Answer Sheet 4

December 8, 2013

[F03] 2.1: Let $\mathbb{U}_\delta$ be the collection of δ-covers of F and $\mathbb{A}_\delta$ the collection of all closed δ- covers of F. Clearly $\mathbb{A}_\delta \subset \mathbb{U}_\delta$ and hence $H^s_\delta(F)_{\mathbb{A}} \geq H^s_\delta(F)_{\mathbb{U}}$. To show $H^s_\delta(F)_{\mathbb{A}} \leq H^s_\delta(F)_{\mathbb{U}}$ we use that for any $\{U_i\} \in \mathbb{U}$ means that $\{cl(U_i)\} \in \mathbb{A}$ and as $|cl(U_i)| = |U_i|$ the result follows.

[F03] 2.2: Assume that F is finite let $\delta' \leq 1/2min_{x,y\in F}\{|x-y|\}$ then we can choose $\{U_i\}$ a $\delta \leq \delta'$-cover such that for each $x_i \in F$ we have a unique set U_i such that $x_i \in U_i$ and $i \neq j \Rightarrow U_i \bigcap U_j = \emptyset$. Hence, as $|U_i|^0 = 1$, the sum $\sum |U_i|^0$ is the number of points in F for all $\delta \leq \delta'$. Thus in the limit $H^0(F)$ is just the counting measure. If F is infinite then choose a sequence of subsets of F denoted $\{F_i\}$ such that each is finite and of cardinalilty i and also:

$$F_1 \subset F_2 \subset ... \subset F_i \subset ...$$

Hence $H^0(F_i) = i$, so $lim_{i\to\infty} H^0(F_i) = \infty$. Thus as $H^0(F_i) \leq H^0(F)$ we have $H^0(F) = \infty$.

[F03] 2.3: Any collection of sets cover the emptyset so clearly $H^s(\emptyset) = 0$. If U_i is a δ-cover of F then U_i is also a δ-cover of E. So the set of all δ covers for F is contained in the set of all δ-covers for E and hence $H^s(E) \leq H^s(F)$. Finally we need to show:

$$H^s(\bigcap_{i=1}^{\infty} F_i) \leq \sum_{i=1}^{\infty} H^s(F_i)$$

Assuming $H^s(F_i) \leq \infty$ for all k. Given $\epsilon > 0$, for each k there exists a δ-cover $\{U_i^k\}$ of F_k such that:

$$\sum_i |U_i^k|^s < H^s_\delta(F_k) + \epsilon/2^k$$

By summing through the k's we get:

$$\sum_k \sum_i |U_i^k|^s < \sum_k H^s_\delta(F_k) + \epsilon$$

The collection $\bigcup_k \{U_i^k\}$ is a cover of $\bigcup_k (F_k)$ and hence:

$$H^s_\delta(\bigcup_k (F_k)) \leq \sum_k \sum_i |U_i^k|^s < \sum_k H^s_\delta(F_k) + \epsilon$$

Letting ϵ tend to zero we have for any $\delta > 0$:

$$H^s_\delta(\bigcup_k (F_k)) \leq \sum_k H^s_\delta(F_k)$$

thus

$$H^s(\bigcup_k (F_k)) \leq \sum_k H^s(F_k)$$

[F03] 2.4: The claim is that $1/4 < H^1([0,1]) < 1$. To see $H^1([0,1]) < 1$ we choose a cover $U^k_j = [(k-1)2^{-j}, k2^{-j}]$. Clearly there are $|\{U^k_j\}| = 2^j$ sets each of size $|U^k_j| = 2^{-j}$. Thus by choosing j large enough such that $2^{-j} < \delta$ we have $H^s_\delta([0,1]) < \sum_{k=0}^{2^j} |(U^k_j)| = \sum_{k=0}^{2^j} 2^{-j} = 2^j 2^{-j} = 1$. Thus $H^s([0,1]) < 1$. To see $1/4 < H^1([0,1])$ we pick any δ-cover $\{U_i\}$. For each U_i there exists k_i such that $2^{-k_i-1} \leq |U_i| < 2^{-k_i}$. In this case U_i can intersect with at most 2 of the intervals $[(k-1)2^{-k_i}, k2^{-k_i}]$. If $j > k_i$ we can count up the maximum number of intervals of diameter 2^{-j} that U_i can intersect which we find to be:

$$2^{1+j-k_i} = 2^{j+1}2^{-k_i} = 2^{j+2}2^{-k_i-1} \leq 2^{j+2}|U_i|$$

hence by choosing j larger than any of the k_i's we can count up the intervals to get:

$$2^j \leq \sum_i 2^{j+2}|U_i| \Rightarrow 2^{-2} \leq \sum_i |U_i|$$

Hence as this is true for any δ-cover we have $1/4 < H^1([0,1])$.

[F03] 2.5: F bounded means for some $R > 0$ and for all $x \in F$ we have $|x| < R$. So we have a closed interval $[-R, R]$ such that $F \subset [-R, R]$ and as f cont.diff. we have f' is bounded in $[-R, R]$, Let $c = sup_{a \in [-R,R]}(|f'(a)|)$. Clearly $c > f'(x)$ for all $x \in F$. For any $a, b \in F$ with $a < b$ we have by the mean value theorem the existance of d such that $d \in [a, b]$ and:

$$\frac{f(b) - f(a)}{b - a} = f'(d) \leq c \Rightarrow |f(b) - f(a)| \leq c|b - a|$$

hence f is lipshitz and we have $dim_H(f(F)) \leq dim_H(F)$. If F is not bounded then we take the sequence of bounded sets F_i defined by $x \in F_i$ iff $x \in F$ and $|x| < i$. Clearly $F_i \subset F_{i+1}$ and as $lim_{i\to\infty} H^s(F_i) = H^s(\bigcup_{i=1}^{\infty})(F_i)$ we have:

$$lim_{n\to\infty} dim_H(f(F_i)) \leq lim_{n\to\infty} dim_H(F_i)$$

taking limits:

$$dim_H(f(F)) \leq dim_H(F)$$

[F03] 2.6: Consider the set $F_i = B_i(0) \bigcap F\ B_{1/i}(0)$ for $i \in \mathbb{N}$. It's clear that $f(x)$ is bi-lipshitz on F_i for all i with $c_1 = 2/i$ and $c_2 = 2i$. Hence we have $Dim_H(f(F_i)) = Dim_H(F_i)$ for a i. Taking limits as in the last question we get $Dim_H(f(F)) = Dim_H(F)$.

[F03] 2.7: If $\{U_i\}$ is a δ-cover for $[0,1]$ then clearly $\{U_i \times f(U_i)\}$ is a cover of $graph(f)$. As f is lipshitz we have $|U_i \times f(U_i)| \leq \sqrt{|U_i^2 + c^2|U_i|^2|} = |U_i|\sqrt{1+c^2}$. So $\sum_i \{U_i \times f(U_i)\} \leq \sqrt{1+c^2}\sum_i(|U_i|)$. Hence $H^1(graph(f)) \leq \sqrt{1+c^2}H^1([0,1]) = \sqrt{1+c^2} \Rightarrow Dim_H(graph(f)) \leq 1$. To see that $Dim_H(graph(f)) \geq 1$ we consider the projection $\pi : \mathbb{R}^{\not{\,}} \to \mathbb{R}$ given by $\pi(x,y) = x$. This is lipshitz and so we have: $1 = H^1([0,1]) = H^1(\pi(graph(f))) \leq H^1(graph(f))$ and the result follows.

[F03] 2.8: In each case given any $s > 0$ choose a δ-cover $\{U_i\}$ such that $x_i \in U_i$ where x_i is just the $i-th$ element of $F_1 = \{0,1,...\}$ or $F_2 = \{0,2,1/2,1/3,...\}$. For $i > 0$ let $|U_i| < \delta/i^{2/s}$, and let $|U_0| < \delta$. Then in each case: $\sum_i |U_i|^s = \sum_{i=1}^{\infty}(\delta^2/i^2) + \delta^2 = \delta^2(1 + \sum_{i=1}^{\infty} 1/i^2) = \delta^2\pi^2/6$. This means that $H^s_\delta(F_k) \leq \delta^2\pi^2/6 \to 0$ as $\delta \to 0$. Hence $dim_H(F_k) = 0$.

[F03] 2.9: F Can be constructed similarly to the cantor set. You partition $[0,1]$ into 10 intervals and remove the 5th. Then repeat for each of the 9 remaining intervals. Call the set resulting form the k^{th} stage in this process the k^{th} layer. Notice that we can shrink F into any of the 9 intervals in the 2^{nd} layer. We then have:

$$H^s(F) = H^s(S_1(F)) + ... + H^s(S_4(F)) + H^s(S_6(F)) + ... + H^s(S_{10}(F))$$

Where S_i is the map of $[0,1]$ onto $[i/10,(i+1)/10]$. Clearly $H^s(s_i(F)) = (10^{-s})H^s(F)$. From above we obtain $H^s(F) = 9.(10^{-s})H^s(F)$. Assuming $H^s(F) \neq 0$ we get $1 = 9.(10^{-s})$. Hence $Dim_H(F) = s = log(9)/log(10)$.

[F03] 2.10: By a similar argument as above we can construct F by partitioning $[0,1] \times [0,1]$ into 100 squares each with side $1/10$ Then remove the 5^{th} column and row. Again repeat this process to obtain F. In this case $S_{i,j}$ contracts $[0,1] \times [0,1]$ onto the $(i,j)^{th}$ square where $i,j \neq 5$. Hence there are 9^2 possible contractions $S_{(i,j)}$. Each contracts by $1/10$ and we get (assuming $H^s(F) \neq 0$):

$$H^s(F) = 9^2(1/10)^s H^s(F) \Rightarrow 1 = 9^2(1/10)^s \Rightarrow Dim_H(F) = 2log(9)/log(10)$$

[F03] 2.11: Again similar to above but this time there are 5 contractions on $[0,1] \times [0,1]$, 4 to each of the smaller squares each a contraction of $1/4$, and 1 to the middle square, a contraction of $1/2$ thus we have (assuming $H^s(F) \neq 0$):

$$H^s(F) = (1/2)^s H^s(F) + 4(1/4)^s H^s(F) \Rightarrow (1/2)^s + 4(1/4)^s - 1 = 0$$

Let $x = 2^{-s}$ then we have $4x^2 + x - 1 = 0$ the solns of which are $x = \frac{-1\pm\sqrt{17}}{8}$. Picking the positive root and solving for S we obtain:

$$s = log(\frac{-1+\sqrt{17}}{8})/log(1/2) \approx 1.35702$$

[F03] 2.12: F satisfies $S(F) = F$ where $S(x) = x+2$. In fact $F = \bigcup_{n=-\infty}^{\infty} S^n(C)$ where C is the normal cantor set on $[0,1]$. Clearly $dim_H(S^n(C)) = dim_H(C)$ then by countable stability we have $dim_H(F) = sup_{n\in\mathbb{Z}}(dim_H(S^n(C))) = dim_H(C)$.

[F03] 2.13: Let F_k be all the $x \in F$ such that for all $l > k$, $x = 0.a_0a_1...a_k...$, $a_l \neq 1$ and $a_{k-1} = 1$. Notice that $F_0 = C$ where C is the Cantor set. Similarly $F_1 = 1/3C + 1/3$ and $F_2 = \{1/9C + 1/9\} \bigcup \{1/9C + 4/9\} \bigcup \{1/9C + 7/9\}$. In general we have:

$$F_i = \bigcup_{j=0}^{3^{i-1}-1} \{\frac{C}{3^i} + \frac{1+3j}{3^i}\}$$

Noticing this we see that you can map F onto any of the three $[0,1/3]$, $[1/3,2/3]$ or $[2/3,1]$ with a contraction. Hence $H^s(F) = 3(1/3)^s H^s(F)$ and so $1 = 3(1/3)^s$ which implies $s = 1$.

[F03] 2.14: By the same argument as questions 2.10,2.11 we find a map S with contraction $\frac{1-\lambda}{2}$ and hence $H^s(S(F)) = (\frac{1-\lambda}{2})^s H^s(F)$. We get $H^s(F) = 2(\frac{1-\lambda}{2})^s H^s(F)$ and assumeing $H^s(F) \neq 0$ we get: $1 = 2(\frac{1-\lambda}{2})^s$ so $dim_H(F) = log2/log(\frac{2}{1-\lambda})$. For E apply similar reasoning to Q2.10 to get $H^s(E) = 4(\frac{1-\lambda}{2})^s H^s(E)$ and hence $dim_H(E) = log4/log(\frac{2}{1-\lambda}) = 2log2/log(\frac{2}{1-\lambda})$.

[F03] 2.15: Partition $[0,1] \times [0,1]$ by 16 squares and shrink each to have sides $1/d$. Then do repeat with each of these smaller squares and so on. The set F obtained at the end of this process has dimension $s(d) = 4log2/log(d)$ where $d \geq 4$. To see this we find an upper bound by looking at the cover of the k^{th} level of squares. Given $\delta > 0$ choose k s.t. $d^{-k} < \delta$ then:

$$H^s(F) \leq \sum_{i=0}^{16^k} |U_i|^s = 16^k (1/d^k)^s \sqrt{2}^s = \sqrt{2}^{4log(2)/log(d)}$$

Next let $\{U_i\}$ be any cover of F. Find k_i such that $\sqrt{2}d^{-k_i-1} \leq |U_i| \leq \sqrt{2}d^{-k_i}$. Thus U_i can intersect at most 4 of the 16^{k_i} k^{th} level sets. If we choose $j > k_i$ we have U_i intersects at most:

$$4.16^{j-k_i} = 4.16^j d^{-sk_i} = 4.16^j d^s d^{-(s(k_i+1))} \leq 4.16^j d^s |U_i|^s$$

By choosing j large enough such that $d^{-(j+1)} \leq |U_i|$ for all U_i we have:

$$16^j \leq \sum_i 4.16^j d^s |U_i|^s \Rightarrow \sum_i |U_i|^s \geq \frac{1}{4d^s} = 1/4^3$$

Hence we have:

$$1/4^3 \leq H^s(F) \leq \sqrt{2}^{4log(2)/log(d)}$$

[F03] 2.16: Let $\varphi : [0,1] \to \mathbb{R}^2$ be the parameterisation of the unit circle around 0. $\varphi(t) = (cos2\pi t, sin2\pi t)$. φ is bi-lipshitz on $R_i = [1/2^i, 1 - 1/2^i]$. write $F_i = \varphi(C \bigcap R_i)$. Clearly $F_i \subset F$, $F_i \to F$ and as $dim_H(F_i) = dim_H(C)$ the result follows in taking the limit.