractal? How does one evaluate the "average" temperature of the coastline of Sweden? Here we learn how to integrate functions with respect to measures. Let $(\mathbf{X}, d)$ be a compact metric space. Let μ be a Borel measure on $\mathbf{X}$. Let $f : \mathbf{X} \to \mathbb{R}$ be a continuous function. We will explain the meaning of integrals such as

$$\int_{\mathbf{X}} f(x)\, d\mu(x).$$

Definition 4.1 *We reserve the notation* χ_A *for the* characteristic function *of a set* $A \subset \mathbf{X}$. *It is defined by*

$$\chi_A(x) = \begin{cases} 1 & \text{for } x \in A, \\ 0 & \text{for } x \in \mathbf{X} \setminus A. \end{cases}$$

A function $f : \mathbf{X} \to \mathbb{R}$ *is called* simple *if it can be written in the form*

$$f(x) = \sum_{i=1}^{N} y_i \chi_{I_i}(x),$$

where N is a positive integer, $I_i \in \mathcal{B}$ and $y_i \in \mathbb{R}$ for $i = 1, 2, \ldots, N$, $\cup_{i=1}^{N} I_i = \mathbf{X}$, and $I_i \cap I_j = \emptyset$ for $i \neq j$.

The graphs of several simple functions, associated with different spaces, are shown in Figures IX.253 and IX.254.

Definition 4.2 *The* integral *(with respect to μ) of the simple function f in Definition 4.1, is*

$$\int_{\mathbf{X}} f(x)\, d\mu(x) = \int_{\mathbf{X}} f\, d\mu = \sum_{i=1}^{N} y_i \mu(I_i).$$

Examples & Exercises

4.1. Let $f : [0, 1] \to \mathbb{R}$ be a piecewise constant function, with finitely many discontinuities. Show that f is a simple function. Let μ denote the Borel measure on $[0, 1]$ such that $\mu(I) =$ length of I, when I is a subinterval of $[0, 1]$. Show that

$$\int_0^1 f(x)\, dx = \int_{[0,1]} f(x)\, d\mu(x),$$

where the right-hand side denotes the area under the graph of f.

4.2. This example takes place in the metric space ($\blacksquare$, Euclidean). Let $\mathcal{G}$ denote the set of rectangular subsets of $\blacksquare$. Let $\mathcal{F}$ denote the field generated by $\mathcal{G}$. Show that there is a unique measure μ on $\mathcal{F}$ such that $\mu(A) =$ area of A, for all $A \in \mathcal{G}$. Notice that the σ-field generated by $\mathcal{F}$ is precisely the Borel field $\mathcal{B}$ associated with ($\blacksquare$, Euclidean). Let $\hat{\mu}$ denote the extension of μ to $\mathcal{B}$. Let $\blacktriangle$ denote a Sierpinski triangle contained in $\blacksquare$. Show that $\blacktriangle \in \mathcal{B}$, and

$$\int_{\blacksquare} \chi_{\blacktriangle}\, d\hat{\mu} = \hat{\mu}(\blacktriangle) = 0.$$

4.3. This example concerns the IFS with probabilities

$$\{\mathbb{C};\ w_1(z), w_2(z), w_3(z);\ p_1 = 0.2,\ p_2 = 0.3,\ p_3 = 0.5\},$$

where

$$w_1(z) = 0.5z, \qquad w_2(z) = 0.5z + (0.5)i, \qquad w_3(z) = 0.5z + 0.5.$$

Let $\blacktriangle$ denote the attractor of the IFS, and $\mathcal{B}$ the Borel subsets of ($\blacktriangle$, Euclidean). Let μ denote the unique measure on $\mathcal{B}$ such that

$$\begin{aligned} \mu(\blacktriangle) &= 1 \\ \mu(w_i(\blacktriangle)) &= p_i \quad \text{for } i \in \{1, 2, 3\}; \\ \mu(w_i \circ w_j(\blacktriangle)) &= p_i p_j \quad \text{for } i, j \in \{1, 2, 3\}; \\ &\vdots \\ \mu(w_i \circ w_j \cdots \circ w_k(\blacktriangle)) &= p_i p_j \ldots p_k \quad \text{for } i, j, \ldots, k \in \{1, 2, 3\}; \end{aligned}$$

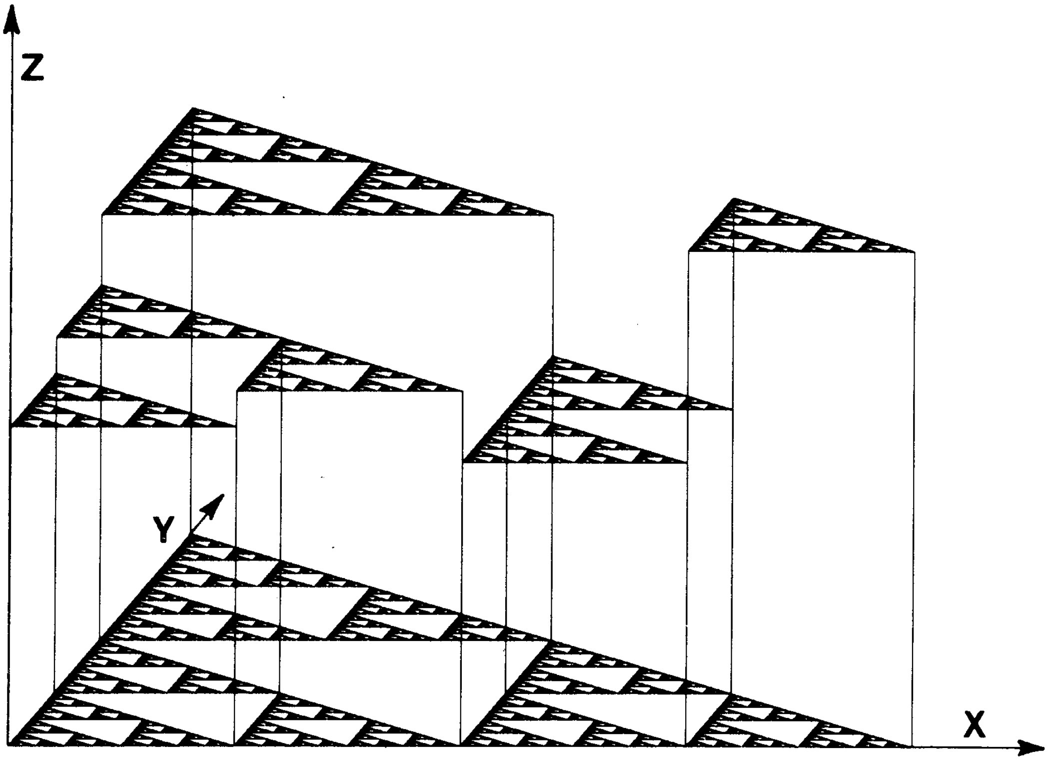


Figure IX.253. The graph of a simple function on a Sierpinski triangle. The domain is a Sierpinski triangle in the (x, y) plane. The function values are represented by the z-coordinates.

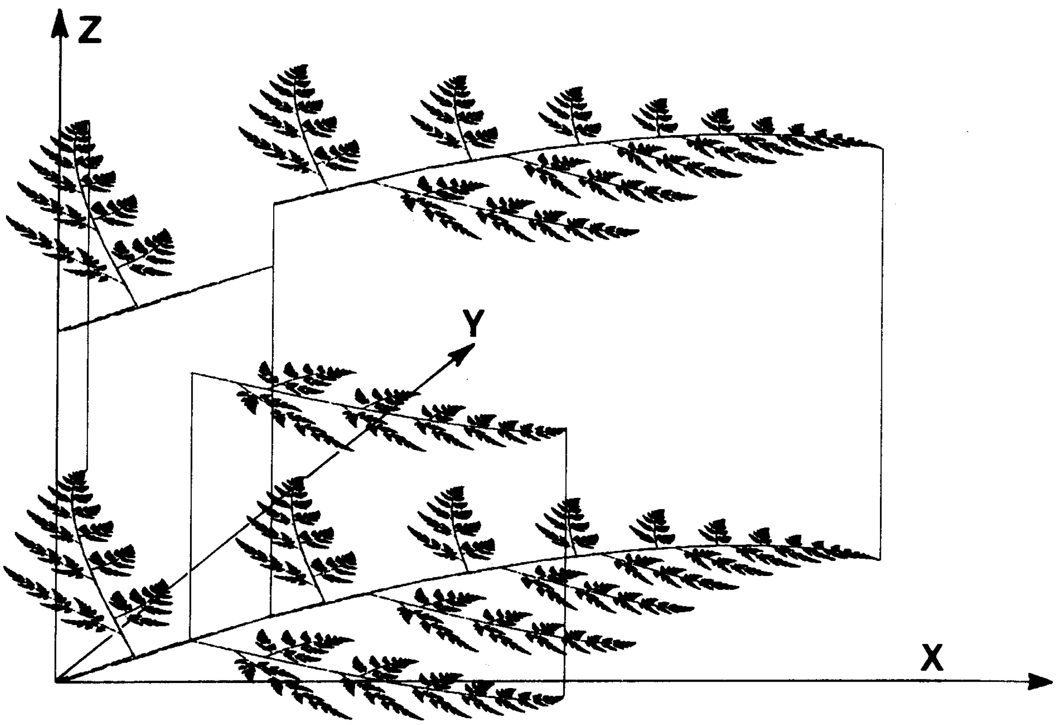


Figure IX.254. The graph of a function whose domain is a fractal fern. If, instead, the function values were represented by colors, a painted fern would replace the graph.

Define a simple function on ▲ by

$$f(x+iy)=\begin{cases}1 & \text{for } x+iy\in ▲ \text{ and } 1/3\le x\le 1,\\ -1 & \text{for } x+iy\in ▲ \text{ and } 0\le x\le 2/3.\end{cases}$$

Calculate $\int_{▲} f(z)d\mu(z)$, accurate to two decimal places.

Based on the ideas in section 1 of this chapter, can you guess a method for calculating the integral that makes use of the Random Iteration Algorithm? Try it!

4.4. Show that if $\alpha, \beta \in \mathbb{R}$ and f, g are simple functions then $\alpha f + \beta g$ is a simple function, and

$$\alpha \int_{\mathbf{X}} f\,d\mu + \beta \int_{\mathbf{X}} g\,d\mu = \int_{\mathbf{X}} (\alpha f + \beta g)\,d\mu.$$

4.5. Black ink is printed to make this page. Let $■ \subset \mathbb{R}^2$ be a model for the page, and represent the ink by means of a Borel measure μ, so that $\mu(A)$ is the mass of ink associated with the set $A \subset ■$. Let $\mathcal{A} \in \mathcal{F}$ denote the smallest Borel set that contains all of the letters "a"on the page. Assume that the total mass of ink on the page is one unit. Estimate $\int_{■} \chi_{\mathcal{A}} d\mu$.

4.6. Let Σ denote code space on two symbols $\{1, 2\}$. $\mathcal{B}$ denotes the Borel field associated with (Σ, code space metric). Consider the IFS $\{\Sigma; w_1(x) = 1x, w_2(x) = 2x; p_1 = 0.4, p_2 = 0.6\}$, where "$1x$" means the string "$1x_1x_2x_3\ldots$" and "$2x$" means the string "$2x_1x_2x_3\ldots$." The attractor of the IFS is Σ. Let μ denote the unique measure on $\mathcal{B}$ such that

$$\mu(w_i \circ w_j \cdots \circ w_k(\Sigma)) = p_i p_j \ldots p_k \qquad \text{for } i, j, \ldots, k \in \{1, 2\}.$$

Define sets A and B in $\mathcal{B}$ by

$$A = \{x \in \mathcal{B} : x_1 = 0\} \text{ and } B = \{x \in \mathcal{B} : x_2 = 1\}.$$

Define $f : \Sigma \to \mathbb{R}$ by

$$f(x) = \chi_A(x) + (2.3)\chi_B(x) \qquad \text{for all } x \in \Sigma.$$

Evaluate the integral

$$\int_{\Sigma} f(x)\,d\mu(x).$$

Definition 4.3 *Let $(\mathbf{X}, d)$ be a compact metric space, and let $\mathcal{B}$ denote the associated Borel field. Let μ be a Borel measure. A* partition *of $\mathbf{X}$ is a finite set of nonempty Borel sets, $\{A_i \in \mathcal{B} : i = 1, 2, \ldots, M\}$, such that $\mathbf{X} = \cup_{i=1}^{M} A_i$, and $A_i \cap A_j = \emptyset$ for $i \neq j$. The* diameter *of the partition is* $\max\{\sup\{d(x, y) : x, y \in A_i\} : i = 1, 2, \ldots, M\}$.

Theorem 4.1 *Let $(\mathbf{X}, d)$ be a compact metric space. Let $\mathcal{B}$ denote the associated Borel field. Let μ be a Borel measure on $\mathbf{X}$. Let $f : \mathbf{X} \to \mathbb{R}$ be continuous. (i) Let n be a positive integer. Then there exists a partition $\mathcal{B}_n = \{A_{n,m} \in \mathcal{B} : m =$*

$1, 2, \ldots, M(n)\}$ of diameter $1/n$. (ii) Let $x_{n,m} \in A_{n,m}$ for $m = 1, 2, 3, \ldots$, and define a sequence of simple functions by

$$f_n(x) = \sum_{m=1}^{M(n)} f(x_{n,m})\chi_{A_{n,m}}(x) \qquad \text{for } n = 1, 2, 3, \ldots.$$

Then $\{f_n\}$ converges uniformly to $f(x)$. (iii) The sequence $\{\int_{\mathbf{X}} f_n d\mu\}$ converges. (iv) The value of the limit is independent of the particular sequence of partitions, and of the choices of $x_{n,m} \in A_{n,m}$.

Sketch of proof

(i) Since $\mathbf{X}$ is compact it is possible to cover $\mathbf{X}$ by a finite set of closed balls of diameter $1/n$, say $b_{n,1}, b_{n,2}, \ldots, b_{n,M(n)}$. We can assume that each ball contains a point not in any of the other balls. Then define $A_{n,1} = b_{n,1}$, and $A_{n,j} = b_{n,j} \setminus \cup_{k=1}^{j-1} A_{n,k}$, for $j = 2, 3, \ldots, M(n)$. Then $\mathcal{B}_n = \{A_{n,m} \in \mathcal{B} : m = 1, 2, \ldots, M(n)\}$ is a partition of $\mathbf{X}$ of diameter $1/n$.

(ii) Let $\epsilon > 0$. f is continuous on a compact space, so it is uniformly continuous. It follows that there exists an integer $N(\epsilon)$ so that if $x, y \in \mathbf{X}$ and $d(x, y) \leq 1/N(\epsilon)$ then $|f(x) - f(y)| \leq \epsilon$. It follows that $|f(x) - f_n(x)| \leq \epsilon$ when $n \geq N(\epsilon)$.

(iii) It is readily proved that $\{\int_{\mathbf{X}} f_n d\mu\}$ is a Cauchy sequence. Indeed, for all $n, m \geq N(\epsilon)$ we have

$$\left| \int_{\mathbf{X}} f_n \, d\mu - \int_{\mathbf{X}} f_m \, d\mu \right| \leq \int_{\mathbf{X}} |f_n - f_m| \, d\mu \leq 2\epsilon\mu(\mathbf{X}).$$

It follows that the sequence converges.

(iv) Let $\{\tilde{f}_n\}$ be a sequence of simple functions, constructed as above. Then there is an integer $\tilde{N}(\epsilon)$ such that $|f(x) - \tilde{f}_n(x)| \leq \epsilon$ when $n \geq \tilde{N}(\epsilon)$. It follows that for all $n \geq \max\{N(\epsilon), \tilde{N}(\epsilon)\}$,

$$\left| \int_{\mathbf{X}} f_n \, d\mu - \int_{\mathbf{X}} \tilde{f}_n \, d\mu \right| \leq \int_{\mathbf{X}} |f_n - \tilde{f}_n| \, d\mu \leq 2\epsilon\mu(\mathbf{X}).$$

This completes the sketch of the proof.

Definition 4.4 *The limit in Theorem 4.1 is called the* integral *of f (with respect to μ). It is denoted by*

$$\lim_{n\to\infty} \int_{\mathbf{X}} f_n d\mu = \int_{\mathbf{X}} f \, d\mu.$$

Examples & Exercises

4.7. Let $(\mathbf{X}, d)$ be a metric space. Let $a \in \mathbf{X}$. Define a Borel measure δ_a by $\delta_a(B) = 1$ if $a \in B$ and $\delta_a(B) = 0$ if $a \notin B$, for all Borel sets $B \subset \mathbf{X}$. This measure is referred to as a "a delta function" and "a point mass at a." Let $f : \mathbf{X} \to \mathbb{R}$ be continuous. Show that

$$\int_{\mathbf{X}} f(x)\, d\delta_a(x) = f(a).$$

4.8. This example takes place in the metric space (■, Euclidean). Let μ be the measure defined in exercise 4.2, and define $f : ■ \to \mathbb{R}$ by $f(x, y) = x^2 + 2xy + 3$. Evaluate

$$\int_{■} f\, d\mu.$$

4.9. Make an approximate evaluation of the integral $\int_{▲} x^2 d\mu(x)$ where μ and ▲ are as defined in exercise 4.3.

4.10. Let $\mathbf{X}$ denote the set of pixels corresponding to a certain computer graphics display device. Define a metric d on $\mathbf{X}$ so that $(\mathbf{X}, d)$ is a compact metric space. Give an example of a Borel subset of $\mathbf{X}$ and of a nontrivial Borel measure on $\mathbf{X}$. Show that any function $f : \mathbf{X} \to \mathbb{R}$ is continuous. Give a specific example of such a function, and evaluate $\int_{\mathbf{X}} f d\mu$.

5 The Compact Metric Space $(\mathcal{P}(\mathbf{X}), d)$

We introduce the most exciting metric space in the book. It is the space where fractals *really* live.

Definition 5.1 *Let* $(\mathbf{X}, d)$ *be a compact metric space. Let* μ *be a Borel measure on* $\mathbf{X}$. *If* $\mu(\mathbf{X}) = 1$, *then* μ *is said to be* normalized.

Definition 5.2 *Let* $(\mathbf{X}, d)$ *be a compact metric space. Let* $\mathcal{P}(\mathbf{X})$ *denote the set of* normalized Borel measures *on* $\mathbf{X}$. *The* Hutchinson metric d_H *on* $\mathcal{P}(\mathbf{X})$ *is defined by*

$$d_H(\mu, \nu) = \sup\left\{ \int_{\mathbf{X}} f\, d\mu - \int_{\mathbf{X}} f\, d\nu : f : \mathbf{X} \to \mathbb{R} f \text{ continuous}, \right.$$
$$\left. |f(x) - f(y)| \le d(x, y) \forall x, y \in \mathbf{X} \right\},$$

for all $\mu, \nu \in \mathcal{P}(\mathbf{X})$.

Theorem 5.1 *Let* $(\mathbf{X}, d)$ *be a compact metric space. Let* $\mathcal{P}(\mathbf{X})$ *denote the set of normalized Borel measures on* $\mathbf{X}$ *and let* d_H *denote the Hutchinson metric. Then* $(\mathcal{P}(\mathbf{X}), d_H)$ *is a compact metric space.*

Sketch of proof A direct proof, using the tools in this book, is cumbersome. It is straightforward to verify that d_H is a metric. It is most efficient to use the concept of the "weak topology" on $\mathcal{P}(\mathbf{X})$ to prove compactness. One shows that this topology is the same as the one induced by the Hutchinson metric, and then applies Alaoglu's theorem. See [Hutchinson 1981] and [Dunford 1966].

Examples & Exercises

5.1. Let K be a positive integer. Let $\mathbf{X} = \{(i, j) : i, j = 1, 2, \ldots, K\}$. Define a metric on $\mathbf{X}$ by $d((i_1, j_1), (i_2, j_2)) = |i_1 - i_2| + |j_1 - j_2|$. Then $(\mathbf{X}, d)$ is a compact metric space. Let $\mu \in \mathcal{P}(\mathbf{X})$ be such that $\mu((i, j)) = (i + j)/(K^3 + K^2)$ and let $\nu \in \mathcal{P}(\mathbf{X})$ be such that $\nu(i, j) = 1/K^2$, for all $i, j \in \{1, 2, \ldots, N\}$. Calculate $d_H(\mu, \nu)$.

5.2. Let $(\mathbf{X}, d)$ be a compact metric space. Let $\mu \in \mathcal{P}(\mathbf{X})$. Prove that the support of μ belongs to $\mathcal{H}(\mathbf{X})$.

6 A Contraction Mapping on $\mathcal{P}(\mathbf{X})$

Let $(\mathbf{X}, d)$ denote a compact metric space. Let $\mathcal{B}$ denote the Borel subsets of $\mathbf{X}$. Let $w : \mathbf{X} \to \mathbf{X}$ be continuous. Then one can prove that $w^{-1} : \mathcal{B} \to \mathcal{B}$. It follows that if ν is a normalized Borel measure on $\mathbf{X}$ then so is $\nu \circ w^{-1}$. In turn, this implies that the function defined next indeed takes $\mathcal{P}(\mathbf{X})$ into itself.

Definition 6.1 *Let $(\mathbf{X}, d)$ be a compact metric space and let $\mathcal{P}(\mathbf{X})$ denote the space of normalized Borel measures on $\mathbf{X}$. Let*

$$\{\mathbf{X}; w_1, w_2, \ldots, w_N; p_1, p_2, \ldots, p_N\}$$

be a hyperbolic IFS with probabilities. The Markov operator *associated with the IFS is the function $M : \mathcal{P}(\mathbf{X}) \to \mathcal{P}(\mathbf{X})$ defined by*

$$M(\nu) = p_1 \nu \circ w_1^{-1} + p_2 \nu \circ w_2^{-1} + \cdots + p_N \nu \circ w_N^{-1}$$

for all $\nu \in \mathcal{P}(\mathbf{X})$.

Lemma 6.1 *Let M denote the Markov operator associated with a hyperbolic IFS, as in Definition 6.1. Let $f : \mathbf{X} \to \mathbb{R}$ be either a simple function or a continuous function . Let $\nu \in \mathcal{P}(\mathbf{X})$. Then*

$$\int_{\mathbf{X}} f d(M(\nu)) = \sum_{i=1}^{N} p_i \int_{\mathbf{X}} f \circ w_i \, d\nu.$$

Proof Suppose that $f : \mathbf{X} \to \mathbb{R}$ is continuous. By Theorem 5.1 we can find a sequence of simple functions $\{f_n\}$ which converges uniformly to f. Let n be a positive integer. It is readily verified that

$$\begin{aligned}\int_{\mathbf{X}} f_n d(M(\nu)) &= \sum_{i=1}^{N} p_i \int_{\mathbf{X}} f_n \, d\nu \circ w_i^{-1} \\ &= \sum_{i=1}^{N} p_i \int_{w_i(\mathbf{X})} f_n \, d\nu \circ w_i^{-1} \\ &= \sum_{i=1}^{N} p_i \int_{\mathbf{X}} f_n \circ w_i \, d\nu.\end{aligned}$$

The sequence $\{\int f_n d(M(\nu))\}$ converges to $\int f d(M(\nu))$.

For each $i \in \{1, 2, \ldots, N\}$ and each positive integer n, $f_n \circ w_i$ is a simple function. The sequence $\{f_n \circ w_i\}_{n=1}^{\infty}$ converges uniformly to $f \circ w_i$. It follows that $\{\int f_n \circ w_i d\nu\}_{n=1}^{\infty}$ converges to $\int f \circ w_i$. It follows that $\{\sum_{i=1}^{N} p_i \int f_n \circ w_i d\nu\}_{n=1}^{\infty}$ converges to $\sum_{i=1}^{N} p_i \int f \circ w_i d\nu$. This completes the proof.

Theorem 6.1 *Let $(\mathbf{X}, d)$ be a compact metric space. Let*

$$\{\mathbf{X}; w_1, w_2, \ldots, w_N; p_1, p_2, \ldots, p_N\}$$

be a hyperbolic IFS with probabilities. Let $s \in (0, 1)$ be a contractivity factor for the IFS. Let $M : \mathcal{P}(\mathbf{X}) \to \mathcal{P}(\mathbf{X})$ be the associated Markov operator. Then M is a contraction mapping, with contractivity factor s, with respect to the Hutchinson metric on $\mathcal{P}(\mathbf{X})$. That is,

$$d_H(M(\nu), M(\mu)) \leq s d_H(\nu, \mu) \qquad \text{for all } \nu, \mu \in \mathcal{P}(\mathbf{X}).$$

In particular, there is a unique measure $\mu \in \mathcal{P}(\mathbf{X})$ such that

$$M\mu = \mu.$$

Proof Let L denote the set of continuous functions $f : \mathbf{X} \to \mathbb{R}$ such that $|f(x) - f(y)| \leq d(x, y) \forall x, y \in \mathbf{X}$. Then

$$\begin{aligned}d_H(M(\nu), M(\mu)) &= \sup\{\int f d(M(\mu)) - \int f d(M(\nu)) : f \in L\} \\ &= \sup\{\int \sum_{i=1}^{N} p_i f \circ w_i \, d\mu - \int \sum_{i=1}^{N} p_i f \circ w_i \, d\nu : f \in L\}.\end{aligned}$$

Let $\tilde{f} = s^{-1} \sum_{i=1}^{N} p_i f \circ w_i$. Then $\tilde{f} \in L$. Let $\tilde{L} = \{\tilde{f} \in L : \tilde{f} = s^{-1} \sum_{i=1}^{N} p_i f \circ w_i$, some $f \in L\}$. Then we can write

$$d_H(M(\nu), M(\mu)) = \sup\{s \int \tilde{f} d\mu - s \int \tilde{f} d\nu : \tilde{f} \in \tilde{L}\}.$$

Since $\tilde{L} \subset L$, it follows that

$$d_H(M(\nu), M(\mu)) \leq s d_H(\nu, \mu).$$

This completes the proof.

Definition 6.2 *Let μ denote the fixed point of the Markov operator, promised by Theorem 6.1. μ is called the* invariant measure *of the IFS with probabilities.*

We have arrived at our goal! This invariant measure is the object we discussed informally in section 1 of this chapter. *Now* we know what fractals are.

Examples & Exercises

6.1. Verify that the Markov operator associated with a hyperbolic IFS on a compact metric space indeed maps the space into itself.

6.2. This example uses the notation in the proof of Theorem 6.1. Let $f \in L$ and let $\tilde{f} = s^{-1} \sum_{i=1}^{N} p_i f \circ w_i$. Prove that $\tilde{f} \in L$.

6.3. Consider the hyperbolic IFS

$$\{\blacksquare \subset \mathbb{R}^2; w_1, w_2, w_3, w_4; p_1, p_2, p_3, p_4\}$$

corresponding to the collage in Figure IX.255(a). Let M be the associated Markov operator. Let $\mu_0 \in \mathcal{P}(\mathbf{X})$, so that $\mu_0(\blacksquare) = 1$. For example, μ_0 could be the uniform measure, for which $\mu_0(S)$ is the area of $S \in \mathcal{P}(\blacksquare)$. We look at the sequence of measures $\{\mu_n = M^{\circ n}(\mu_0)\}$. The measure $\mu_1 = M(\mu_0)$ is such that $\mu(w_i(\blacksquare)) = p_i$ for $i = 1, 2, 3, 4$, as illustrated in Figure IX.255(b). It follows that $\mu_2 = M^{\circ 2}(\mu_0)$ obeys $\mu(w_i \circ w_j(\blacksquare)) = p_i p_j$ for $i, j = 1, 2, 3, 4$, as illustrated in Figure IX.255(c). We quickly get the idea. When the Markov operator is applied, the "mass" in a cell $\blacksquare_{ij\ldots k} = w_i \circ w_j \circ \cdots \circ w_k(\blacksquare)$ is redistributed among the four smaller cells $w_1(\blacksquare_{ij\ldots k})$, $w_2(\blacksquare_{ij\ldots k})$, $w_3(\blacksquare_{ij\ldots k})$, and $w_4(\blacksquare_{ij\ldots k})$. Also, mass from other cells is mapped into subcells of $\blacksquare_{ij\ldots k}$ in such a way that the total mass of $\blacksquare_{ij\ldots k}$ remains the same as before the Markov operator was applied. In this manner the distribution of "mass" is defined on finer and finer scales as the Markov operator is repeatedly applied. What a wonderful idea. We have also illustrated this idea in Figures IX.256 and IX.257.

6.4. Apply the Random Iteration Algorithm to an IFS of the form considered in example 6.3. Choose the probabilities p_1, p_2, p_3, and p_4 so as to obtain a "picture" of the invariant measure that would occur at the end of the sequence that commences in Figure IX.257(a), (b), (c), and (d).

6.5. Consider the IFS

$$\{[0, 1] \subset \mathbb{R}; w_1(x) = (0.5)x, w_2(x) = (0.7)x + 0.3; p_1 = 0.45, p_2 = 0.55\}.$$

The attractor of the IFS is $[0, 1]$. Let M denote the associated Markov operator. Let $\mu_0 \in \mathcal{P}([0, 1])$ be the uniform measure on $[0, 1]$. In Figure IX.258(a), μ_0 is represented by a rectangle, whose base is $[0, 1]$ and whose area is 1. The successive iterates $M(\mu_0)$, $M^{\circ 2}(\mu_0)$, $M^{\circ 3}(\mu_0)$ are represented in Figure IX.258(b), (c) and (d). Each measure is represented by a collection of rectangles whose bases are contained in the interval $[0, 1]$. The area of a rectangle equals the measure of the base of the rectangle. Although the sequence of measures converges $\{M^{\circ n}(\mu_0)\}$ in the metric

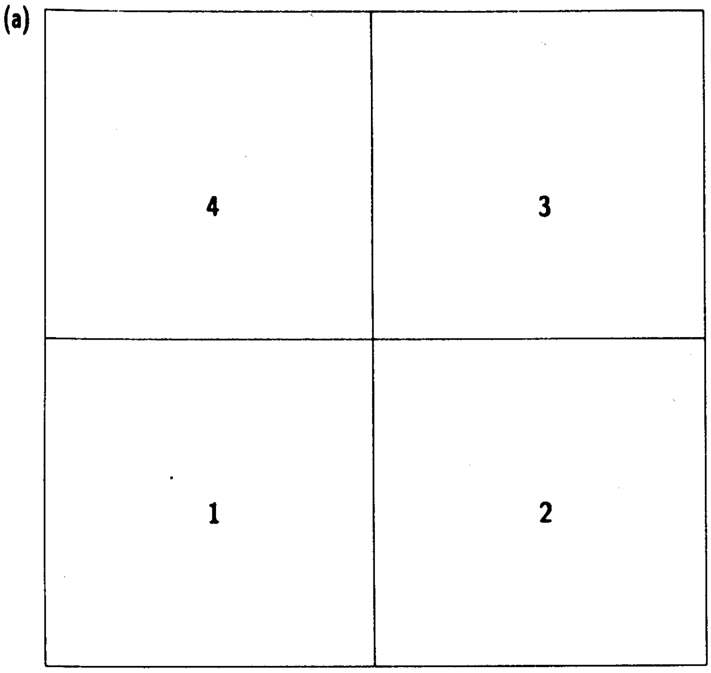


Figure IX.255. A collage for an IFS of four maps. The attractor of the IFS is ■, and the probability of the map w_i is p_i for $i = 1, 2, 3, 4$. Let M denote the associated Markov operator. Let $\mu_0 = 1$. Then $\mu_1 = M(\mu_0)$ is a measure such that $\mu(w_i(■)) = p_i$ for $i = 1, 2, 3, 4$, as illustrated in (b). The measure $\mu_2 = M^{\circ 2}(\mu_0)$ is such that $\mu(w_i \circ w_j(■)) = p_i p_j$ for $i, j = 1, 2, 3, 4$, as illustrated in (c). See also Figures IX.256 and IX.257.

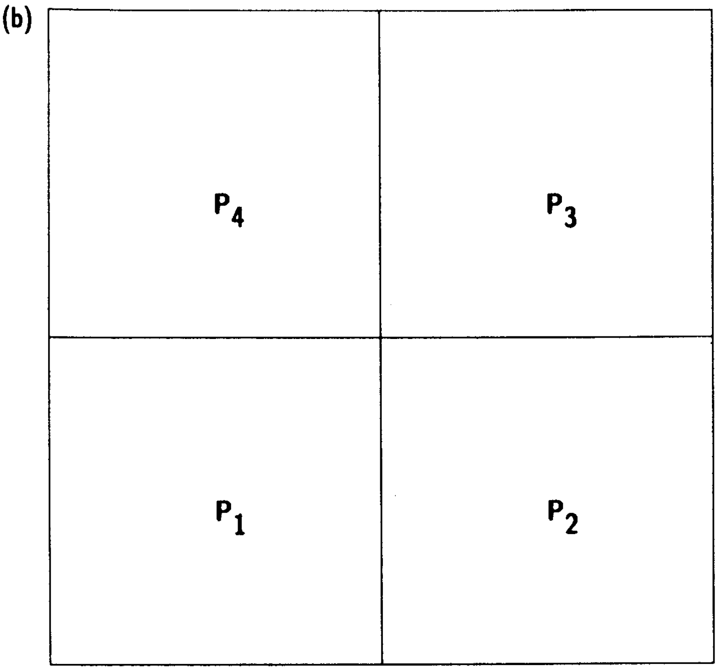


Figure IX.255. (b)

Figure IX.255. (c)

(c)

P_4P_4	P_4P_3	P_3P_4	P_3P_3
P_4P_1	P_4P_2	P_3P_1	P_3P_2
P_1P_4	P_1P_3	P_2P_4	P_2P_3
P_1P_1	P_1P_2	P_2P_1	P_2P_2

space$\{\mathcal{P}([0,1], d_H\}$, some of the rectangles would become infinitely tall as n tends to infinity.

6.6. Make a sequence of figures, analagous to Figure IX.258(a)–(d), to represent the Markov operator applied to the uniform measure μ_0, for each of the following IFS's with probabilities:

$(i)\{[0,1] \subset \mathbb{R};\ w_1(x) = (0.5)x,\ w_2(x) = (0.5)x + 0.5;\ p_1 = 0.5,\ p_2 = 0.5\}$;

$(ii)\{[0,1] \subset \mathbb{R};\ w_1(x) = (0.5)x,\ w_2(x) = (0.5)x + 0.5;$
$p_1 = 0.99,\ p_2 = 0.01\}$;

$(iii)\{[0,1] \subset \mathbb{R};\ w_1(x) = (0.9)x,\ w_2(x) = (0.9)x + 0.1;\ p_1 = 0.45,\ p_2 = 0.55\}$.

In each case describe the associated invariant measure.

6.7. Let $\mathbf{X} = \{A, B, C\}$ denote a space that consists of three points. Let $\mathcal{B}$ denote the σ-field that consists of all subsets of $\mathbf{X}$. Consider the IFS with probabilities

$$\{\mathbf{X};\ w_1, w_2;\ p_1 = 0.6,\ p_2 = 0.4\},$$

where $w_1 : \mathbf{X} \to \mathbf{X}$ is defined by $w_1(A) = B$, $w_1(B) = B$, $w_1(C) = B$, and $w_2 : \mathbf{X} \to \mathbf{X}$ is defined by $w_2(A) = C$, $w_2(B) = A$, and $w_2(C) = C$. Let $\mathcal{P}(\mathbf{X})$ denote the set of normalized measures on $\mathcal{B}$. Let $\mu_0 \in \mathcal{P}(\mathbf{X})$ be defined by $\mu_0(A) = \mu_0(B) = \mu_0(C) = \frac{1}{3}$. Let M denote the Markov operator associated with the IFS, and let $\mu_n = M^{\circ n}(\mu_0)$ for $n = 1, 2, 3, \ldots$. Determine real numbers $a, b, c, d, e, f, g, h, i$ such that for each n,

$$\begin{bmatrix} \mu_n(A) \\ \mu_n(B) \\ \mu_n(C) \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} \mu_{n-1}(A) \\ \mu_{n-1}(B) \\ \mu_{n-1}(C) \end{bmatrix}.$$

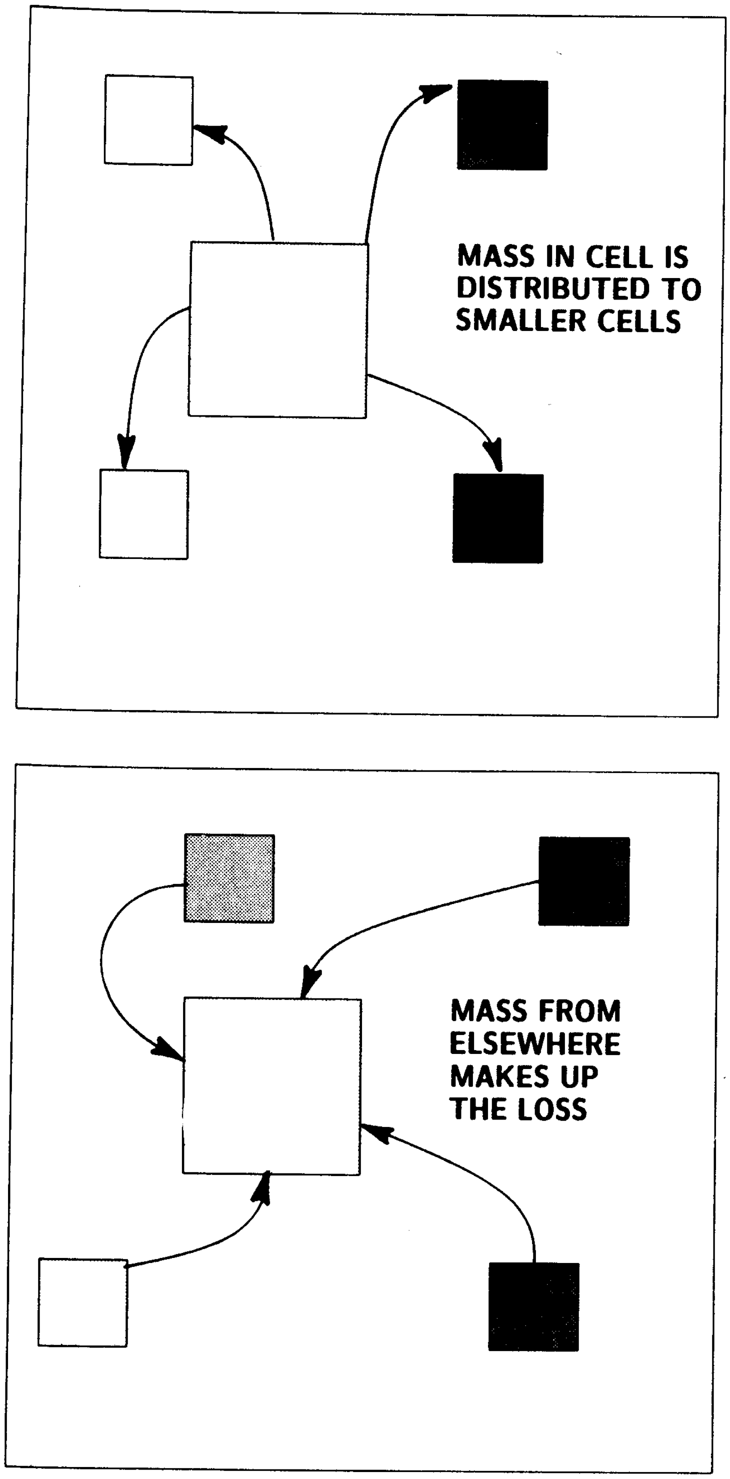
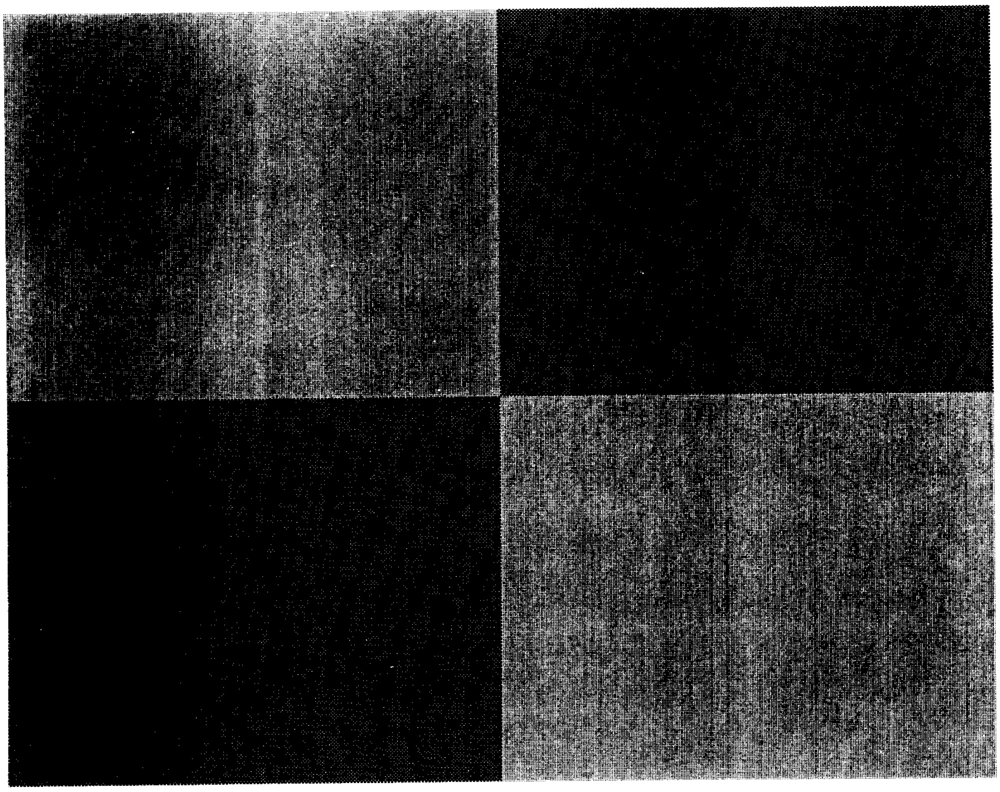


Figure IX.256. This illustrates the action of the Markov operator on one of the sequence of measures $\{M^{\circ n}(\mu_0)\}$, where $\mu_0(\blacksquare) = 1$. When the Markov operator is applied, the "mass" in a cell $\blacksquare_{ij\ldots k} = w_i \circ w_j \circ \cdots \circ w_k(\blacksquare)$ is redistributed among the four cells $w_1(\blacksquare_{ij\ldots k})$, $w_2(\blacksquare_{ij\ldots k})$, $w_3(\blacksquare_{ij\ldots k})$, and $w_4(\blacksquare_{ij\ldots k})$. Also, mass from other cells is mapped into subcells of $\blacksquare_{ij\ldots k}$ in such a way that the total mass of $\blacksquare_{ij\ldots k}$ remains the same as before the Markov operator was applied. In this manner the distribution of "mass" is defined on finer and finer scales as the Markov operator is repeatedly applied.

Figure IX.257. This sequence of figures represents successive measures produced by iterative applications of a Markov operator of the type considered in Figures IX.255 and IX.256. The result of one application of the operator to the uniform measure on ■ is represented in (a). Figures (b), (c), and (d) show the results of further successive applications of the Markov operator. The measures are represented in such a way as to keep the total number of dots constant. The measure of a set corresponds approximately to the number of dots it contains. This represents the first few of a sequence of measures that converges in the metric space $(\mathcal{P}(\blacksquare), d_H)$ to the invariant measure of the IFS.



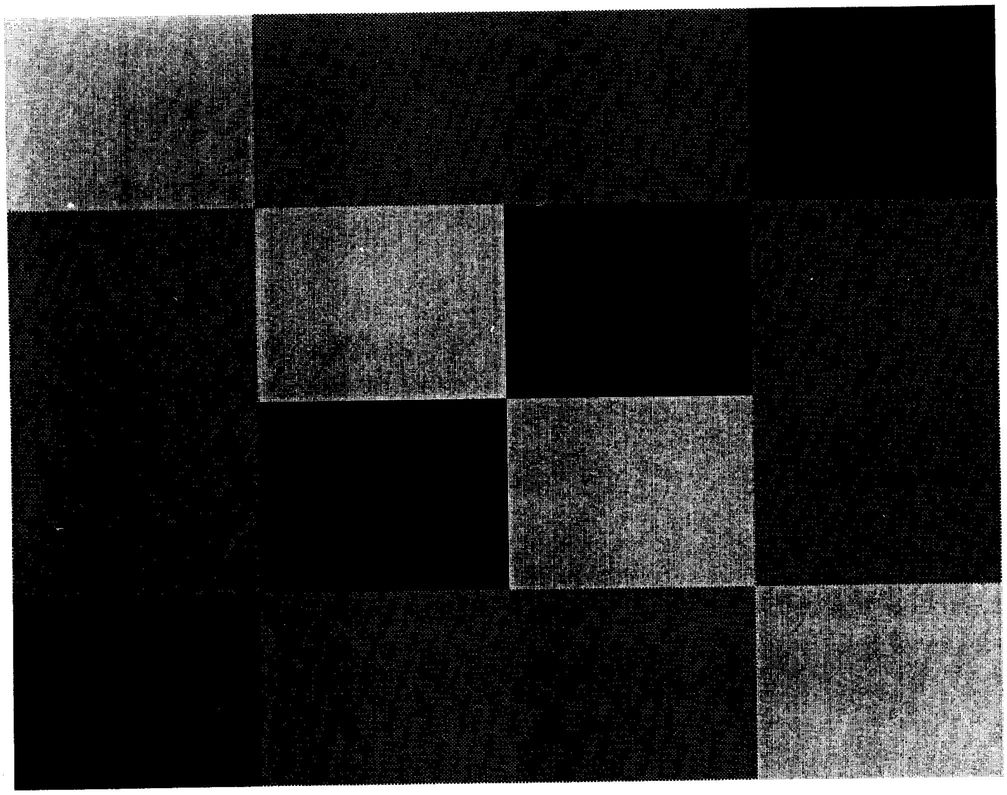


Figure IX.257. (b)

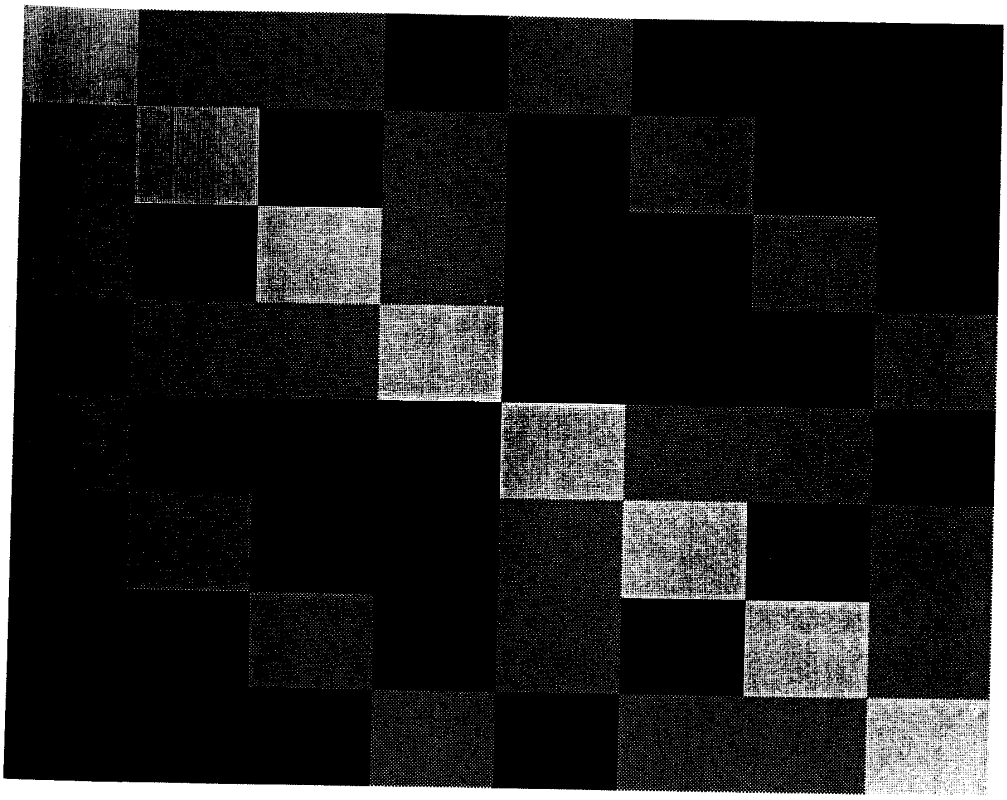


Figure IX.257. (c)

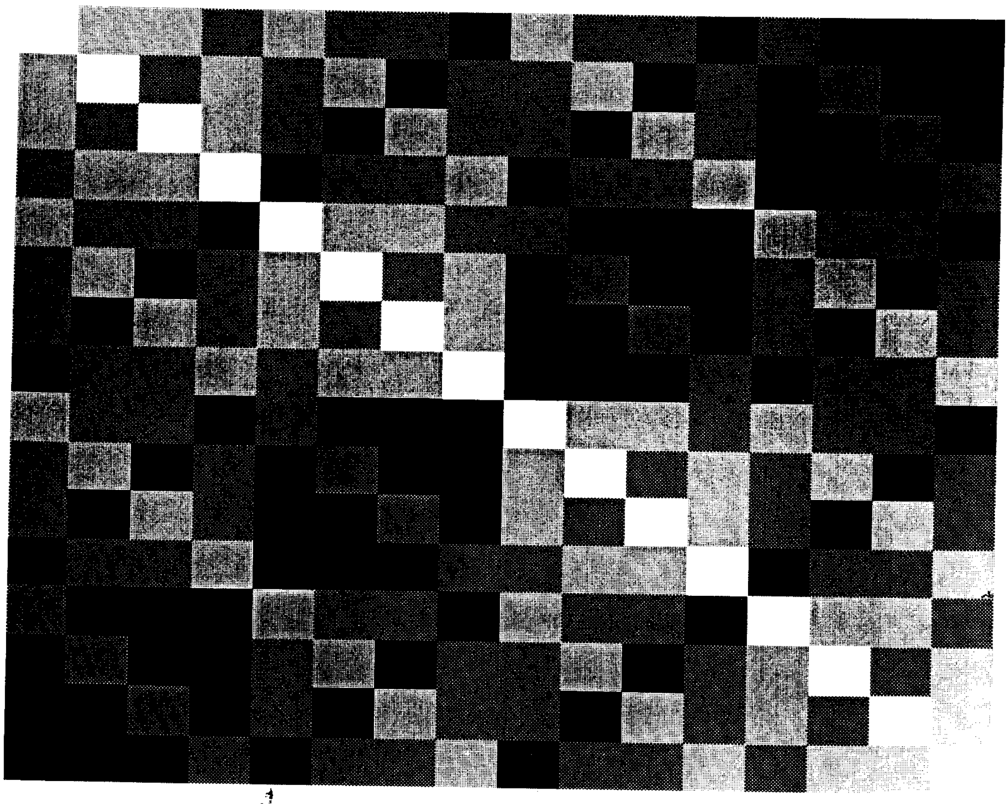
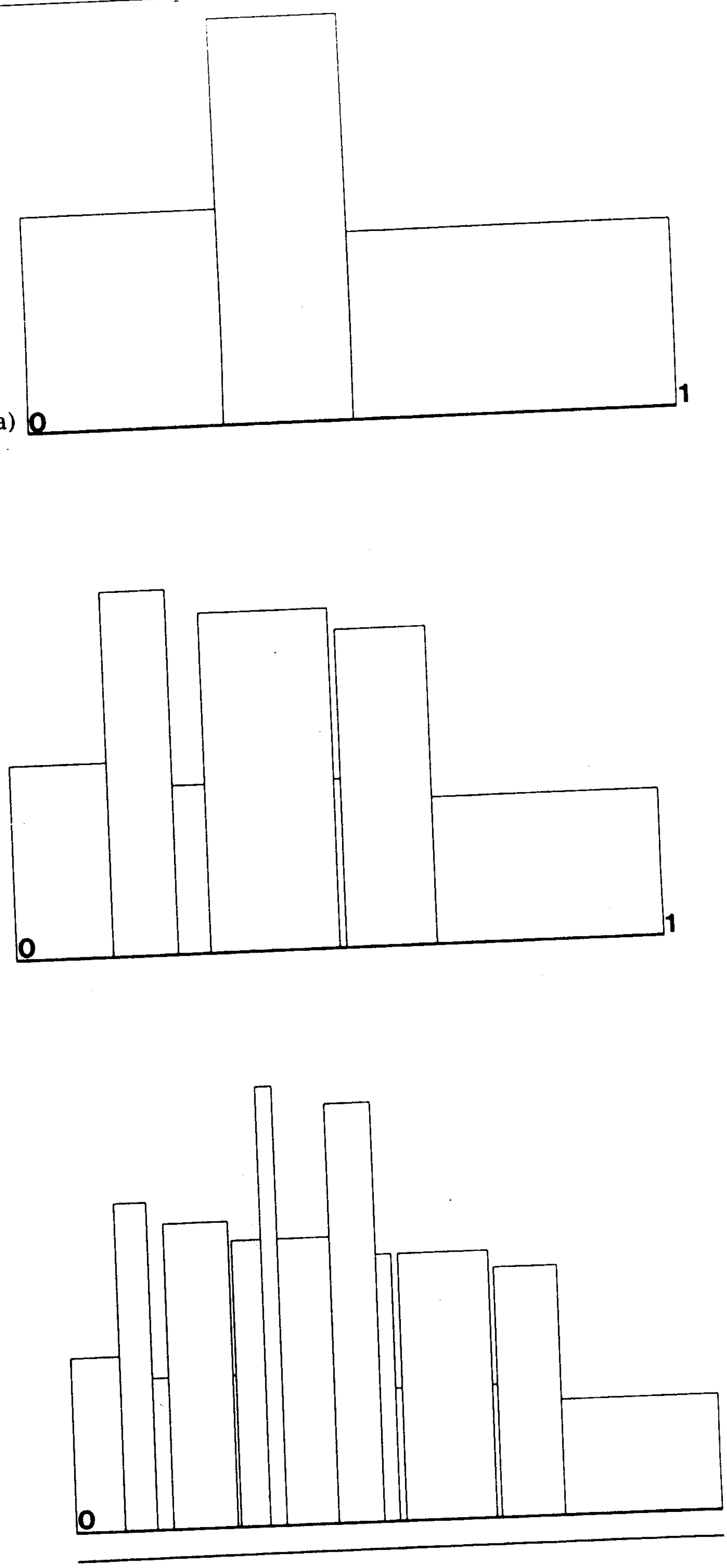


Figure IX.257. (d)

Figure IX.258. This sequence of images relates to the IFS $\{[0, 1] \subset \mathbb{R}; w_1(x) = (0.5)x, w_2(x) = (0.7)x + 0.3, p_1 = 0.45, p_2 = 0.55\}$. The attractor of the IFS is $[0, 1]$. Let M denote the associated Markov operator. Let $\mu_0 \in \mathcal{P}([0, 1])$ be the uniform measure on $[0, 1]$. The successive iterates $M(\mu_0)$, $M^{\circ 2}(\mu_0)$, $M^{\circ 3}(\mu_0)$, and $M^{\circ 4}(\mu_0)$ are represented in parts (a),(b),(c), and (d). Each measure is represented by a collection of rectangles whose bases are contained in the interval $[0, 1]$. The area of a rectangle equals the measure of the base of the rectangle.



(a)

(b)

(c)

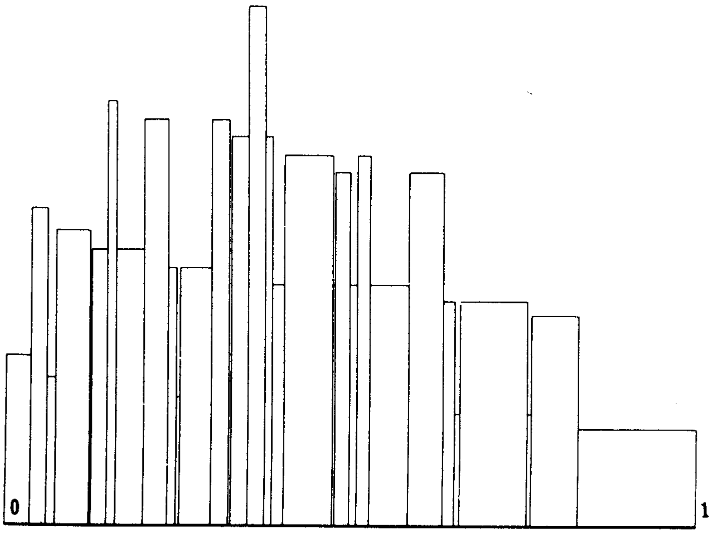


Figure IX.258. (d)

Let $\tilde{M}$ denote the 3×3 matrix here. Explain how $\tilde{M}$ is related to M, and show that the invariant measure of the IFS can be described in terms of an eigenvector of $\tilde{M}$.

6.8. Let

$$\{\mathbf{X}; w_1, w_2, \ldots, w_N; p_1, p_2, \ldots, p_N\}$$

be a hyperbolic IFS with probabilities. Let μ denote the associated invariant measure. Let A denote the attractor of the IFS. Let $\mu_0 \in \mathcal{P}(\mathbf{X})$ be such that $\mu_0(A) = 1$. By considering the sequence of measures $\{\mu_n = M^{\circ n}(\mu_0)\}$, prove that

$$\mu(w_i \circ w_j \circ \cdots \circ w_k(A)) \geq p_i p_j \ldots p_k, \qquad \text{for all } i, j, \ldots, k \in \{1, 2, \ldots, N\}.$$

Show that if the IFS is totally disconnected then the equality sign holds.

Theorem 6.2 *Let* $(\mathbf{X}, d)$ *be a compact metric space. Let*

$$\{\mathbf{X}; w_1, w_2, \ldots, w_N; p_1, p_2, \ldots, p_N\}$$

be a hyperbolic IFS with probabilities. Let μ *be the associated invariant measure. Then the support of* μ *is the attractor of the IFS* $\{\mathbf{X}; w_1, w_2, \ldots, w_N\}$.

Proof Let B denote the support of μ. Then B is a nonempty compact subset of $\mathbf{X}$. Let A denote the attractor of the IFS. Then

$$\{A; w_1, w_2, \ldots, w_N; p_1, p_2, \ldots, p_N\}$$

is a hyperbolic IFS. Let ν denote the invariant measure of the latter. Then ν is also an invariant measure for the original IFS. So, since μ is unique, $\nu = \mu$. It follows that $B \subset A$.

Let $a \in A$. Let $\mathcal{O}$ be an open set that contains a. We will use the notation of Theorem 2.1 in Chapter IV. Let Σ denote the code space associated with the IFS and let $\sigma \in \Sigma$ denote the address of a. It follows from Theorem 2.1 in Chapter IV that $\lim_{n\to\infty} \phi(\sigma, n, A) = a$, where the convergence is in the Hausdorff metric. It follows that there is a positive integer n so that $\phi(\sigma, n, A) \subset \mathcal{O}$. But

$$\mu(\phi(\sigma, n, A)) \geq p_{\sigma_1} p_{\sigma_2} \cdots p_{\sigma_n} > 0.$$

It follows that $\mu(\mathcal{O}) > 0$. It follows that a is in the support of μ. It follows that $a \in B$. It follows that $A \subset B$. This completes the proof.

Theorem 6.3 The Collage Theorem for Measures.. *Let*

$$\{\mathbf{X}; w_1, w_2, \ldots, w_N; p_1, p_2, \ldots, p_N\}$$

be a hyperbolic IFS with probabilities. Let μ be the associated invariant measure. Let $s \in (0, 1)$ be a contractivity factor for the IFS. Let $M : \mathcal{P}(\mathbf{X}) \to \mathcal{P}(\mathbf{X})$ be the associated Markov operator. Let $\nu \in \mathcal{P}(\mathbf{X})$. Then

$$d_H(\nu, \mu) \leq \frac{d_H(\nu, M(\nu))}{(1 - s)}.$$

Proof This is a corollary of Theorem 6.1.

We conclude this section with a description of the application of Theorem 6.3 to an inverse problem. The problem is to find an IFS with probabilities whose invariant measure, when represented by a set of dots, looks like a given texture.

A measure supported on a subset of $\mathbb{R}^2$ such as ■ can be represented by a lot of black dots on a piece of white paper. Figures IX.248 and IX.250 provide examples. The dots may be granules of carbon attached to the paper by means of a laser printer. The number of dots inside any circle of radius $\frac{1}{2}$ inch, say, should be approximately proportional to the measure of the corresponding ball in $\mathbb{R}^2$. A gray-tone image in a newspaper is made of small dots and can be thought of as representing a measure.

Let two such images, each consisting of the same number of points, be given. Then we expect that the degree to which they look alike corresponds to the Hutchinson distance between the corresponding measures. Let such an image, L, be given. We imagine that it corresponds to a measure ν. Theorem 6.3 can be used to help to find a hyperbolic IFS with probabilities whose invariant measure, represented with dots, approximates the given image. Let N be a positive integer. Let $w_i : \mathbb{R}^2 \to \mathbb{R}^2$ be an affine transformation, for $i = 1, 2, \ldots, N$. Let

$$\{\mathbb{R}^2; w_1, w_2, \ldots, w_N; p_1, p_2, \ldots, p_N\}$$

denote the sought-after IFS. Let M denote the associated Markov operator.

Let $p_i \& L$ mean the set of dots L after the "density of dots" has been decreased by a factor p_i. For example $0.5\&L$ means L after "every second dot" in L has been removed. The action of the Markov operator on ν is represented by $\cup_{i=1}^N w_i(p_i \& L)$. This set consists of approximately the same number of dots as L. Then we seek contractive affine transformations and probabilities such that

$$\cup_{i=1}^N w_i(p_i \& L) \approx L. \tag{1}$$

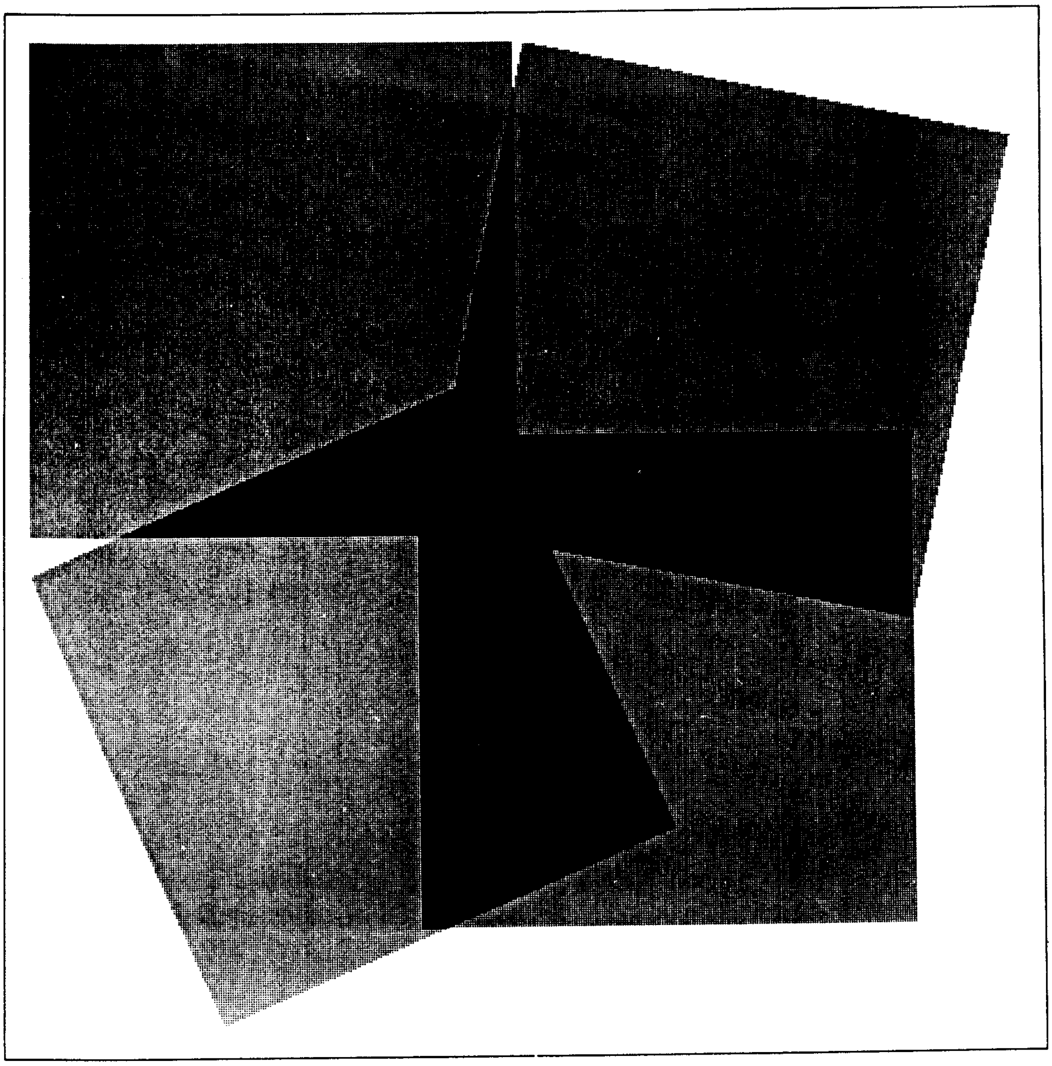


Figure IX.259. This illustration relates to the Collage Theorem for Measures. The shades of gray "add up" in the overlapping regions.

That is, the coefficients that define the affine transformations and the probabilities must be adjusted so that the left-hand side "looks like" the original image.

Suppose we have found an IFS with probabilities so that equation 1 is true. Then generate an image $\tilde{L}$ of the invariant measure of the IFS, containing the same number of points as L. We expect that

$$\tilde{L} \approx L. \tag{2}$$

If the maps are sufficiently contractive, then the meaning of "$\approx$" should be the same in both equations 1 and 2. These ideas are illustrated in Figure IX.259.

Examples & Exercises

6.9. Use the Collage Theorem for Measures to help find an IFS with probabilities for each of the images in Figures IX.260, IX.261, and IX.262.

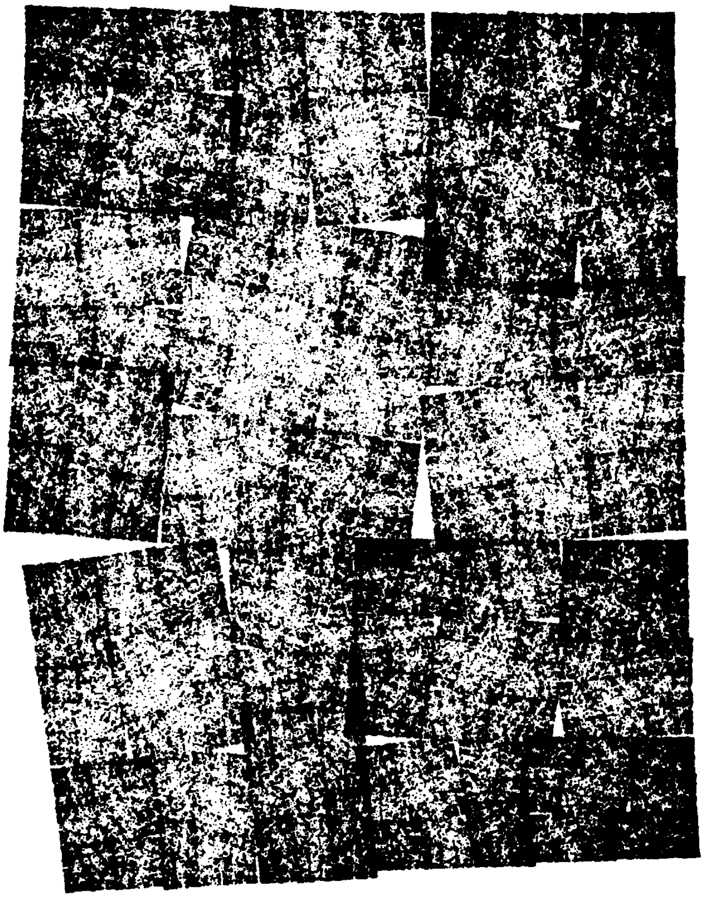


Figure IX.260. Can you find the IFS and probabilities corresponding to this texture?

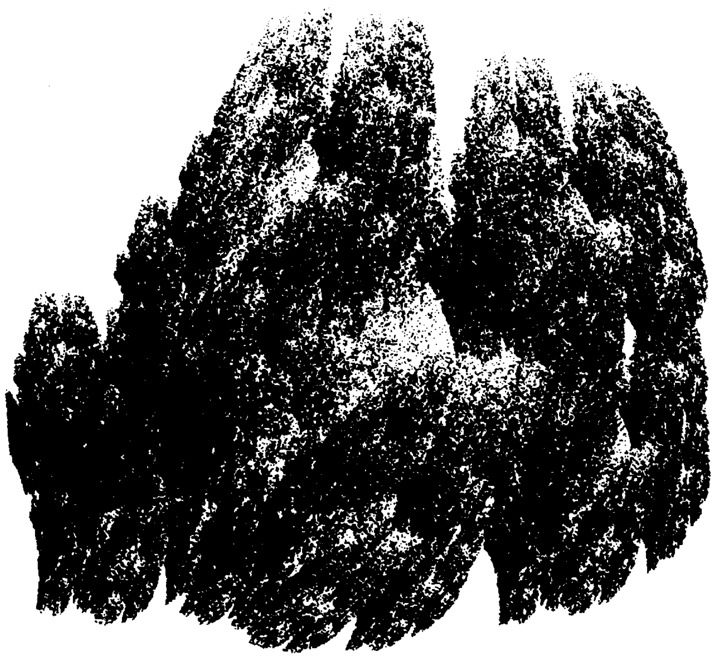


Figure IX.261. Determine the IFS and probabilities for this cloud texture.

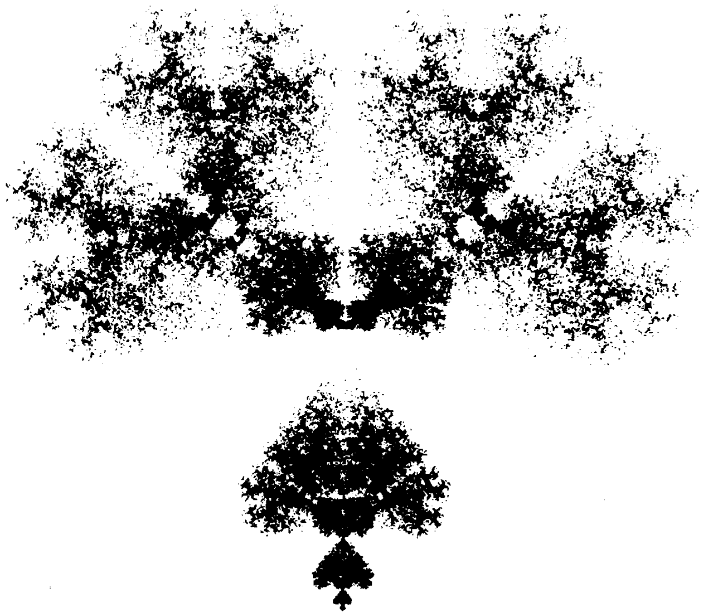


Figure IX.262. Find the four affine maps and probabilities for this texture.

6.10. Estimate the probabilities and transformations used to make each part of Figure IX.248.

6.11. Let

$$\{\mathbf{X}; w_1, w_2, \ldots, w_N; p_1, p_2, \ldots, p_N\}$$

be a hyperbolic IFS. Let μ denote the invariant measure. Let A denote the attractor. Let Σ denote the associated code space on the N symbols $\{1, 2, \ldots, N\}$. Let $T_i : \Sigma \to \Sigma$ be defined by $T_i(\sigma) = i\sigma$, for all $\sigma \in \Sigma$, for $i = 1, 2, 3, 4$. Let ρ denote the invariant measure for the hyperbolic IFS

$$\{\Sigma; T_1, T_2, T_3, T_4; p_1, p_2, p_3, p_4\}.$$

Let $\phi : \Sigma \to A$ denote the continuous map between code space and the attractor of the IFS intoduced in Theorem 4.2.1. Prove that $\rho(\phi^{-1}(B)) = \mu(B)$ for all Borel subsets B of $\mathbf{X}$.

6.12. Figure IX.263 depicts the invariant measure for the IFS $\{[0, 1] \subset \mathbb{R}; w_1(x) = a_1 x, w_2(x) = a_2 x + e_2; p_1,$
$p_2\}$, where a_1, a_2, and e_2 are real constants such that the attractor is contained in $[0, 1]$. The measure of a Borel subset of $[0, 1]$ is approximately the amount of black that lies "vertically" above it. Find a_1, a_2, and e_2.

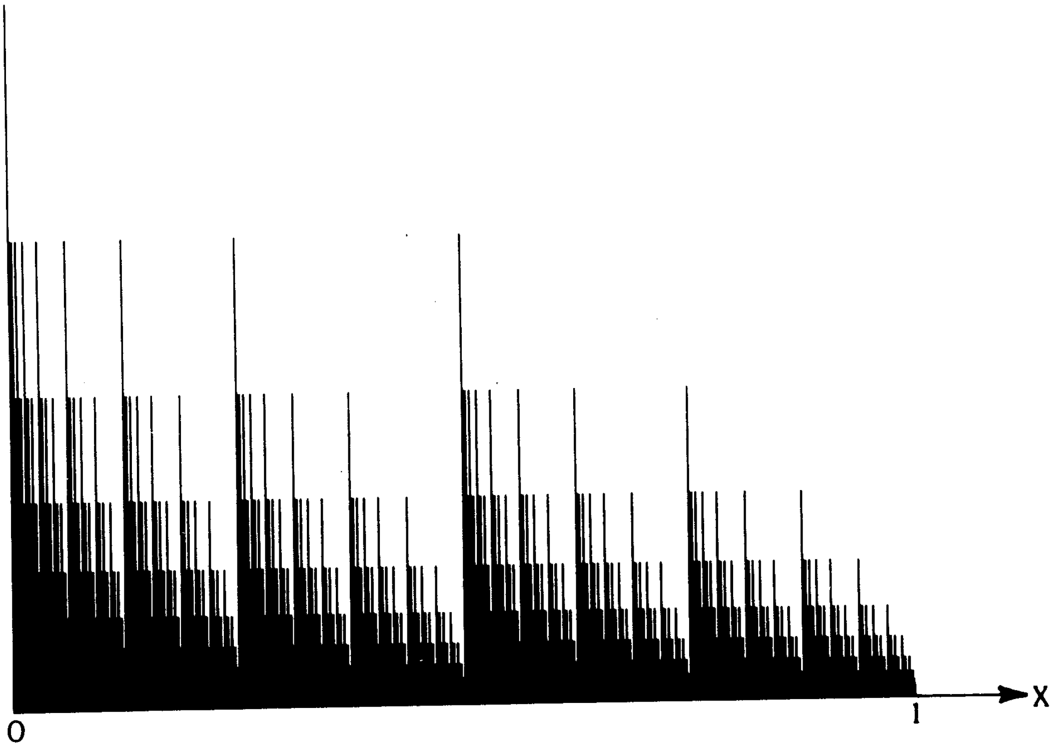


Figure IX.263. This figure depicts the invariant measure for the IFS $\{[0,1] \subset \mathbb{R}; w_1(x) = a_1x, w_2 = a_2x + e_2; p_1, p_2\}$, where a_1, a_2, and e_2 are real constants such that the attractor is contained in $[0, 1]$. The measure of a Borel subset of $[0, 1]$ is approximately the amount of black that lies "vertically" above it. Can you find a_1, a_2, and e_2?

7 Elton's Theorem

Both the following theorem and its corollary claim that certain events occur "with probability one." Although this has a very precise technical meaning, it is fine to interpret it in the same way as you would interpret the statement "There is a 100% chance of rain tomorrow." After the statements we mention the mathematical framework used for dealing with probabilistic statements. To go further we recommend reading parts of [Eisen 1969].

The theorem below is actually true when the p_i's are functions of x, the w_i's are only contraction mappings "on the average," and the space is "locally" compact.

Theorem 7.1 *Let* $(\mathbf{X}, d)$ *be a compact metric space. Let*

$$\{\mathbf{X}; w_1, w_2, \ldots, w_N; p_1, p_2, \ldots, p_N\}$$

be a hyperbolic IFS with probabilities. Let $(\mathbf{X}, d)$ *be a compact metric space. Let* $\{x_n\}_{n=0}^{\infty}$ *denote an orbit of the IFS produced by the Random Iteration Algorithm, starting at* x_0. *That is,*

$$x_n = w_{\sigma_n} \circ w_{\sigma_{n-1}} \circ \cdots \circ w_{\sigma_1}(x_0),$$

where the maps are chosen independently according to the probabilities

$$p_1, p_2, \ldots, p_N, \qquad \text{for } n = 1, 2, 3, \ldots.$$

Let μ be the unique invariant measure for the IFS. Then with probability one (that is, for all code sequences $\sigma_1, \sigma_2, \ldots$ except for a set of sequences having probability zero),

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} f(x_k) = \int_{\mathbf{X}} f(x)\, d\mu(x)$$

for all continuous functions $f : \mathbf{X} \to \mathbb{R}$ and all x_0.

Proof See [Elton 1987].

Corollary 7.1 *Let B be a Borel subset of $\mathbf{X}$ and let μ(boundary of B) $= 0$. Let*

$\mathcal{N}(B, n) =$ number of points in $\{x_0, x_1, x_2, x_3, \ldots, x_n\} \cap B$, for $n = 0, 1, 2, \ldots$.

Then, with probability one,

$$\mu(B) = \lim_{n\to\infty} \left\{ \frac{\mathcal{N}(B, n)}{(n+1)} \right\}$$

for all starting points x_0. That is, the "mass" of B is the proportion of iteration steps, when running the Random Iteration Algorithm, which produce points in B.

Let's explain more deeply the context of the statement "with probability one." Let Σ denote the code space on the N symbols $\{1, 2, \ldots, N\}$. Let ρ denote the unique Borel measure on Σ such that

$$\rho(C(\sigma_1, \sigma_2, \ldots, \sigma_m)) = p_{\sigma_1} p_{\sigma_2} \ldots p_{\sigma_m}$$

for each positive integer m and all $\sigma_1, \sigma_2, \ldots, \sigma_m \in \{1, 2, \ldots, N\}$, where

$$C(\sigma_1, \sigma_2, \ldots, \sigma_m) = \{\omega \in \Sigma : \omega_1 = \sigma_1, \omega_2 = \sigma_2, \ldots, \omega_m = \sigma_m\}.$$

Then $\rho \in \mathcal{P}(\Sigma)$. This measure provides a means for assigning probabilities to sets of possible outcomes of applying the Random Iteration Algorithm. Let us see how this works.

When the Random Iteration Algorithm is applied, an infinite sequence of symbols $\omega_1, \omega_2, \omega_3, \ldots$, namely a code $\omega = \omega_1\omega_2\omega_3 \ldots \in \Sigma$, is generated. Provided that we keep $x_0 \in \mathbf{X}$ fixed, we can describe the probabilities of orbits $\{x_n\}$ in terms of the probabilities of codes ω. So we examine how probabilities are associated to sets of codes.

The Random Iteration Algorithm is applied and produces a code $\omega \in \Sigma$. What is the probability that $\omega_1 = 1$? Clearly it is $p_1 = \rho(C(1))$. What is the probability that $\omega_1 = \sigma_1, \omega_2 = \sigma_2, \ldots$, and $\omega_n = \sigma_n$? Because the symbols are chosen independently, it is

$$\rho(C(\sigma_1, \sigma_2, \ldots, \sigma_m)) = p_{\sigma_1} p_{\sigma_2} \ldots p_{\sigma_m}.$$

Let B denote a Borel subset of Σ. What is the probability that the Random Iteration Algorithm produces a code $\sigma \in B$? It is at least intuitively reasonable that it is

$\rho(B)$. This can be formalized, see, for example [Eisen 1969]. The measure ρ provides a means of describing the probabilities of outcomes of the Random Iteration Algorithm.

Here is a heavy way of stating the central part of Theorem 7.1. "... Let $x_0 \in \mathbf{X}$. Let $B \subset \Sigma$ denote the set of codes $\sigma \in \Sigma$ such that

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} f(x_k) = \int_{\mathbf{X}} f(x)\, d\mu(x),$$

for all $x_0 \in \mathbf{X}$ and all continuous functions $f : \mathbf{X} \to \mathbb{R}$, where

$$x_n = w_{\sigma_n} \circ w_{\sigma_{n-1}} \circ \ldots \circ w_{\sigma_1}(x_0).$$

Then B is a Borel subset of Σ and $\rho(B) = 1$." A similar heavy restatement of the corollary can be made.

Examples & Exercises

7.1. This example concerns the IFS

$$\{[0,1]; \frac{1}{2}x, \frac{1}{2}x + \frac{1}{2}; 0.5, 0.5\}.$$

Show that the invariant measure μ is such that $\mu([x, x+\delta]) = \Delta$ when $[x, x+\delta]$ is a subinterval of $[0, 1]$. Deduce that if $f : [0, 1] \to \mathbb{R}$ is a continuous function then

$$\int_0^1 f(x)\, dx = \int_{[0,1]} f\, d\mu.$$

Let $f(x) = 1 + x^2$. Compute approximations to the latter integral with the aid of Elton's theorem and the Random Iteration Algorithm. Compare your results with the exact value $\frac{4}{3}$.

7.2. This example concerns the IFS

$$\{\blacksquare \subset \mathbb{R}^2; w_1, w_2, w_3, w_4; 0.25, 0.25, 0.25, 0.25\}$$

corresponding to the collage in Figure IX.255(a). Let μ denote the invariant measure. Argue that μ is the uniform measure that assigns "measure" $dxdy$ to an infinitesimal rectangular cell of side lengths dx and dy. Use Elton's theorem and the Random Iteration Algorithm to evaluate approximations to

$$\int_{\blacksquare} (x^2 + 2xy + 3y^2)\, dxdy.$$

Compare your approximations with the exact value.

7.3. This example concerns the IFS

$$\{\blacktriangle \subset \mathbb{R}^2; w_1, w_2, w_3; \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\},$$

where

$$w_1(x,y) = (\frac{1}{2}x, \frac{1}{2}y), \quad w_2(x,y) = (\frac{1}{2}x, \frac{1}{2}y + \frac{1}{2}), \quad w_3(x,y) = (\frac{1}{2}x + \frac{1}{2}, \frac{1}{2}y),$$

and $\mathbb{A}$ is the attractor of the IFS, our old friend. Let μ denote the invariant measure of the IFS. Argue that μ provides a good concept of a "uniform" measure on $\mathbb{A}$. Use Elton's theorem and the Random Iteration Algorithm to compute approximations to

$$\int_{\mathbb{A}} (x^2 + 2xy + 3y^2)\, dxdy.$$

In Chapters II, III, and IV, we introduced the space Σ_N of shifts on N symbols. It was mentioned in passing in Chapter IV that any invertible mixing function could be represented by a baker's transformation with "uneven cutting and stretching." We are now in a position to show how this comes about using an example involving two simple IFS. The same model with some necessary refinements yields the code space mixing model used to justify the representations. It is easier to visualize without the refinements, as we present it here. It is one of the most important properties involved with the modelling of physical chaos.

We begin with perhaps the most simple of all IFS with probabilities. On the interval [0, 1], we define the just-touching IFS with N maps and with probabilities

$$\{[0, 1]; w_1, w_2, \ldots, w_N; p_1, p_2, \ldots, p_N\},$$

where

$$\begin{aligned} w_1(x) &= \frac{1}{N}x \\ w_2(x) &= \frac{1}{N}x + \frac{1}{N} \\ w_3(x) &= \frac{1}{N}x + \frac{3}{N} \\ &\vdots \\ w_N(x) &= \frac{1}{N}x + \frac{N-1}{N}, \end{aligned}$$

and the probabilities are arbitrary, subject to the usual condition

$$\sum_{i=1}^{N} p_i = 1.$$

Associated with this IFS there is an invariant measure on [0, 1], which we denote by ν.

Now we define another IFS on [0, 1], this time without probabilities, using the p_i from above. On [0, 1] define the IFS

$$\{[0, 1]; v_1, v_2, \ldots, v_N\},$$

where

$$
\begin{aligned}
v_1(x) &= p_1 x \\
v_2(x) &= p_2 x + p_1 \\
v_3(x) &= p_3 x + (p_1 + p_2) \\
&\vdots \\
v_k(x) &= p_k x + \sum_{i=1}^{k-1} p_i \\
&\vdots \\
v_N(x) &= p_N x + \sum_{i=1}^{N-1} p_i.
\end{aligned}
$$

This IFS is also just-touching by construction and, because the probabilities from the first IFS sum to one, has as its attractor the interval $[0, 1]$ as well. We are going to use it to define an equivalent metric on $[0, 1]$ as follows:

Each point has a unique address under this IFS in code space, except the points $v_i(A)$, whose multiple addresses correspond to

$$\sigma = i\overline{N-1} = (i+1)\overline{0}.$$

These are precisely the points in a base N expansion of a real number which are equated to form the real line. We denote the *value* of a point x with address $x_1x_2x_3\ldots$ in this new metric space to be the real number with N-ary expansion $.x_1x_2x_3\ldots$. In effect we have given each point the numeric value that would correspond to having measured its distance from say 0 with a ruler on which the spacing of the tick marks had been made uneven in a very specific way by the IFS.

With these values, the space is still $[0, 1]$, but we put a metric on it by assigning the distance between two real numbers to be the distance measured with a "normal" ruler. Another way to put it is that we take the normal interval $[0, 1]$ and assign the distance between two points to be the distance between the addresses corresponding to their N-ary expansions in the above IFS. Thus if $N = 10$ for instance, the distance between $.251$ and $.137$ is not $.251 - .137$, but rather the distance between the points with addresses $251\overline{0}$ and $137\overline{0}$ in the IFS $\{[0, 1]; v_1, v_2, \ldots, v_10\}$. We will call this space $[0, 1]_p$, and the distance function d_p to avoid confusion.

We have a metric space, so we will now assign a Borel measure to it by defining $\mu([a, b]) = \mu((a, b)) = d_p(a, b)$, which is uniform for this metric space. And to proceed with the example, we need a function, $f : [0, 1] \to [0, 1]_p$ which we define by $f(x) =$(point with value x in $[0, 1]_p$). Because the definition was very careful to preserve the ordering of the real line and its conventions about multiple addressing, f is both a homeomorphism and a metric equivalence. Because it is continuous, it is also what is called a *measurable function* in that if $A \in \mathcal{B}([0, 1]_p)$ then $f^{-1}(A) \in \mathcal{B}([0, 1])$.

7.4. Show that f is *measure-preserving* with respect to the invariant measure on v

associated with

$$\{[0, 1]; w_1, w_2, \ldots, w_N; p_1, p_2, \ldots, p_N\};$$

that is, that for any Borel subset $A \in [0, 1]$, we have $\nu(A) = \mu(f(A))$.

We now have the machinery to cast the Random Iteration Algorithm entirely in terms of IFS with no recourse to randomness. It is really a deterministic model, with the random part coming in to help when a very simple statement made all the time in mathematics turns out to be something a computer cannot do.

The exact transfer of the Random Iteration Algorithm to the model using the space $([0, 1]_p, d_p)$ looks like this: define the function $g : [0, 1] \to [0, 1]_p$ by

$$d_p(g(x), 0) = x.$$

Define the map $h : [0, 1]_p \to \{0, 1, 2, \ldots, N-1\}$ defined by $h(p) = [Np]$ where $[\cdot]$ is the greatest integer function. In other words, take the first N-ary digit of the value of the point $p \in [0, 1]_p$. Define the map $y : [0, 1] \to [0, 1]$ given by $y(x) = Nx \bmod N$. Then the Random Iteration Algorithm is precisely the iteration of the map $R : [0, 1] \times \mathbf{X}$ given by

$$R(p, x) = (y(p), w_{h \circ g(p)}(x)).$$

Where does the random part of the algorithm come in? We need it to "pick a real number." One can think of the random number at each iteration as a function to get the next digit of the real number we "picked." In the above expression, we get a random number and find out which function to use via $h(g(p))$. Then we iterate the IFS using $w_{h \circ g(p)}(x)$, and in order to have a new "random number" the next time, we advance p to the next digit using $y(p)$.

Now, consider the space $[0, 1]_p \times [0, 1]$. Think of it as a square with coordinates spaced unevenly in the x direction and evenly in the y direction. Your "usual" point in the square (where here usual means with probability one) has an N-ary expansion for y in which every digit occurs with equal probability, while the x value has an N-ary expansion in which 0 occurs with probability p_1, 1 occurs with probability p_2, etc.

7.5. Draw a diagonal from $(0, 0)$ to $(1, 1)$ on this square. Show that this statement is still true if we pick a "usual point" from this diagonal.

7.6. Draw a smooth curve from $(0, 0)$ to $(1, 1)$ on the square. Then the statement is still true if we pick a "usual point" from this curve.

By using the diagonal in exercise 7.5, we can take a point x in $[0, 1]$ and map it to a new point $\tilde{x}$, by putting x along the vertical coordinate and reading the horizontal coordinate like a web diagram. In terms of all the functions we have defined, this operation is $\tilde{x} = f^{-1}(g(x))$. Under the original IFS with probabilities, this new point will, with probability 1, have an orbit under the shift dynamical system $\{A; S\}$ with a distribution of dots identical to the one we would get by using the Random Iteration Algorithm with probabilities $\{p_1, p_2, \ldots, p_N\}$.

There seems to be a lot of mileage in this square with the strange coordinates. There should be; the uneven coordinates correspond to future cuts and stretches for the baker's transformation with uneven stretches and cuts. (A real baker's transformation would not use the just-touching IFS used here, but it's easier to visualize, and for the most general case N is allowed to be infinite.) It is a mixing function, so it automatically satisfies the equation that results from Elton's theorem (a property called *ergodicity*). The theorem takes care of how little "hyperbolicity" an IFS can have and still retain this property. Alternatively, Elton's theorem can be viewed as a set of minimal requirements on the w_i such that the baker's transformation as set up here accurately reflects the behavior of the IFS on addresses.

8 Application to Computer Graphics

We begin by illustrating how a color image of the invariant measure of an IFS with probabilities can be produced. The idea is very simple. We start from an IFS such as

$$\{\mathbb{C}; 0.5z + 24 + 24i, 0.5z + 24i, 0.5z; 0.25, 0.25, 0.5\}.$$

A viewing window and a corresponding array of pixels P_{ij} is specified. The Random Iteration Algorithm is applied to the IFS, to produce an orbit $\{z_n : n = 0, 1, \ldots, numits\}$, where *numits* is the number of iterations. For each (i, j) the number of points, $\mathcal{N}(P_{ij})$, which lie in the pixel P_{ij} are counted. The pixel P_{ij} is assigned the value $\mathcal{N}(P_{ij})/numits$. By Elton's theorem, if *numits* is large, this value should be a good approximation to the measure of the pixel. The pixels are plotted on the screen in colors determined from their measures.

The following program implements this procedure. It is is written in BASIC. It runs without modification on an IBM PC with Enhanced Graphics Adaptor and Turbobasic.

Program 1. (Uses the Random Iteration Algorithm to Make a "Picture" of the Invariant Measure Associated with an IFS with Probabilities)

```
screen 9 : cls  'Initialize graphics.

dim s(51,51)  'Allocate array of pixels.

'IFS code for a Sierpinski triangle.
a(1)=0.5 : b(1)=0 : c(1)=0 : d(1)=0.5 : e(1)=24 : f(1)=24

a(2)=0.5 : b(2)=0 : c(2)=0 : d(2)=0.5 : e(2)=0 : f(2)=24
```

```
a(3)=0.5 : b(3)=0 : c(3)=0 : d(3)=0.5 : e(3)=0 : f(3)=0

'Probabilities for the IFS; they must add to one!
p(1)=0.25 : p(2)=.25 : p(3)=.5

mag=1 'Magnification factor.

'Increase the number of iterations as you magnify.
numits=5000

factor =100 'Scales pixel values to color values.

'This is the number of colors you are able to use.
numcols=8

for n=1 to numits 'Random iteration begins!

r=rnd : k=1 'Pick a number in [0,1] at random.

if r > p(1) then k=2

if r > p(1)+p(2) then k=3

'Map k is picked with probability p(k).
newx=a[k]*x + b[k]*y + e[k]
newy=c[k]*x + d[k]*y + f[k]

x=newx : y=newy

i=int(mag*x) : j=int(mag*y) 'Scale by magnification factor.

if (((i < 50) and (i>=0))and((0=<j) and (j<50))) then

'If the scaled value is...

s(i,j)=s(i,j)+1
'...in the array add one to pixel (i,j).

end if

pset(i,j) 'Plot the point.

if instat then end 'Stop if a key is pressed.

next
```

```
for i=0 to 49 'Normalize values in pixel array, and plot...

for j=0 to 49 '...in colors corresponding to the normalized...

'...values of the numbers s(i,j).
col=s(i,j)*numcols*factor*mag*mag/numits

'Plot the pixel (i,j) in the color determined by...
pset(i,j),col

next j '...its measure.

next i

end
```

The program allows the user to zoom in on a piece of the rendered measure by altering the value of the magnification parameter *mag*. The result of running an adaptation of this program on a Masscomp workstation and then printing the contents of the graphics screen is shown in Figure IX.264.

Rendered invariant measures for IFS's acting in $\mathbb{R}^2$ are also shown in Figure IX.265.

By carrying out some simple computergraphical experiments, using a program such as the one above, we discover that "pictures" of invariant measures of IFS's possess a number of properties. (i) Once the viewing window and color assignments have been fixed, the image produced is stable with respect to the number of iterations, provided that the number of iterations is sufficiently large. (ii) Images vary consistently with respect to translation and rotation of the viewing window, and with respect to changes in resolution. In particular they vary consistently when they are magnified. (iii) The images depend continuously on the IFS code, including the probabilities. Property (i) ensures that the images are well defined. The properties in (ii) are also true for views of the real world seen through the viewfinder of a camera. Property (iii) means that images can be controlled interactively. These properties suggest that IFS theory is applicable to computer graphics.

We should, if we have done our measure theory homework, understand the reasons for (i) and (ii). They are consequences of corresponding properties of Borel of measures on $\mathbb{R}^2$. Property (iii) follows from a theorem by Withers [Withers 1987].

Examples & Exercises

8.1. Rewrite Program 1, section 8, in a form suitable for your own computer environment. Adjust *numits* and *factor* to ensure that a stable image results. Then make experiments to verify that the conditions (i)–(iii) above are verified. For example, to test the consistency of images with respect to changes in resolution you should try *mag* $= 0.5$, 1, and 1.5. Unless you have a very powerful system, do not make ex-

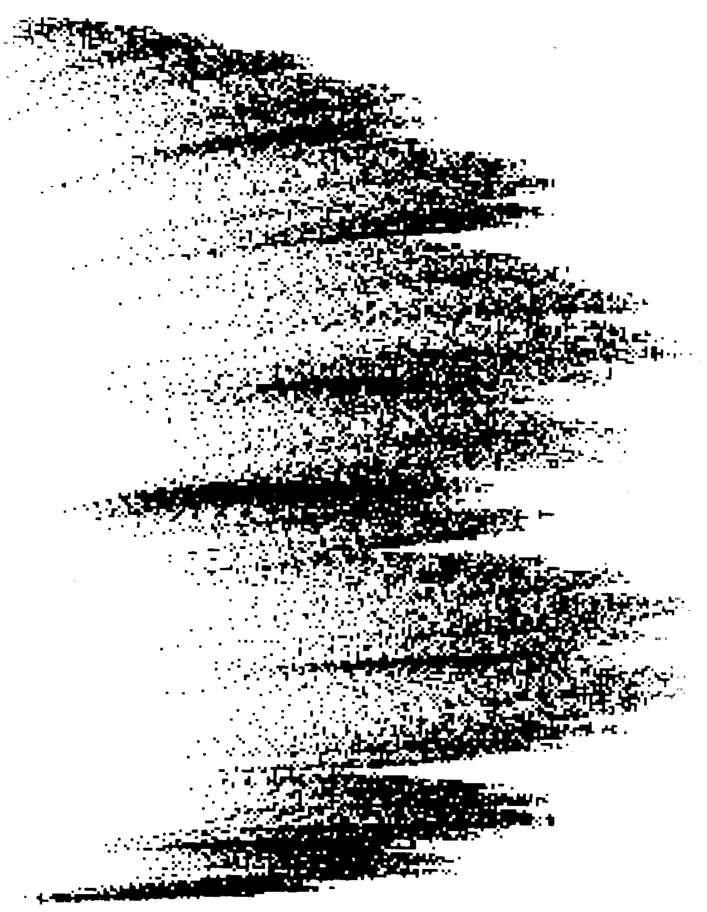


Figure IX.264. The result of running a modified version of Program 9.8.1 and then printing the contents of the graphics screen in gray tones. A rendered picture of a measure is the result.

treme adjustments. For example, do not choose *mag* too small, otherwise you will need a very large value for *numits*.

Applications of fractal geometry to computer graphics have been investigated by a number of authors including Mandelbrot [Mandelbrot 1982], Kawaguchi [Kawaguchi 1982], Oppenheimer [Oppenheimer 1986], Fournier *et al.* [Fournier 1982], Smith [Smith 1984], Miller [Miller 1986], and Amburn *et al.* [Amburn 1986]. In all cases the focus has been on the modelling of natural objects and scenes. Both deterministic and random geometries have been used. The application of IFS theory to computer graphics was first reviewed in [Demk 85]. It provides a single framework that can reach an unlimited range of images. It is distinguished from other fractal approaches because it is the only one that uses measure theory.

The modelling of natural scenes is an important area of computer graphics. Photographs of natural scenes contain redundant information in the form of subtle patterns and variations. There are two characteristic features: (i) the presence of complex geometrical structure and distributions of color and brightness at many scales; and (ii) the hierarchical layout of objects. (i) Natural boundaries and textures are not

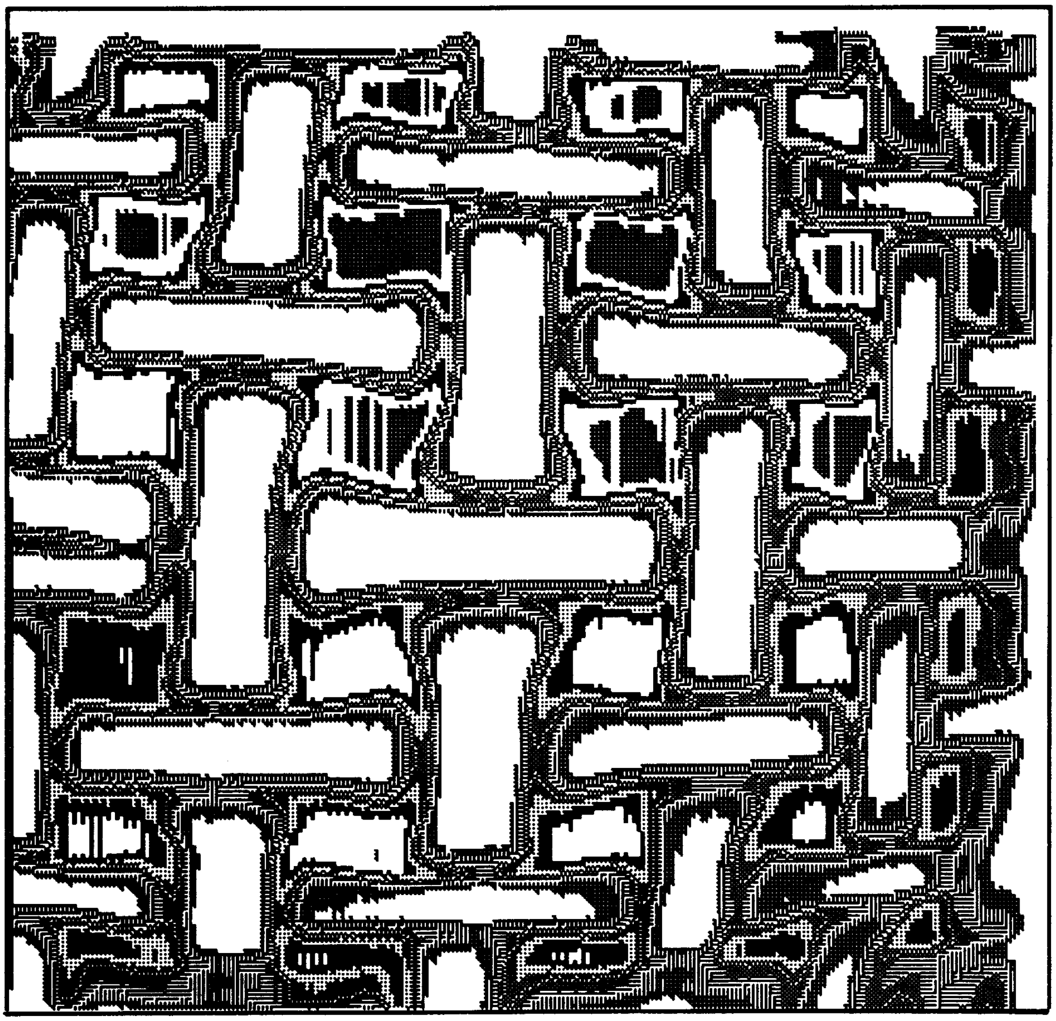


Figure IX.265. Rendered invariant measures for IFS's of two maps.

(a)

smoothed out under magnification; they preserve some degree of geometrical complexity. (ii) Natural scenes are organized in hierarchical structures. For example, a forest is made of trees; a tree is a collection of boughs and limbs along a trunk; on each branch there are clusters of leaves; and a single leaf is filled with veins and covered with fine hairs. It appears often in a natural scene that a recognizable entity is built up from numerous near repetitions of some smaller structure. These two observations can be integrated into systems for modelling images using IFS theory.

Examples & Exercises

8.2. Examine a good-quality color photograph of a natural scene, such as can be found in a *Sierra Club* calender, or an issue of *National Geographic*. Discuss the extent to which (i) and (ii) are true for that photograph. Be specific.

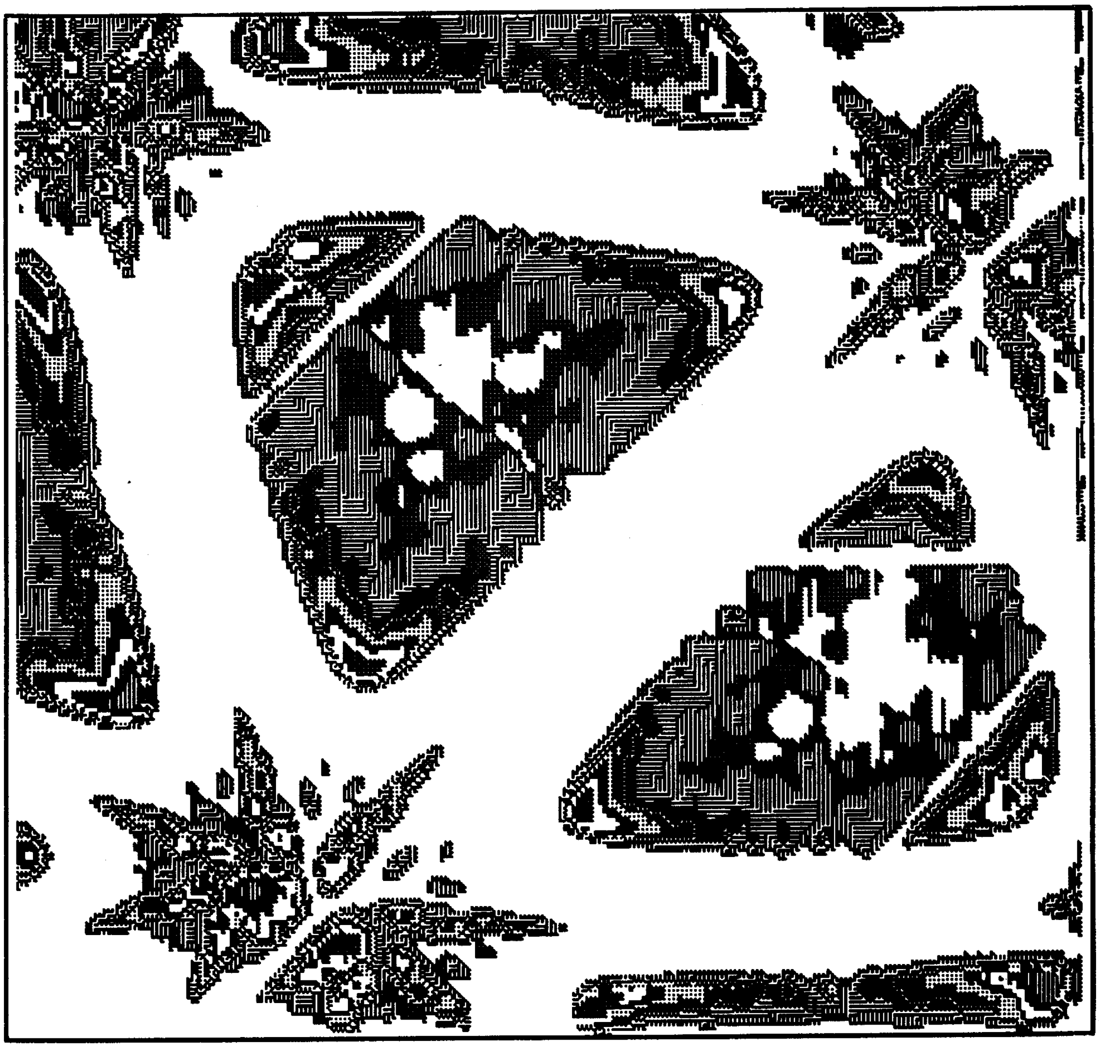


Figure IX.265. (b)

In [Barnsley 1988a] it is reported that IFS theory can be used efficiently to model photographs of clouds, mountains, ferns, a field of sunflowers, a forest, seascapes and landscapes, a hat, the face of a girl, and a glaring arctic wolf.

There are two parts to making any computer graphics image: geometrical modelling and rendering. Consider an architect making a computergraphical house: first she defines the dimensions of the floor, the roof, the windows, the shapes of the gables, and so on, to produce the geometrical model. Traditionally this is specified in terms of polygons, circles, and other classical geometrical objects that can be conveniently input to the computer. This model is not a picture. To make a picture, the model must be projected into two dimensions from a certain point of view and distance, discretized so that it can be represented with pixels, and finally rendered in colors on a display device.

Here we describe briefly the software system designed by the author, Alan Sloan, and Laurie Reuter, which was used to produce the color images that accompany this section. More details can be found in [Reuter 1987] and [Barnsley 1988a]. The system consists of two subsystems known as *Collage* and *Seurat*. *Collage* is used for geometrical modelling, while *Seurat* is used for rendering.

Collage and *Seurat* process IFS structures of the form

$$\{\mathbb{R}^2; w_1, w_2, \ldots, w_N; p_1, p_2, \ldots, p_N : n = 1, 2, \ldots, N\},$$

where the maps are affine transformations in $\mathbb{R}^2$. An IFS is represented by a file that consists of an IFS code, where each coefficient is written with a fixed number of bits Let μ denote the invariant measure of such an IFS and let A denote the attractor. The pair (A, μ) is referred to as an *underlying model*. The attractor A carries the geometry while μ carries the rendering information. One can think of the IFS code, or equivalently (A, μ), as being analagous to the plans of an architect. It corresponds to many different pictures.

Collage is a geometrical modelling system used to determine the coefficients of the affine transformations $w_1, w_2, \ldots, w_N$. It is based on the Collage Theorem. *Seurat* is a software system for rendering images starting from an IFS code. An image is produced once a viewing window, color table, and resolution have been specified. This is achieved using the Random Iteration Algorithm. Its mathematical basis is Elton's Theorem. *Seurat* is also used in an interactive mode to determine the probabilities and color values.

The input to *Collage* is a target image, which we denote here by T. For example, T may be a polygonal approximation to a leaf. We suppose that

$$T \subset \blacksquare = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq 1\},$$

and that the screen of the computer display device corresponds to $\blacksquare$. T is rendered on the graphics workstation monitor. An affine transformation

$$w_1(x, y) = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_1 \\ f_1 \end{pmatrix} = A_1 x + t_1$$

is introduced, with coefficients initialized at $a_1 = d_1 = 0.25$, $b_1 = c_1 = e_1 = f_1$. The image $w_1(T)$ is displayed on the monitor in a different color from T. $w_1(T)$ is a quarter-sized copy of T, centered closer to the point $(0, 0)$. The user now interactively adjusts the coefficients with a mouse or some other interaction technique, so that the image $w_1(T)$ is variously translated, rotated, and sheared on the screen. The goal of the user is to transform $w_1(T)$ so that it lies over part of T. It is important that the dimensions of $w_1(T)$ are smaller than those of T, to ensure that w_1 is a contraction. Once $w_1(T)$ is suitably positioned, it is fixed, and a new subcopy of the target, $w_2(T)$, is introduced. w_2 is adjusted until $w_2(T)$ covers a subset of those pixels in T that are not in $w_1(T)$. Overlap between $w_1(T)$ and $w_2(T)$ is allowed, but in general it should be made as small as possible, for efficiency. New maps are added and adjusted until $\cup_{j=1}^{N} w_j(T)$ is a good approximation to T. The output from *Collage* is the resulting IFS code. The probability p_j is chosen proportional to $|a_j d_j - b_j c_j|$ if this number is nonzero, and equal to a small positive number if the determinant of A_j equals zero.

The input to *Seurat* is one or more IFS codes generated by *Collage*. The viewing window and the number of iterations are specified by the user. The measures of the pixels are computed. The resulting numbers are multiplied by the inverse of the maximum value so that all of them lie in [0, 1]. Colors are assigned to numbers in [0, 1] using a color assignment function. The default is a gray scale where the intensity is proportional to the number, such as 0 corresponds to black and 1 corresponds to brightest white. The coloring and texture of the image can be controlled through the probabilities and the color assignment function. Although one does not explicitly use it, Theorem 9.6.3 lies in the background and can help in the adjustment of the probabilities.

Color Plate 21 shows some smoking chimneys in a landscape. We obtained the IFS codes for the elements of this image we obtained using *Collage*. Different color assignment functions are associated to different elements in the image. The image was rendered using *Seurat*.

The consistency of images with respect to changes in resolution is illustrated in Color Plate 22, which shows a zoom on one of the smokestacks in Color Plate 21. The number of iterations must be increased with magnification to keep the number of points landing within the viewing window constant. This requirement ensures the consistency of the textures in an image throughout the magnification process.

Color Plates 23 and 24 show various renderings of leaves produced by *Seurat*.

Color Plate 25 shows a sequence of frames taken from an IFS encoded movie entitled *A Cloud Study* [Barnsley 1987]. The smooth transition from frame to frame is a consequence of the continuous dependence on parameters of the invariant measure of the IFS for the cloud.

Color Plates 26, 27, and 28 were encoded from color photographs. Segmentation according to color was performed on the originals to define textured pieces. IFS codes for these components were obtained using *Collage*. The IFS data base contained less than 180 maps for the Monterey seascape, and less than 160 maps for the Andes Indian girl.

The two primitives, a leaf and a flower, in Color Plate 29 were used as condensation sets in the picture *Sunflower Field*, Color Plate 30. Here we see the hierarchical structure: the leaf is itself the attractor of an IFS; and the flower is an overlay of four IFS attractors. The leaf is a condensation set for the IFS that generates all of the leaves. The flower is a condensation set to an IFS that generates many flowers, converging to the horizon. In the pictures *Sunflower Field* and *Black Forest*, shown in Color Plates 31–34, the primitives were displayed from back to front. The data bases for the *Sunflower Field* and *Black Forest* contain less than 100 and 120 maps, respectively. Notice the shadows behind the little trees in the background in Color Plate 32. The winter forest pictures were obtained by adjusting the color assignment function. The important point is that once the adjustment has been made, the image and the zoom are consistent.

Examples & Exercises

8.3. Use the Collage Theorem to help you find an IFS code for a leaf. Adjust your version of Program 1 in section 8 to allow you to render images of associated invariant measures. Assign colors in the range from red through orange to green. Adjust the probabilities. Obtain a spectacular color picture of the leaf showing the veins. Make a color slide of the output. To photograph a picture on the screen of a computergraphics monitor, use a telephoto lens. Mount the camera on a tripod, and take the photograph in a darkened room, on Ectachrome 64 ASA color slide film, 0.1 sec exposure, f-stop 5.6. For possible publication, submit the color slide, together with a letter of copyright assignment, to Michael Barnsley, Iterated Systems, Inc., 5550-A Peachtree Parkway, Suite 650, Norcross GA 30092 USA. Include a self-addressed envelope.

8.4. Obtain a very powerful computer with good graphics. Find the heirarchical IFS codes for the *Sunflower Field*. Replace the sunflowers by roses. Fly into your picture, to explore forever that scent-filled horizon. You are on your own.