Problem [HK] 7.1.1

Take any rational point x=p/q with gcd(p,q)=1. Then for any integer m, m*x mod 1 is inside the finite set {0,1/q, …,(q-1)/q}. Hence, since E_m is a function (every input yields a unique output), there must be positive integers n and p such that (E_m)^n(x)=(E_m)^(n+p)(x)=:y=(E_m)^p(y) (and hence that y= (E_m)^(p*k)(y) for all integer k positive.

7.1.2

A rational point is periodic under E_m $\Leftrightarrow$ it has a periodic base-m expansion.

7.1.3 Let $m \in \mathbb{N}$, $m \geq 2$

We need to show, that $P_n(E_{-m}) = |(-m)^n - 1|$

Now, if z is periodic ~~w~~ with period n then

$$z = E_{-m}^n(z) = z^{(-m)^n} \quad \text{and so}$$
$$z^{(-m)^n - 1} = 1$$

If n is even, then $(-m)^n = m^n > 0$ and so we find that our periodic points are the roots of unity of order $m^n - 1$ just as in the notes.

When n is odd, we have $(-m)^n = -(m^n)$, so

$$z^{-(m^n+1)} = 1 \iff \frac{1}{z^{m^n+1}} = 1$$

But z solves this equation iff $\bar{z}$ solves

$$\bar{z}^{m^n+1} = 1.$$

To see this first let $\bar{z}^{m^n+1} = 1$. Then, as:

$$z \cdot \bar{z} = |z|^2 = 1$$
$$\Rightarrow \bar{z} = \frac{1}{z} \Rightarrow \bar{z}^{m^n+1} = \frac{1}{z^{m^n+1}} = 1$$

The other implication is similar. Hence the number of periodic points is the same as the number of roots of unity of order $m^n + 1$, that is, $P_n(E_{-m}) = m^n + 1$. But, since we have $(-m)^n = -(m^n)$,

$$m^n + 1 = |-(m^n) - 1| = |(-m)^n - 1|$$

and we are done.

7.1.4

Write $MP_n(f_\lambda)$ for the number of periodic points of minimal period n. Then,

$$MP_n(f_\lambda) = P_n(f_\lambda) - \sum_{i|n} MP_i(f_\lambda)$$

(where $i|n$ means "i divides n").

Using the result $P_n(f_\lambda) = 2^n$, we can proceed as follows:

$$MP_n(f_\lambda) = 2^n - \sum_{i|n} MP_i(f_\lambda)$$

$$\geq 2^n - \sum_{i|n} P_i(f_\lambda) \geq 2^n - \sum_{i=1}^{\lfloor n/2 \rfloor} P_i(f_\lambda)$$

$$\geq 2^n - \lfloor n/2 \rfloor \, 2^{\lfloor n/2 \rfloor}$$

If n is even, we have

$$\geq 2^n - n\, 2^{n/2 - 1} = \underbrace{2^{n/2}\left(2^{n/2} - n/2\right)}_{\geq 2}$$

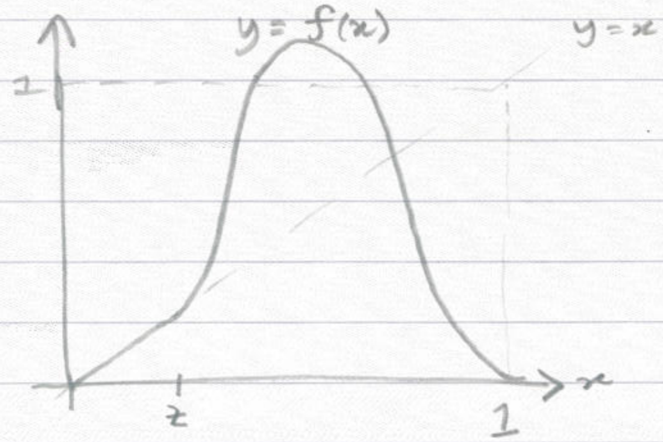
If n is odd and $n \geq 3$, we have

$$MP_n(f_\lambda) \geq 2^n - \frac{n-1}{2} 2^{\frac{n-1}{2}}$$

$$= 2^{\frac{n-1}{2}}\left(2 \cdot 2^{\frac{n-1}{2}} - \frac{n-1}{2}\right) \geq 6$$

And $MP_1(f_\lambda) = P_1(f_\lambda) = 2$, So we have $MP_n(f_\lambda) \geq 1 \;\forall\, n \in \mathbb{N}$, as desired.

7.1.5

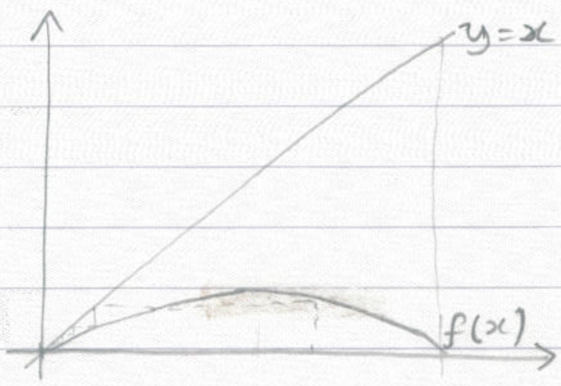


This map has uncountably many fixed points, which are periodic with period n for all $n \in \mathbb{N}$, so $P_n(f) > 2^n$ for every n.

Remark: Note that f fails to be 'expanding' on $[0, z]$; this is why proposition 7.1.5 does not apply.

7.1.6

Let $f(x) = \frac{1}{2}x(1-x)$



f has only the fixed point at zero. All other orbits tend to zero monotonically (as can be seen via 'cobweb' diagrams) and so are not periodic.

7.1.7

Let $H : [0,1] \times S^1 \to S^1$ be defined by

$$H(t,x) = t\,E_{\deg(f)}(x) + (1-t)f(x) \pmod 1$$

H is clearly cts as it is built from cts functions, and

$$H(0,x) = f(x),$$
$$H(1,x) = E_{\deg(f)}.$$

Let us check that $H(t,\cdot)$ defines a map from $S^1 \to S^1$ for each fixed t.

$$H(t,0) = t \times 0 + (1-t)f(0) \pmod 1,$$
$$H(t,1) = t \times \deg(f) + (1-t)f(1). \pmod 1$$

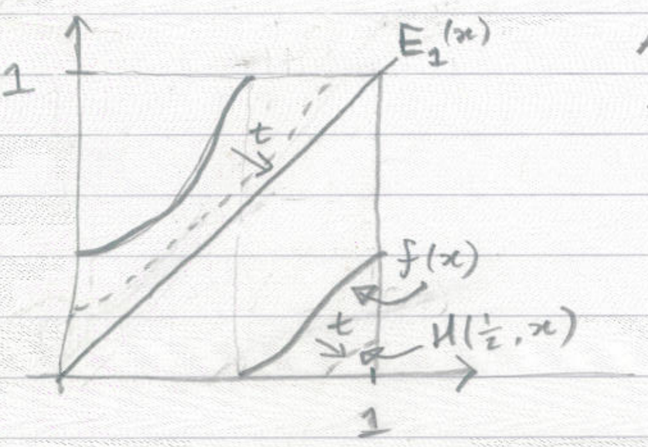
Recalling the definition of degree, let F be a lift of f, and write

$$H(t,1) = t(F(1) - F(0)) + (1-t)f(1) \pmod 1$$
$$= t(F(1) - f(1)) + f(1) - tF(0) \pmod 1$$

but $[F(x)] = f([x])$, and $f(1) = f(0)$, we have

$$H(t,1) = f(0)(1-t) = H(t,0), \text{ so}$$

$$H(t,\cdot) : S^1 \to S^1 \text{ as desired}$$



A homotopy H taking f to $E_{\deg(f)}$, in the case $\deg(f) = 1$

7.1.8

Assume for contradiction that $\deg(f) = m \neq n = \deg(g)$, & $H(t,x) : [0,1] \times S^1 \to S^1$ a homotopy (cts deformation) from f to g.

Consider $d(t) := \deg(H(t,x)) : [0,1] \to \mathbb{Z}$.
Since $\mathbb{Z}$ is discrete, d must have at least one point of discontinuity $\tau \in [0,1)$ (if d were cts, it would be constant, but $d(0) \neq d(1)$).
Fix $0 < \varepsilon < \frac{1}{2}$.
For arbitrarily small $\delta > 0$, $\exists\ t_1, t_2 \in B_\delta(\tau)$ with

$$d(t_i) = d_i \quad , \quad |d_1 - d_2| \geq 1 .$$

By continuity of H at (τ, x) there exists lifts L_1, L_2 of $H(t_1, x)$, $H(t_2, x)$ with

$$\|L_1 - L_2\| := \sup_{x \in [0,1]} |L_1(x) - L_2(x)| < \varepsilon .$$

But, since $L_1(1) - L_1(0) = d_1$, $L_2(1) - L_2(0) = d_2$,

$$\begin{aligned} 1 \leq |d_1 - d_2| &= |L_1(1) - L_2(1) - L_1(0) + L_2(0)| \\ &\leq |L_1(1) - L_2(1)| + \underbrace{|L_2(0) - L_1(0)|}_{< \varepsilon} \end{aligned}$$

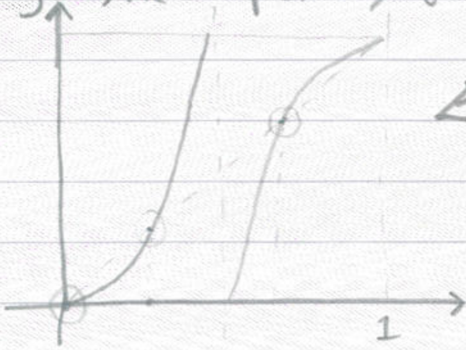
So we have

$$\varepsilon < 1 - \varepsilon \leq |L_1(1) - L_2(1)| \leq \|L_1 - L_2\| < \varepsilon ,$$

a contradiction.
We've shown that H __must__ have a discontinuity, so we can't find a cts deformation from f to g.

7.1.9

If f is C^1, and has degree 2, with 0 an attracting fixed point, then $P_1(f) \geq 3$:



An example with $P_1(f) = 3$. If f had more 'bumps', it could intersect the diagonal at other points too, but no fewer!

Applying this reasoning to the iterates of f, (noting that $\deg(f^n) = (\deg(f))^n$, and using the chain rule to see that 0 is still an attracting fixed point), we obtain $P_n(f) \geq 2^n + 2$, and hence the result.