1

Measure and dimension

1.1 Basic measure theory

This section contains a condensed account of the basic measure theory we require. More complete treatments may be found in Kingman & Taylor (1966) or Rogers (1970).

Let X be any set. (We shall shortly take X to be n-dimensional Euclidean space $\mathbb{R}^n$.) A non-empty collection $\mathscr{S}$ of subsets of X is termed a *sigma-field* (or *σ-field*) if $\mathscr{S}$ is closed under complementation and under countable union (so if $E \in \mathscr{S}$, then $X \backslash E \in \mathscr{S}$ and if $E_1, E_2, \ldots \in \mathscr{S}$, then $\bigcup_{j=1}^{\infty} E_j \in \mathscr{S}$). A little elementary set theory shows that a σ-field is also closed under countable intersection and under set difference and, further, that X and the null set $\varnothing$ are in $\mathscr{S}$.

The *lower* and *upper limits* of a sequence of sets $\{E_j\}$ are defined as

$$\varliminf_{j \to \infty} E_j = \bigcup_{k=1}^{\infty} \bigcap_{j=k}^{\infty} E_j$$

and

$$\varlimsup_{j \to \infty} E_j = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} E_j.$$

Thus $\varliminf E_j$ consists of those points lying in all but finitely many E_j, and $\varlimsup E_j$ consists of those points in infinitely many E_j. From the form of these definitions it is clear that if E_j lies in the σ-field $\mathscr{S}$ for each j, then $\varliminf E_j$, $\varlimsup E_j \in \mathscr{S}$. If $\varliminf E_j = \varlimsup E_j$, then we write $\lim E_j$ for the common value; this always happens if $\{E_j\}$ is either an increasing or a decreasing sequence of sets.

Let $\mathscr{C}$ be any collection of subsets of X. Then the σ-field *generated by* $\mathscr{C}$, written $\mathscr{S}(\mathscr{C})$, is the intersection of all σ-fields containing $\mathscr{C}$. A straightforward check shows that $\mathscr{S}(\mathscr{C})$ is itself a σ-field which may be thought of as the 'smallest' σ-field containing $\mathscr{C}$.

A *measure* μ is a function defined on some σ-field $\mathscr{S}$ of subsets of X and taking values in the range $[0, \infty]$ such that

$$\mu(\varnothing) = 0 \tag{1.1}$$

and

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j) \tag{1.2}$$

for every countable sequence of disjoint sets $\{E_j\}$ in $\mathscr{S}$.

It follows from (1.2) that μ is an increasing set function, that is, if $E \subset E'$ and $E, E' \in \mathscr{S}$, then

$$\mu(E) \leq \mu(E').$$

Theorem 1.1 (continuity of measures)
Let μ be a measure on a σ-field $\mathscr{S}$ of subsets of X.
(a) *If $E_1 \subset E_2 \subset \ldots$ is an increasing sequence of sets in $\mathscr{S}$, then*

$$\mu(\lim_{j\to\infty} E_j) = \lim_{j\to\infty} \mu(E_j).$$

(b) *If $F_1 \supset F_2 \supset \ldots$ is a decreasing sequence of sets in $\mathscr{S}$ and $\mu(F_1) < \infty$, then*

$$\mu(\lim_{j\to\infty} F_j) = \lim_{j\to\infty} \mu(F_j).$$

(c) *For any sequence of sets $\{F_j\}$ in $\mathscr{S}$,*

$$\mu(\underline{\lim}_{j\to\infty} F_j) \leq \underline{\lim}_{j\to\infty} \mu(F_j).$$

Proof. (a) We may express $\bigcup_{j=1}^{\infty} E_j$ as the disjoint union $E_1 \cup \bigcup_{j=2}^{\infty}(E_j \backslash E_{j-1})$. Thus by (1.2),

$$\begin{aligned}
\mu(\lim_{j\to\infty} E_j) &= \mu\left(\bigcup_{j=1}^{\infty} E_j\right) \\
&= \mu(E_1) + \sum_2^{\infty} \mu(E_j \backslash E_{j-1}) \\
&= \lim_{k\to\infty}\left[\mu(E_1) + \sum_2^{k} \mu(E_j \backslash E_{j-1})\right] \\
&= \lim_{k\to\infty} \mu\left(E_1 \cup \bigcup_2^{k}(E_j \backslash E_{j-1})\right) \\
&= \lim_{k\to\infty} \mu(E_k).
\end{aligned}$$

(b) If $E_j = F_1 \backslash F_j$, then $\{E_j\}$ is as in (a). Since $\bigcap_j F_j = F_1 \backslash \bigcup_j E_j$,

$$\begin{aligned}
\mu(\lim_{j\to\infty} F_j) &= \mu\left(\bigcap_{j=1}^{\infty} F_j\right) \\
&= \mu(F_1) - \mu(\bigcup_j E_j) \\
&= \mu(F_1) - \lim_{j\to\infty} \mu(E_j)
\end{aligned}$$

$$= \lim_{j\to\infty} (\mu(F_1) - \mu(E_j))$$

$$= \lim_{j\to\infty} \mu(F_j).$$

(c) Now let $E_k = \bigcap_{j=k}^{\infty} F_j$. Then $\{E_k\}$ is an increasing sequence of sets in $\mathscr{S}$, so by (a),

$$\mu(\varliminf_{j\to\infty} F_j) = \mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \lim_{k\to\infty} \mu(E_k) \le \varliminf_{j\to\infty} \mu(F_j). \qquad \square$$

Next we introduce outer measures which are essentially measures with property (1.2) weakened to subadditivity. Formally, an *outer measure* ν on a set X is a function defined on *all* subsets of X taking values in $[0, \infty]$ such that

$$\nu(\varnothing) = 0, \tag{1.3}$$

$$\nu(A) \le \nu(A') \quad \text{if } A \subset A' \tag{1.4}$$

and

$$\nu\left(\bigcup_1^{\infty} A_j\right) \le \sum_1^{\infty} \nu(A_j) \quad \text{for any subsets } \{A_j\} \text{ of } X. \tag{1.5}$$

Outer measures are useful since there is always a σ-field of subsets on which they behave as measures; for reasonably defined outer measures this σ-field can be quite large.

A subset E of X is called *ν-measurable* or *measurable with respect to the outer measure ν* if it decomposes every subset of X additively, that is, if

$$\nu(A) = \nu(A \cap E) + \nu(A \backslash E) \tag{1.6}$$

for all 'test sets' $A \subset X$. Note that to show that a set E is ν-measurable, it is enough to check that

$$\nu(A) \ge \nu(A \cap E) + \nu(A \backslash E), \tag{1.7}$$

since the opposite inequality is included in (1.5). It is trivial to verify that if $\nu(E) = 0$, then E is ν-measurable.

Theorem 1.2

Let ν be an outer measure. The collection $\mathscr{M}$ of ν-measurable sets forms a σ-field, and the restriction of ν to $\mathscr{M}$ is a measure.

Proof. Clearly, $\varnothing \in \mathscr{M}$, so $\mathscr{M}$ is non-empty. Also, by the symmetry of (1.6), $A \in \mathscr{M}$ if and only if $X \backslash A \in \mathscr{M}$. Hence $\mathscr{M}$ is closed under taking complements.

To prove that $\mathscr{M}$ is closed under countable union, suppose that $E_1, E_2, \ldots \in \mathscr{M}$ and let A be any set. Then applying (1.6) to $E_1, E_2, \ldots$ in turn with appropriate test sets,

$$\begin{aligned} \nu(A) &= \nu(A\cap E_1)+\nu(A\backslash E_1) \\ &= \nu(A\cap E_1)+\nu((A\backslash E_1)\cap E_2)+\nu(A\backslash E_1\backslash E_2) \\ &= \cdots \\ &= \sum_{j=1}^{k}\nu\left(\left(A\backslash\bigcup_{i=1}^{j-1}E_i\right)\cap E_j\right)+\nu\left(A\backslash\bigcup_{j=1}^{k}E_j\right). \end{aligned}$$

Hence

$$\nu(A)\geq \sum_{j=1}^{k}\nu\left(\left(A\backslash\bigcup_{i=1}^{j-1}E_i\right)\cap E_j\right)+\nu\left(A\backslash\bigcup_{j=1}^{\infty}E_j\right)$$

for all k and so

$$\nu(A)\geq \sum_{j=1}^{\infty}\nu\left(\left(A\backslash\bigcup_{i=1}^{j-1}E_i\right)\cap E_j\right)+\nu\left(A\backslash\bigcup_{j=1}^{\infty}E_j\right). \tag{1.8}$$

On the other hand,

$$A\cap\bigcup_{j=1}^{\infty}E_j=\bigcup_{j=1}^{\infty}\left(\left(A\backslash\bigcup_{i=1}^{j-1}E_i\right)\cap E_j\right),$$

so, using (1.5),

$$\begin{aligned} \nu(A) &\leq \nu\left(A\cap\bigcup_{j=1}^{\infty}E_j\right)+\nu\left(A\backslash\bigcup_{j=1}^{\infty}E_j\right) \\ &\leq \sum_{j=1}^{\infty}\nu\left(\left(A\backslash\bigcup_{i=1}^{j-1}E_i\right)\cap E_j\right)+\nu\left(A\backslash\bigcup_{j=1}^{\infty}E_j\right)\leq \nu(A), \end{aligned}$$

by (1.8). It follows that $\bigcup_{j=1}^{\infty}E_j\in\mathscr{M}$, so $\mathscr{M}$ is a σ-field.

Now let $E_1,E_2,\ldots$ be *disjoint* sets of $\mathscr{M}$. Taking $A=\bigcup_{j=1}^{\infty}E_j$ in (1.8),

$$\nu\left(\bigcup_{j=1}^{\infty}E_j\right)\geq\sum_{j=1}^{\infty}\nu(E_j)$$

and combining this with (1.5) we see that ν is a measure on $\mathscr{M}$. □

We say that the outer measure ν is *regular* if for every set A there is a ν-measurable set E containing A with $\nu(A)=\nu(E)$.

Lemma 1.3

If ν is a regular outer measure and $\{A_j\}$ is any increasing sequence of sets,

$$\lim_{j\to\infty}\nu(A_j)=\nu(\lim_{j\to\infty}A_j).$$

Proof. Choose a ν-measurable E_j with $E_j\supset A_j$ and $\nu(E_j)=\nu(A_j)$ for each j. Then, using (1.4) and Theorem 1.1(c),

$$\nu(\lim A_j)=\nu(\underline{\lim}\,A_j)\leq\nu(\underline{\lim}\,E_j)\leq\underline{\lim}\,\nu(E_j)=\lim\nu(A_j).$$

The opposite inequality follows from (1.4). □

Now let (X, d) be a metric space. (For our purposes X will usually be n-dimensional Euclidean space, $\mathbb{R}^n$, with d the usual distance function.) The sets belonging to the σ-field generated by the closed subsets of X are called the *Borel sets* of the space. The Borel sets include the open sets (as complements of the closed sets), the *F_σ-sets* (that is, countable unions of closed sets), the *G_δ-sets* (countable intersections of open sets), etc.

An outer measure ν on X is termed a *metric outer measure* if

$$\nu(E \cup F) = \nu(E) + \nu(F) \tag{1.9}$$

whenever E and F are *positively separated*, that is, whenever

$$d(E, F) = \inf\{d(x, y) : x \in E, y \in F\} > 0.$$

We show that if ν is a metric outer measure, then the collection of ν-measurable sets includes the Borel sets. The proof is based on the following version of 'Carathéodory's lemma'.

Lemma 1.4

Let ν be a metric outer measure on (X, d). Let $\{A_j\}_1^\infty$ be an increasing sequence of subsets of X with $A = \lim_{j\to\infty} A_j$, and suppose that $d(A_j, A \backslash A_{j+1}) > 0$ for each j. Then $\nu(A) = \lim_{j\to\infty} \nu(A_j)$.

Proof. It is enough to prove that

$$\nu(A) \le \lim_{j\to\infty} \nu(A_j), \tag{1.10}$$

since the opposite inequality follows from (1.4). Let $B_1 = A_1$ and $B_j = A_j \backslash A_{j-1}$ for $j \ge 2$. If $j + 2 \le i$, then $B_j \subset A_j$ and $B_i \subset A \backslash A_{i-1} \subset A \backslash A_{j+1}$, so B_i and B_j are positively separated. Thus, applying (1.9) $(m-1)$ times,

$$\nu\left(\bigcup_{k=1}^{m} B_{2k-1}\right) = \sum_{k=1}^{m} \nu(B_{2k-1}),$$

$$\nu\left(\bigcup_{k=1}^{m} B_{2k}\right) = \sum_{k=1}^{m} \nu(B_{2k}).$$

We may assume that both these series converge – if not we would have $\lim_{j\to\infty} \nu(A_j) = \infty$, since $\bigcup_{k=1}^m B_{2k-1}$ and $\bigcup_{k=1}^m B_{2k}$ are both contained in A_{2m}. Hence

$$\nu(A) = \nu\left(\bigcup_{j=1}^{\infty} A_j\right) = \nu\left(A_j \cup \bigcup_{k=j+1}^{\infty} B_k\right)$$

$$\le \nu(A_j) + \sum_{k=j+1}^{\infty} \nu(B_k)$$

$$\leq \lim_{i\to\infty} \nu(A_i) + \sum_{k=j+1}^{\infty} \nu(B_k).$$

Since the sum tends to 0 as $j \to \infty$, (1.10) follows. □

Theorem 1.5

If ν is a metric outer measure on (X,d), then all Borel subsets of X are ν-measurable.

Proof. Since the ν-measurable sets form a σ-field, and the Borel sets form the smallest σ-field containing the closed subsets of X, it is enough to show that (1.7) holds when E is closed and A is arbitrary.

Let A_j be the set of points in $A\backslash E$ at a distance at least $1/j$ from E. Then $d(A \cap E, A_j) \geq 1/j$, so

$$\nu(A \cap E) + \nu(A_j) = \nu((A \cap E) \cup A_j) \leq \nu(A) \tag{1.11}$$

for each j, as ν is a metric outer measure. The sequence of sets $\{A_j\}$ is increasing and, since E is closed, $A\backslash E = \bigcup_{j=1}^{\infty} A_j$. Hence, provided that $d(A_j, A\backslash E\backslash A_{j+1}) > 0$ for all j, Lemma 1.4 gives $\nu(A\backslash E) \leq \lim_{j\to\infty} \nu(A_j)$ and (1.7) follows from (1.11). But if $x \in A\backslash E\backslash A_{j+1}$ there exists $z \in E$ with $d(x,z) < 1/(j+1)$, so if $y \in A_j$ then $d(x,y) \geq d(y,z) - d(x,z) > 1/j - 1/(j+1) > 0$. Thus $d(A_j, A\backslash E\backslash A_{j+1}) > 0$, as required. □

There is another important class of sets which, unlike the Borel sets, are defined explicitly in terms of unions and intersections of closed sets. If (X,d) is a metric space, the *Souslin sets* are the sets of the form

$$E = \bigcup_{i_1 i_2 \ldots} \bigcap_{k=1}^{\infty} E_{i_1 i_2 \ldots i_k},$$

where $E_{i_1 i_2 \ldots i_k}$ is a closed set for each finite sequence $\{i_1, i_2, \ldots, i_k\}$ of positive integers. Note that, although E is built up from a countable collection of closed sets, the union is over continuum-many infinite sequences of integers. (Each closed set appears in the expression in many places.)

It may be shown that every Borel set is a Souslin set and that, if the underlying metric spaces are complete, then any continuous image of a Souslin set is Souslin. Further, if ν is an outer measure on a metric space (X,d), then the Souslin sets are ν-measurable provided that the closed sets are ν-measurable. It follows from Theorem 1.5 that if ν is a metric outer measure on (X,d), then the Souslin sets are ν-measurable. We shall only make passing reference to Souslin sets. Measure-theoretic aspects are described in greater detail by Rogers (1970), and the connoisseur might also consult Rogers *et al.* (1980).

1.2 Hausdorff measure

For the remainder of this book we work in Euclidean n-space, $\mathbb{R}^n$, although it should be emphasized that much of what is said is valid in a general metric space setting.

If U is a non-empty subset of $\mathbb{R}^n$ we define the *diameter* of U as $|U| = \sup\{|x-y| : x, y \in U\}$. If $E \subset \bigcup_i U_i$ and $0 < |U_i| \leq \delta$ for each i, we say that $\{U_i\}$ is a *δ-cover* of E.

Let E be a subset of $\mathbb{R}^n$ and let s be a non-negative number. For $\delta > 0$ define

$$\mathscr{H}^s_\delta(E) = \inf \sum_{i=1}^{\infty} |U_i|^s, \tag{1.12}$$

where the infimum is over all (countable) δ-covers $\{U_i\}$ of E. A trivial check establishes that $\mathscr{H}^s_\delta$ is an outer measure on $\mathbb{R}^n$.

To get the *Hausdorff s-dimensional outer measure* of E we let $\delta \to 0$. Thus

$$\mathscr{H}^s(E) = \lim_{\delta \to 0} \mathscr{H}^s_\delta(E) = \sup_{\delta > 0} \mathscr{H}^s_\delta(E). \tag{1.13}$$

The limit exists, but may be infinite, since $\mathscr{H}^s_\delta$ increases as δ decreases. $\mathscr{H}^s$ is easily seen to be an outer measure, but it is also a *metric* outer measure. For if δ is less than the distance between positively separated sets E and F, no set in a δ-cover of $E \cup F$ can intersect both E and F, so that

$$\mathscr{H}^s_\delta(E \cup F) = \mathscr{H}^s_\delta(E) + \mathscr{H}^s_\delta(F),$$

leading to a similar equality for $\mathscr{H}^s$. The restriction of $\mathscr{H}^s$ to the σ-field of $\mathscr{H}^s$-measurable sets, which by Theorem 1.5 includes the Borel sets (and, indeed, the Souslin sets) is called *Hausdorff s-dimensional measure.*

Note that an equivalent definition of Hausdorff measure is obtained if the infimum in (1.12) is taken over δ-covers of E by convex sets rather than by arbitrary sets since any set lies in a convex set of the same diameter. Similarly, it is sometimes convenient to consider δ-covers of open, or alternatively of closed, sets. In each case, although a different value of $\mathscr{H}^s_\delta$ may be obtained for $\delta > 0$, the value of the limit $\mathscr{H}^s$ is the same, see Davies (1956). (If however, the infimum is taken over δ-covers by balls, a different measure is obtained; Besicovitch (1928*a*, Chapter 3) compares such 'spherical Hausdorff measures' with Hausdorff measures.)

For any E it is clear that $\mathscr{H}^s(E)$ is non-increasing as s increases from 0 to ∞. Furthermore, if $s < t$, then

$$\mathscr{H}^s_\delta(E) \geq \delta^{s-t} \mathscr{H}^t_\delta(E),$$

which implies that if $\mathscr{H}^t(E)$ is positive, then $\mathscr{H}^s(E)$ is infinite. Thus there is a unique value, dim E, called the *Hausdorff dimension* of E, such that

$$\mathscr{H}^s(E) = \infty \text{ if } 0 \leq s < \dim E, \mathscr{H}^s(E) = 0 \text{ if } \dim E < s < \infty. \tag{1.14}$$

If C is a cube of unit side in $\mathbb{R}^n$, then by dividing C into k^n subcubes of side $1/k$ in the obvious way, we see that if $\delta \geq k^{-1}n^{\frac{1}{2}}$ then $\mathscr{H}^n_\delta(C) \leq k^n(k^{-1}n^{\frac{1}{2}})^n \leq n^{\frac{1}{2}n}$, so that $\mathscr{H}^n(C) < \infty$. Thus if $s > n$, then $\mathscr{H}^s(C) = 0$ and $\mathscr{H}^s(\mathbb{R}^n) = 0$, since $\mathbb{R}^n$ is expressible as a countable union of such cubes. It follows that $0 \leq \dim E \leq n$ for any $E \subset \mathbb{R}^n$. It is also clear that if $E \subset E'$ then $\dim E \leq \dim E'$.

An $\mathscr{H}^s$-measurable set $E \subset \mathbb{R}^n$ for which $0 < \mathscr{H}^s(E) < \infty$ is termed an *s-set*; a 1-set is sometimes called a *linearly measurable set*. Clearly, the Hausdorff dimension of an s-set equals s, but it is important to realize that an s-set is something much more specific than a measurable set of Hausdorff dimension s. Indeed, Besicovitch (1942) shows that any set can be expressed as a disjoint union of continuum-many sets of the same dimension. Most of this book is devoted to studying the geometric properties of s-sets.

The definition of Hausdorff measure may be generalized by replacing $|U_i|^s$ in (1.12) by $h(|U_i|)$, where h is some positive function, increasing and continuous on the right. Many of our results have direct analogues for these more general measures, though sometimes at the expense of algebraic simplicity. The Hausdorff 'dimension' of a set E may then be identified more precisely as a partition of the functions which measure E as zero or infinity (see Rogers (1970)). Some progress is even possible if $|U_i|^s$ is replaced by $h(U_i)$, where h is simply a function of the set U_i (see Davies (1969) and Davies & Samuels (1974)).

We next prove that $\mathscr{H}^s$ is a regular measure, together with the useful consequence that we may approximate to s-sets from below by closed subsets. This proof is given by Besicovitch (1938) who also demonstrates (1954) the necessity of the finiteness condition in Theorem 1.6(b).

Theorem 1.6

(a) *If E is any subset of $\mathbb{R}^n$ there is a G_δ-set G containing E with $\mathscr{H}^s(G) = \mathscr{H}^s(E)$. In particular, $\mathscr{H}^s$ is a regular outer measure.*

(b) *Any $\mathscr{H}^s$-measurable set of finite $\mathscr{H}^s$-measure contains an F_σ-set of equal measure, and so contains a closed set differing from it by arbitrarily small measure.*

Proof. (a) If $\mathscr{H}^s(E) = \infty$, then $\mathbb{R}^n$ is an open set of equal measure, so suppose that $\mathscr{H}^s(E) < \infty$. For each $i = 1, 2 \ldots$ choose an *open* $2/i$-cover of E, $\{U_{ij}\}_j$, such that

$$\sum_{j=1}^{\infty} |U_{ij}|^s < \mathscr{H}^s_{1/i}(E) + 1/i.$$

Then $E \subset G$, where $G = \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{\infty} U_{ij}$ is a G_δ-set. Since $\{U_{ij}\}_j$ is a $2/i$-cover of G, $\mathscr{H}^s_{2/i}(G) \leq \mathscr{H}^s_{1/i}(E) + 1/i$, and it follows on letting $i \to \infty$ that $\mathscr{H}^s(E) = \mathscr{H}^s(G)$. Since G_δ-sets are $\mathscr{H}^s$-measurable, $\mathscr{H}^s$ is a regular outer measure.

(b) Let E be $\mathscr{H}^s$-measurable with $\mathscr{H}^s(E) < \infty$. Using (a) we may find open sets $O_1, O_2, \ldots$ containing E, with $\mathscr{H}^s(\bigcap_{i=1}^{\infty} O_i \backslash E) = \mathscr{H}^s(\bigcap_{i=1}^{\infty} O_i) - \mathscr{H}^s(E) = 0$. Any open subset of $\mathbb{R}^n$ is an F_σ-set, so suppose $O_i = \bigcup_{j=1}^{\infty} F_{ij}$ for each i, where $\{F_{ij}\}_j$ is an increasing sequence of closed sets. Then by continuity of $\mathscr{H}^s$,

$$\lim_{j\to\infty} \mathscr{H}^s(E \cap F_{ij}) = \mathscr{H}^s(E \cap O_i) = \mathscr{H}^s(E).$$

Hence, given $\varepsilon > 0$, we may find j_i such that

$$\mathscr{H}^s(E \backslash F_{ij_i}) < 2^{-i}\varepsilon \qquad (i = 1, 2, \ldots).$$

If F is the closed set $\bigcap_{i=1}^{\infty} F_{ij_i}$, then

$$\mathscr{H}^s(F) \geq \mathscr{H}^s(E \cap F) \geq \mathscr{H}^s(E) - \sum_{i=1}^{\infty} \mathscr{H}^s(E \backslash F_{ij_i}) > \mathscr{H}^s(E) - \varepsilon.$$

Since $F \subset \bigcap_{i=1}^{\infty} O_i$, then $\mathscr{H}^s(F \backslash E) \leq \mathscr{H}^s(\bigcap_{i=1}^{\infty} O_i \backslash E) = 0$. By (a) $F \backslash E$ is contained in some G_δ-set G with $\mathscr{H}^s(G) = 0$. Thus $F \backslash G$ is an F_σ-set contained in E with

$$\mathscr{H}^s(F \backslash G) \geq \mathscr{H}^s(F) - \mathscr{H}^s(G) > \mathscr{H}^s(E) - \varepsilon.$$

Taking a countable union of such F_σ-sets over $\varepsilon = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ gives an F_σ-set contained in E and of equal measure to E. □

The next lemma states that any attempt to estimate the Hausdorff measure of a set using a cover of sufficiently small sets gives an answer not much smaller than the actual Hausdorff measure.

Lemma 1.7

Let E be $\mathscr{H}^s$-measurable with $\mathscr{H}^s(E) < \infty$, and let ε be positive. Then there exists $\rho > 0$, dependent only on E and ε, such that for any collection of Borel sets $\{U_i\}_{i=1}^{\infty}$ with $0 < |U_i| \leq \rho$ we have

$$\mathscr{H}^s(E \cap \bigcup_i U_i) < \sum_i |U_i|^s + \varepsilon.$$

Proof. From the definition of $\mathscr{H}^s$ as the limit of $\mathscr{H}^s_\delta$ as $\delta \to 0$, we may choose ρ such that

$$\mathscr{H}^s(E) < \sum |W_i|^s + \tfrac{1}{2}\varepsilon \tag{1.15}$$

for any ρ-cover $\{W_i\}$ of E. Given Borel sets $\{U_i\}$ with $0 < |U_i| \leq \rho$, we may find a ρ-cover $\{V_i\}$ of $E \backslash \bigcup_i U_i$ such that

$$\mathscr{H}^s\left(E \backslash \bigcup_i U_i\right) + \tfrac{1}{2}\varepsilon > \sum |V_i|^s.$$

Since $\{U_i\} \cup \{V_i\}$ is then a ρ-cover of E,

$$\mathscr{H}^s(E) < \sum |U_i|^s + \sum |V_i|^s + \tfrac{1}{2}\varepsilon,$$

by (1.15). Hence

$$\begin{aligned}\mathcal{H}^s\Big(E\cap\bigcup_i U_i\Big)&=\mathcal{H}^s(E)-\mathcal{H}^s\Big(E\backslash\bigcup_i U_i\Big)\\&<\sum|U_i|^s+\sum|V_i|^s+\tfrac{1}{2}\varepsilon-\sum|V_i|^s+\tfrac{1}{2}\varepsilon\\&=\sum|U_i|^s+\varepsilon.\qquad\square\end{aligned}$$

Finally in this section, we prove a simple lemma on the measure of sets related by a 'uniformly Lipschitz' mapping

Lemma 1.8

Let $\psi:E\to F$ be a surjective mapping such that

$$|\psi(x)-\psi(y)|\le c|x-y|\qquad(x,y\in E)$$

for a constant c. Then $\mathcal{H}^s(F)\le c^s\mathcal{H}^s(E)$.

Proof. For each i, $|\psi(U_i\cap E)|\le c|U_i|$. Thus if $\{U_i\}$ is a δ-cover of E, then $\{\psi(U_i\cap E)\}$ is a $c\delta$-cover of F. Also $\sum_i|\psi(U_i\cap E)|^s\le c^s\sum_i|U_i|^s$ so that $\mathcal{H}^s_{c\delta}(F)\le c^s\mathcal{H}^s_\delta(E)$, and the result follows on letting $\delta\to 0$. $\square$

1.3 Covering results

The Vitali covering theorem is one of the most useful tools of geometric measure theory. Given a 'sufficiently large' collection of sets that cover some set E, the Vitali theorem selects a *disjoint* subcollection that covers almost all of E.

We include the following lemma at this point because it illustrates the basic principle embodied in the proof of Vitali's result, but in a simplified setting. A collection of sets is termed *semidisjoint* if no member of the collection is contained in any different member.

Lemma 1.9

Let $\mathscr{C}$ be a collection of balls contained in a bounded subset of $\mathbb{R}^n$. Then we may find a finite or countably infinite disjoint subcollection $\{B_i\}$ such that

$$\bigcup_{B\in\mathscr{C}}B\subset\bigcup_i B_i',\tag{1.16}$$

where B_i' is the ball concentric with B_i and of five times the radius. Further, we may take the collection $\{B_i'\}$ to be semidisjoint.

Proof. We select the $\{B_i\}$ inductively. Let $d_0=\sup\{|B|:B\in\mathscr{C}\}$ and choose B_1 from $\mathscr{C}$ with $|B_1|\ge\frac{1}{2}d_0$. If $B_1,\ldots,B_m$ have been chosen let $d_m=\sup\{|B|:B\in\mathscr{C}$, B disjoint from $\bigcup_1^m B_i\}$. If $d_m=0$ the process terminates. Otherwise choose B_{m+1} from $\mathscr{C}$ disjoint from $\bigcup_1^m B_i$ with $|B_{m+1}|\ge\frac{1}{2}d_m$. Certainly, these balls are disjoint; we claim that they also have the required covering property. If $B\in\mathscr{C}$, then either $B=B_i$ for some i, or B intersects some B_i with $2|B_i| \geq |B|$. If this was not the case B would have been selected in preference to the first ball B_m for which $2\ |B_m| < |B|$. (Note that, by summing volumes, $\sum |B_i|^2 < \infty$ so that $|B_i| \to 0$ as $i \to \infty$ if infinitely many balls are selected.) In either case, $B \subset B'_i$, giving (1.16). To get the $\{B'_i\}$ semidisjoint, simply remove B_i from the subcollection if $B'_i \subset B'_j$ for any $j \neq i$ noting that B'_i can only be contained in finitely many B'_j. □

A collection of sets $\mathscr{V}$ is called a *Vitali class* for E if for each $x \in E$ and $\delta > 0$ there exists $U \in \mathscr{V}$ with $x \in U$ and $0 < |U| \leq \delta$.

Theorem 1.10 (Vitali covering theorem)
(a) *Let E be an $\mathscr{H}^s$-measurable subset of $\mathbb{R}^n$ and let $\mathscr{V}$ be a Vitali class of closed sets for E. Then we may select a (finite or countable) disjoint sequence $\{U_i\}$ from $\mathscr{V}$ such that either $\sum |U_i|^s = \infty$ or $\mathscr{H}^s(E \backslash \bigcup_i U_i) = 0$.*
(b) *If $\mathscr{H}^s(E) < \infty$, then, given $\varepsilon > 0$, we may also require that*

$$\mathscr{H}^s(E) \leq \sum_i |U_i|^s + \varepsilon.$$

Proof. Fix $\rho > 0$; we may assume that $|U| \leq \rho$ for all $U \in \mathscr{V}$. We choose the $\{U_i\}$ inductively. Let U_1 be any member of $\mathscr{V}$. Suppose that $U_1, \ldots, U_m$ have been chosen, and let d_m be the supremum of $|U|$ taken over those U in $\mathscr{V}$ which do not intersect $U_1, \ldots, U_m$. If $d_m = 0$, then $E \subset \bigcup_1^m U_i$ so that (a) follows and the process terminates. Otherwise let U_{m+1} be a set in $\mathscr{V}$ disjoint from $\bigcup_1^m U_i$ such that $|U_{m+1}| \geq \frac{1}{2} d_m$.

Suppose that the process continues indefinitely and that $\sum |U_i|^s < \infty$. For each i let B_i be a ball with centre in U_i and with radius $3|U_i|$. We claim that for every $k > 1$

$$E \backslash \bigcup_1^k U_i \subset \bigcup_{k+1}^{\infty} B_i. \tag{1.17}$$

For if $x \in E\ \bigcup_1^k U_i$ there exists $U \in \mathscr{V}$ not intersecting $U_1, \ldots, U_k$ with $x \in U$. Since $|U_i| \to 0$, $|U| > 2|U_m|$ for some m. By virtue of the method of selection of $\{U_i\}$, U must intersect U_i for some i with $k < i < m$ for which $|U| \leq 2|U_i|$. By elementary geometry $U \subset B_i$, so (1.17) follows. Thus if $\delta > 0$,

$$\mathscr{H}^s_\delta\left(E \backslash \bigcup_1^{\infty} U_i \right) \leq \mathscr{H}^s_\delta\left(E \backslash \bigcup_1^{k} U_i \right) \leq \sum_{k+1}^{\infty} |B_i|^s = 6^s \sum_{k+1}^{\infty} |U_i|^s,$$

provided k is large enough to ensure that $|B_i| \leq \delta$ for $i > k$. Hence $\mathscr{H}^s_\delta(E \backslash \bigcup_1^\infty U_i) = 0$ for all $\delta > 0$, so $\mathscr{H}^s(E \backslash \bigcup_1^\infty U_i) = 0$, which proves (a).

To get (b), we may suppose that ρ chosen at the beginning of the proof is the number corresponding to ε and E given by Lemma 1.7. If $\sum |U_i|^s = \infty$, then (b) is obvious. Otherwise, by (a) and Lemma 1.7,

$$\mathscr{H}^s(E) = \mathscr{H}^s(E \setminus \bigcup_i U_i) + \mathscr{H}^s(E \cap \bigcup_i U_i)$$

$$= 0 + \mathscr{H}^s(E \cap \bigcup_i U_i)$$

$$< \sum |U_i|^s + \varepsilon. \qquad \square$$

Covering theorems are studied extensively in their own right, and are of particular importance in harmonic analysis, as well as in geometric measure theory. Results for very general classes of sets and measures are described in the two books by de Guzmán (1975, 1981) which also contain further references. One approach to covering principles is due to Besicovitch (1945*a*, 1946, 1947); the first of these papers includes applications to densities such as described in Section 2.2 of this book.

1.4 Lebesgue measure

We obtain n-dimensional Lebesgue measure as an extension of the usual definition of the volume in $\mathbb{R}^n$ (we take 'volume' to mean length in $\mathbb{R}^1$ and area in $\mathbb{R}^2$).

Let C be a coordinate block in $\mathbb{R}^n$ of the form

$$C = [a_1, b_1) \times [a_2, b_2) \times \cdots \times [a_n, b_n),$$

where $a_i < b_i$ for each i. Define the volume of C as

$$V(C) = (b_1 - a_1)(b_2 - a_2)\ldots(b_n - a_n)$$

in the obvious way. If $E \subset \mathbb{R}^n$ let

$$\mathscr{L}^n(E) = \inf \sum_i V(C_i), \tag{1.18}$$

where the infimum is taken over all coverings of E by a sequence $\{C_i\}$ of blocks. It is easy to see that $\mathscr{L}^n$ is an outer measure on $\mathbb{R}^n$, known as *Lebesgue n-dimensional outer measure*. Further, $\mathscr{L}^n(E)$ coincides with the volume of E if E is any block; this follows by approximating the sum in (1.18) by a finite sum and then by subdividing E by the planes containing the faces of the C_i. Since any block C_i may be decomposed into small subblocks leaving the sum in (1.18) unaltered, it is enough to take the infimum over δ-covers of E for any $\delta > 0$. Thus $\mathscr{L}^n$ is a metric outer measure on $\mathbb{R}^n$. The restriction of $\mathscr{L}^n$ to the $\mathscr{L}^n$-measurable sets or *Lebesgue-measurable* sets, which, by Theorem 1.5, include the Borel sets, is called *Lebesgue n-dimensional measure* or *volume*.

Clearly, the definitions of $\mathscr{L}^1$ and $\mathscr{H}^1$ on $\mathbb{R}^1$ coincide. As might be expected, the outer measures $\mathscr{L}^n$ and $\mathscr{H}^n$ on $\mathbb{R}^n$ are related if $n > 1$, in fact they differ only by a constant multiple. To show this we require the following well-known geometric result, the 'isodiametric inequality', which says that the set of maximal volume of a given diameter is a sphere. Proofs, using symmetrization or other methods, may be found in any text on convexity, e.g. Eggleston (1958), see also Exercise 1.6.

Theorem 1.11
The n-dimensional volume of a closed convex set of diameter d is, at most, $\pi^{\frac{1}{2}n}(\frac{1}{2}d)^n/(\frac{1}{2}n)!$, *the volume of a ball of diameter d.*

Theorem 1.12
If $E \subset \mathbb{R}^n$, *then* $\mathscr{L}^n(E) = c_n \mathscr{H}^n(E)$, *where* $c_n = \pi^{\frac{1}{2}n}/2^n(\frac{1}{2}n)!$. *In particular,* $c_1 = 1$ *and* $c_2 = \pi/4$.

Proof. Given $\varepsilon > 0$ we may cover E by a collection of closed convex sets $\{U_i\}$ such that $\sum |U_i|^n < \mathscr{H}^n(E) + \varepsilon$. By Theorem 1.11 $\mathscr{L}^n(U_i) \leq c_n |U_i|^n$, so $\mathscr{L}^n(E) \leq \sum \mathscr{L}^n(U_i) < c_n \mathscr{H}^n(E) + c_n\varepsilon$, giving $\mathscr{L}^n(E) \leq c_n \mathscr{H}^n(E)$.

Conversely, let $\{C_i\}$ be a collection of coordinate blocks covering E with

$$\sum_i V(C_i) < \mathscr{L}^n(E) + \varepsilon. \tag{1.19}$$

We may suppose these blocks to be open by expanding them slightly whilst retaining this inequality. For each i the closed balls contained in C_i of radius, at most, δ form a Vitali class for C_i. By the Vitali covering theorem, Theorem 1.10(a), there exist disjoint balls $\{B_{ij}\}_j$ in C_i of diameter, at most, δ, with $\mathscr{H}^n(C_i \backslash \bigcup_{j=1}^{\infty} B_{ij}) = 0$ and so with $\mathscr{H}^n_\delta(C_i \backslash \bigcup_{j=1}^{\infty} B_{ij}) = 0$. Since $\mathscr{L}^n$ is a Borel measure, $\sum_{j=1}^{\infty} \mathscr{L}^n(B_{ij}) = \mathscr{L}^n(\bigcup_{j=1}^{\infty} B_{ij}) \leq \mathscr{L}^n(C_i)$. Thus

$$\begin{aligned}
\mathscr{H}^n_\delta(E) \leq \sum_{i=1}^{\infty} \mathscr{H}^n_\delta(C_i) &\leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mathscr{H}^n_\delta(B_{ij}) + \sum_{i=1}^{\infty} \mathscr{H}^n_\delta\left(C_i \backslash \bigcup_{j=1}^{\infty} B_{ij}\right) \\
&\leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |B_{ij}|^n = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c_n^{-1} \mathscr{L}^n(B_{ij}) \\
&\leq c_n^{-1} \sum_{i=1}^{\infty} \mathscr{L}^n(C_i) < c_n^{-1} \mathscr{L}^n(E) + c_n^{-1}\varepsilon,
\end{aligned}$$

by (1.19). Thus $c_n \mathscr{H}^n_\delta(E) \leq \mathscr{L}^n(E) + \varepsilon$ for all ε and δ, giving $c_n \mathscr{H}^n_\delta(E) \leq \mathscr{L}^n(E)$. □

One of the classical results in the theory of Lebesgue measure is the Lebesgue density theorem. Much of our later work stems from attempts to formulate such a theorem for Hausdorff measures. The reader may care to furnish a proof as an exercise in the use of the Vitali covering theorem. Alternatively, the theorem is a simple consequence of Theorem 2.2.

Theorem 1.13 (Lebesgue density theorem)
Let E be an $\mathscr{L}^n$-measurable subset of $\mathbb{R}^n$. Then the Lebesgue density of E at x,

$$\lim_{r \to 0} \frac{\mathscr{L}^n(E \cap B_r(x))}{\mathscr{L}^n(B_r(x))}, \tag{1.20}$$

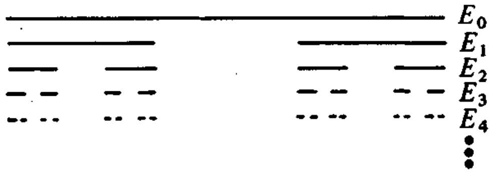
exists and equals 1 if $x \in E$ and 0 if $x \notin E$, except for a set of x of $\mathscr{L}^n$-measure 0. ($B_r(x)$ denotes the closed ball of centre x and radius r, and, as always, r tends to 0 through positive values.)

1.5 Calculation of Hausdorff dimensions and measures

It is often difficult to determine the Hausdorff dimension of a set and harder still to find or even to estimate its Hausdorff measure. In the cases that have been considered it is usually the lower estimates that are awkward to obtain. We conclude this chapter by analysing the dimension and measure of certain sets; further examples will be found throughout the book. It should become apparent that there is a vast range of s-sets in $\mathbb{R}^n$ for all values of s and n, so that the general theory to be described is widely applicable.

The most familiar set of real numbers of non-integral Hausdorff dimension is the Cantor set. Let $E_0 = [0, 1]$, $E_1 = [0, 1/3] \cup [2/3, 1]$, $E_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$, etc., where E_{j+1} is obtained by removing the (open) middle third of each interval in E_j; see Figure 1.1. Then E_j consists of 2^j intervals, each of length 3^{-j}. Cantor's set is the perfect (closed and dense in itself) set $E = \bigcap_{j=0}^{\infty} E_j$. (The collection of closed intervals that occur in this construction form a 'net', that is, any two such intervals are either disjoint or else one is contained in the other. The idea of a net of sets crops up frequently in this book.) Equivalently, E is, to within a countable set of points, the set of numbers in $[0, 1]$ whose base three expansions do not contain the digit 1. We calculate explicitly the Hausdorff dimension and measure of E; this basic type of computation extends to rather more complicated sets.

Fig. 1.1



Theorem 1.14
The Hausdorff dimension of the Cantor set E is $s = \log 2/\log 3 = 0.6309\ldots$. Moreover, $\mathscr{H}^s(E) = 1$.

Proof. Since E may be covered by the 2^j intervals of length 3^{-j} that form

E_j, we see at once that $\mathscr{H}^s_{3^{-j}}(E) \leq 2^j 3^{-sj} = 2^j 2^{-j} = 1$. Letting $j \to \infty$, $\mathscr{H}^s(E) \leq 1$.

To prove the opposite inequality we show that if $\mathscr{I}$ is any collection of intervals covering E, then

$$1 \leq \sum_{I \in \mathscr{I}} |I|^s. \tag{1.21}$$

By expanding each interval slightly and using the compactness of E, it is enough to prove (1.21) when $\mathscr{I}$ is a finite collection of closed intervals. By a further reduction we may take each $I \in \mathscr{I}$ to be the smallest interval that contains some pair of net intervals, J and J', that occur in the construction of E. (J and J' need not be intervals of the same E_j.) If J and J' are the largest such intervals, then I is made up of J, followed by an interval K in the complement of E, followed by J'. From the construction of the E_j we see that

$$|J|, |J'| \leq |K|. \tag{1.22}$$

Then

$$\begin{aligned} |I|^s &= (|J| + |K| + |J'|)^s \\ &\geq (\tfrac{3}{2}(|J| + |J'|))^s = 2(\tfrac{1}{2}|J|^s + \tfrac{1}{2}|J'|^s) \geq |J|^s + |J'|^s, \end{aligned}$$

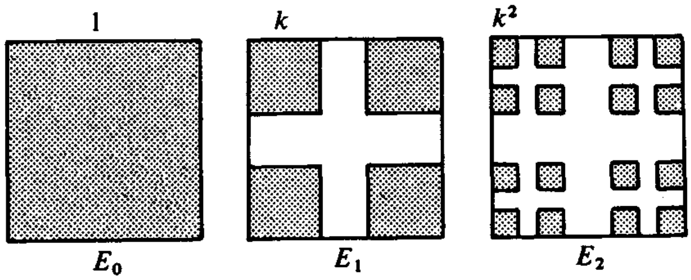
using the concavity of the function t^s and the fact that $3^s = 2$. Thus replacing I by the two subintervals J and J' does not increase the sum in (1.21). We proceed in this way until, after a finite number of steps, we reach a covering of E by equal intervals of length 3^{-j}, say. These must include all the intervals of E_j, so as (1.21) holds for this covering it holds for the original covering $\mathscr{I}$. □

There is nothing special about the factor $\frac{1}{3}$ used in the construction of the Cantor set. If we let E_0 be the unit interval and obtain E_{j+1} by removing a proportion $1 - 2k$ from the centre of each interval of E_j, then by an argument similar to the above (with (1.22) replaced by $|J|$, $|J'| \leq |K|k/(1 - 2k)$ we may show that $\mathscr{H}^s(\bigcap_1^\infty E_j) = 1$, where $s = \log 2/\log(1/k)$.

We may construct irregular subsets in higher dimensions in a similar fashion. For example, take E_0 to be the unit square in $\mathbb{R}^2$ and delete all but the four corner squares of side k to obtain E_1. Continue in this way, at the jth stage replacing each square of E_{j-1} by four corner squares of side k^j to get E_j (see Figure 1.2 for the first few stages of construction). Then the same sort of calculation gives positive upper and lower bounds for $\mathscr{H}^s(\bigcap_1^\infty E_j)$, where $s = \log 4/\log(1/k)$. More precision is required to find the exact value of the measure in such cases, and we do not discuss this further.

Instead, we describe a generalization of the Cantor construction on the

Fig. 1.2



real line. Let s be a number strictly between 0 and 1; the set constructed will have dimension s. Let E_0 denote the unit interval; we define inductively sets $E_0 \supset E_1 \supset E_2 \ldots$, each a finite union of closed intervals, by specifying $E_{j+1} \cap I$ for each interval I of E_j. If I is such an interval, let $m \geq 2$ be an integer, and let $J_1, J_2, \ldots, J_m$ be equal and equally spaced closed subintervals of I with lengths given by

$$|J_i|^s = \frac{1}{m}|I|^s, \tag{1.23}$$

and such that the left end of J_1 coincides with the left end of I and the right end of J_m with the right end of I. Thus

$$m|J_i| + (m-1)d = |I| \quad (1 \leq i \leq m), \tag{1.24}$$

where d is the spacing between two consecutive intervals J_i. Define E_{j+1} by requiring that $E_{j+1} \cap I = \bigcup_1^m J_i$. Note that the value of m may vary over different intervals I in E_j, so that the sets E_j can contain intervals of many different lengths.

The set $E = \bigcap_{j=0}^{\infty} E_j$ is a perfect nowhere dense subset of the unit interval. The following analysis is to appear in a forthcoming paper of Davies.

Theorem 1.15
If E is the set described above, then $\mathcal{H}^s(E) = 1$.

Proof. An interval used in the construction of E, that is, a component subinterval of some E_j, is called a net interval. For $F \subset E$ let

$$\mu(\mathrm{F}) = \inf \sum_{I \in \mathcal{I}} |I|^s, \tag{1.25}$$

where the infimum is taken over all possible coverings of F by collections $\mathcal{I}$ of net intervals. Then μ is an outer measure (and, indeed, a Borel measure) on the subsets of E. Note that the value of μ is unaltered if we insist that $\mathcal{I}$ be a δ-cover of F for case $\delta > 0$, since using (1.23) we may always replace a net interval I by a number of smaller net intervals without altering the sum in (1.25).

Let $\mathcal{I}$ be a cover of E by net intervals. To find a lower bound for $\sum_{I \in \mathcal{I}} |I|^s$ we may assume that the collection $\mathcal{I}$ is finite (since each net interval is open relative to the compact set E) and also that the intervals in $\mathscr{I}$ are pairwise disjoint (we may remove those intervals contained in any others by virtue of the net property). Let J be one of the shortest intervals of $\mathscr{I}$; suppose that J is a component interval of E_j, say. Then $J \subset I$ for some interval I in E_{j-1}. Since $\mathscr{I}$ is a disjoint cover of E, all the other intervals of $E_j \cap I$ must be in $\mathscr{I}$. If we replace these intervals by the single interval I, the value of $\sum |I|^s$ is unaltered by (1.23). We may proceed in this way, replacing sets of net intervals by larger intervals without altering the value of the sum, until we reach the single interval $[0,1]$. It follows that $\sum_{I \in \mathscr{I}} |I|^s = |[0,1]|^s = 1$, so, in particular,

$$\mu(E) = 1.$$

In exactly the same manner we see that if J is any net interval, then

$$\mu(J \cap E) = |J|^s. \tag{1.26}$$

Next we show that

$$\mu(J \cap E) \le |J|^s \tag{1.27}$$

for an *arbitrary* interval J. Contracting J if necessary, it is enough to prove this on the assumptions that $J \subset [0,1]$ and that the endpoints of J lie in E, and, by approximating, coincide with endpoints of net intervals contained in J. Let I be the smallest net interval containing J; say I is an interval of E_j. Suppose that J intersects the intervals $J_q, J_{q+1}, \ldots, J_r$ among the component intervals of $E_{j+1} \cap I$, where $1 \le q < r \le m$. (There must be at least two such intervals by the minimality of I.) We claim that

$$|J_q \cap J|^s + |J_{q+1}|^s + \cdots + |J_{r-1}|^s + |J_r \cap J|^s \le |J|^s. \tag{1.28}$$

If $J_q \cap J$ is not the whole of J_q or if $J_r \cap J$ is not the whole of J_r, then on increasing J slightly the left-hand side of inequality (1.28) increases faster than the right-hand side. Hence it is enough to prove (1.28) when J is the smallest interval containing J_q and J_r. Under such circumstances (1.28) becomes

$$k|J_i|^s \le |J|^s = (k|J_i| + (k-1)d)^s, \tag{1.29}$$

where $k = r - q + 1$. This is true if $k = m$ by (1.23) and (1.24), and is trivial if $k = 1$, with equality holding in both cases. Differentiating twice, we see that the right-hand expression of (1.29) is a convex function of k, so (1.29) holds for $1 \le k \le m$, and the validity of (1.28) follows.

Finally, if either $J_q \cap J$ or $J_r \cap J$ is not a single net interval, we may repeat the process, replacing $J_q \cap J$ and $J_r \cap J$ by smaller net intervals to obtain an expression similar to (1.28) but involving intervals of E_{j+2} rather than of E_{j+1}. We continue in this way to find eventually that $|J|^s$ is at least the sum of the sth powers of the lengths of disjoint net intervals covering $J \cap E$ and contained in J. Thus (1.27) follows from (1.26) for any interval J.

As (1.25) remains true if the infimum is taken over δ-covers $\mathscr{I}$ for any $\delta > 0$, $\mathscr{H}^s(E) \leq \mu(E)$. On the other hand, by (1.27),

$$\mu(E) \leq \sum \mu(J_i \cap E) \leq \sum |J_i|^s$$

for any cover $\{J_i\}$ of E, so $\mu(E) \leq \mathscr{H}^s(E)$. We conclude that $\mathscr{H}^s(E) = \mu(E) = 1$. □

Similar constructions in higher dimensions involve nested sequences of squares or cubes rather than intervals. The same method allows the Hausdorff dimension to be found and the corresponding Hausdorff measure to be estimated.

The basic method of Theorem 1.15 may also be applied to find the dimensions of other sets of related types. For example, if in the construction of E the intervals $J_1, \ldots, J_m$ in each I are just 'nearly equal' or 'nearly equally spaced', the method may be adapted to find the dimension of E. Similarly, if in obtaining E_{j+1} from E_j equations (1.23) and (1.24) only hold 'in the limit as $j \to \infty$', it may still be possible to find the dimension of E.

Another technique useful for finding the dimension of a set is to 'distort' it slightly to give a set of known dimension and to apply Lemma 1.8. The reader may wish to refer to Theorem 8.15(a) where this is illustrated.

Eggleston (1952) finds the Hausdorff dimension of very general sets formed by intersection processes; his results have been generalized by Peyrière (1977). Recently an interesting and powerful method has been described by Davies & Fast (1978). Other related constructions are given by Randolph (1941), Erdös (1946), Ravetz (1954), Besicovitch & Taylor (1954), Beardon (1965), Best (1942), Cigler & Volkmann (1963) and Wegmann (1971*b*), these last three papers continuing earlier works of the same authors. A further method of estimating Hausdorff measures is described in Section 8.3.

Exercises on Chapter 1

1.1 Show that if μ is a measure on a σ-field of sets $\mathscr{M}$ and $E_j \in \mathscr{M} (1 \leq j < \infty)$, then $\mu(\overline{\lim}_{j\to\infty} E_j) \geq \overline{\lim}_{j\to\infty} \mu(E_j)$ provided that $\mu(\bigcup_1^\infty E_j) < \infty$.

1.2 Let ν be an outer measure on a metric space (X, d) such that every Borel set is ν-measurable. Show that ν is a metric outer measure.

1.3 Show that the outer measure $\mathscr{H}^s$ on $\mathbb{R}^n$ is translation invariant, that is, $\mathscr{H}^s(x + E) = \mathscr{H}^s(E)$, where $x + E = \{x + y : y \in E\}$. Deduce that $x + E$ is $\mathscr{H}^s$-measurable if and only if E is $\mathscr{H}^s$-measurable. Similarly, show that $\mathscr{H}^s(cE) = c^s \mathscr{H}^s(E)$, where $cE = \{cy : y \in E\}$.

1.4 Prove the following version of the Vitali covering theorem for a general measure μ: let E be a μ-measurable subset of $\mathbb{R}^n$ with $\mu(E) < \infty$. If $\mathscr{V}$ is a

Vitali class of (measurable) sets for E, then there exist disjoint sets $U_1, U_2, \ldots \in \mathscr{V}$ such that $\mu(E \backslash \bigcup_i U_i) = 0$.

1.5 Use the Vitali covering theorem to prove the Lebesgue density theorem. (Consider the class of balls $\mathscr{V} = \{B_r(x): x \in E, r \leq \rho$ and $\mathscr{L}^n(B_r(x) \cap E) \leq \alpha \mathscr{L}^n(B_r(x))\}$ for each $\alpha < 1$ and $\rho > 0$.)

1.6 Prove that the area of a plane convex set U of diameter d is, at most, $\frac{1}{4}\pi d^2$. (For one method take a point on the boundary of U as origin for polar coordinates so that the area of U is $\frac{1}{2}\int r(\phi)^2 d\phi$, and observe that $r(\phi)^2 + r(\phi + \frac{1}{2}\pi)^2 \leq d^2$ for each ϕ.)

1.7 Use the Lebesgue density theorem to deduce the result of Steinhaus, that if E is a Lebesgue-measurable set of real numbers of positive measure, then the difference set $\{y - x : x, y \in E\}$ contains an interval $(-h, h)$. Show more generally that if E and E' are measurable with positive Lebesgue measure, then $\{y - x : x \in E, y \in E'\}$ contains an interval.

1.8 Let μ be a Borel measure on $\mathbb{R}^n$ and let E be a μ-measurable set with $0 < \mu(E) < \infty$. Show that

(a) if $\overline{\lim}_{r \to 0} r^{-s}\mu(B_r(x) \cap E) < c < \infty$ for $x \in E$, then $\mathscr{H}^s(E) > 0$,

(b) if $\overline{\lim}_{r \to 0} r^{-s}\mu(B_r(x) \cap E) > c > 0$ for $x \in E$, then $\mathscr{H}^s(E) < \infty$.

(For (a) use the definition of Hausdorff measure, for (b) use the version of the Vitali covering theorem in Exercise 1.4.)

1.9 Let E be the set of numbers between 0 and 1 that contain no odd digit in their decimal expansion. Obtain the best upper and lower estimates that you can for the Hausdorff dimension and measure of E. (In fact E is an s-set where $s = \log 5/\log 10$. This example is intended to illustrate some of the difficulties that can arise in finding Hausdorff measures, being a little more awkward than the Cantor set. One approach to such questions is described in Section 8.3.)