7.2.1

Firstly, let us observe that $d(x,y) \leq \frac{1}{2}$ $\forall x,y \in S^1$, so $\Delta_{max} \leq \frac{1}{2}$.

In fact, $\Delta_{max} = \frac{1}{2}$: for any $x \in S^1$, $\varepsilon > 0$, $\exists y \in B_\varepsilon(x)$ s.t $d(x,y) = \frac{1}{2^N}$ for some $N \in \mathbb{N}$.

Then $d(f(x), f(y)) = \frac{2}{2^N} = \frac{1}{2^{N-1}}$ and so on,

and we see that $d(f^N(x), f^N(x)) = \frac{1}{2}$.

7.2.3

$\boxed{\text{NB: Question should read } \Delta < \text{diam}(X)/2}$

Let $0 < \Delta < \text{diam}(X)/2$. Let $\delta > 0$ be such that $\text{diam}(X) - 2\Delta > \delta$. Now let $x \in X$, $\varepsilon > 0$ be arbitrary. Write $\mathcal{U} = B_\varepsilon(x)$.

Now let $z_1, z_2 \in X$ satisfy $d(z_1, z_2) \geq 2\Delta + \delta$. Write $V_1 = B_{\delta/2}(z_1)$, $V_2 = B_{\delta/2}(z_2)$. The def. of topological mixing gives us $N_1, N_2 \in \mathbb{N}$ s.t $f^n(\mathcal{U}) \cap V_1 \neq \emptyset$, $f^m(\mathcal{U}) \cap V_2 \neq \emptyset$ $\forall$ $n > N_1$, $m > N_2$. Let $N = \max\{N_1, N_2\}$. Then $\exists y_1, y_2 \in \mathcal{U}$ s.t $f^N(y_1) \in V_1$, $f^N(y_2) \in V_2$ respectively, and we have

$$2\Delta + \cancel{\delta} \leq d(z_1, z_2) \leq d(z_1, f^N(y_1)) + d(f^N(y_1), f^N(x)) + d(f^N(y_2), f^N(x)) + d(f^N(y_2), z_2)$$
$$\leq \cancel{\delta/2} + d(f^N(y_1), f^N(x)) + d(f^N(y_2), f^N(x)) + \cancel{\delta/2}$$

Hence either $d(f^N(y_1), f^N(x))$ or $d(f^N(y_2), f^N(x))$ is greater than or equal to Δ.

7.3.1

Looking at binary expansions of elements from S^1, we see that the even iterates of $x = 0.x_1x_2x_3x_4\ldots \in S^1$ lie in $[0,\frac{1}{2})$ iff $x_{2k+1} = 0$, $\forall k \in \mathbb{N}$.

We are now free to choose the remaining digits in the expansion to ensure the orbit is non-periodic. One way to do this would be to take something of the form

$$(x_2, x_4, x_6, \ldots) = (1,0,1,1,0,1,1,1,0,1,1,1,1,0,\ldots)$$

that is, blocks of 1's of increasing length separated by single 0's.

7.3.2

Assume that the collection of orbits for which no segment of length 10 has more than one point in the left half of the unit interval is countable:

$$\mathcal{C} = \{x^1, x^2, x^3, \ldots\}$$

We can construct another element of $\mathcal{C}$ which is different from every x^i, using a 'diagonal argument' (popularised by Cantor), contradicting countability: to do this, build $x = 0.x_1x_2x_3x_4\ldots$ as follows: examine the first 10 entries of the binary expansion of $x^1 = 0.x_1^1x_2^1x_3^1\ldots x_{10}^1\ldots$

If $x_i^1 = 0$ for some $i \in \{1,\ldots,10\}$, then choose $x_i \equiv 1$ for $i \in \{1,\ldots,10\}$. If $x_i^1 \equiv 1$, then choose $x_1 = 0$, $x_i = 1$, $i \in \{2,\ldots,10\}$.

Repeat with the second 10 entries of x^2, and so on.

7.3.2 (continued)

The x constructed in this manner is different from every x^i in at least 1 place, and the only possible 0's are in the places x_{10k+1}, $k \in \mathbb{N} \cup \{0\}$, so $x \in \mathcal{C}$.

Hence $\mathcal{C}$ is uncountable.