APPENDIX

A.1 METRIC SPACES

Some interesting dynamical systems do not naturally "live" in Euclidean space, and there are occasions where the study of a dynamical system benefits from considerations in an auxiliary space. Therefore we use metric spaces in some generality.

A.1.1 Definitions

Definition A.1.1 If X is a set, then $d\colon X \times X \to \mathbb{R}$ is said to be a *metric* or *distance function* if

(1) $d(x, y) = d(y, x)$ *(symmetry)*,
(2) $d(x, y) = 0 \Leftrightarrow x = y$ *(positivity)*,
(3) $d(x, y) + d(y, z) \geq d(x, z)$ *(triangle inequality)*.

If d is a metric, then (X, d) is said to be a *metric space*.

Remark A.1.2 Putting $z = x$ in (3) and using (1) and (2) shows that $d(x, y) \geq 0$.

Remark A.1.3 A subset of a metric space is itself a metric space by using the metric of the space (this is then called the *induced metric*).

The following notions generalize familiar concepts from Euclidean space.

Definition A.1.4 The set $B(x, r) := \{y \in X \mid d(x, y) < r\}$ is called the *(open) r-ball* around x. A set A c X is said to be *bounded* if it is contained in a ball.

A set $O \subset X$ is said to be *open* if for every $x \in O$ there exists $r > 0$ such that $B(x, r) \subset O$. (This immediately implies that any union of open sets is open.) The *interior* of a set S is the union $\operatorname{Int} S$ of all open sets contained in it. Equivalently, it is the set of $x \in S$ such that $B(x, r) \subset S$ for some $r > 0$. If $x \in X$ and O is an open set containing x, then O is said to be a *neighborhood* of x. A point $x \in X$ is called a *boundary point* of $S \subset X$ if for every neighborhood U of x we have $U \cap S \neq \varnothing$ and $U \setminus S \neq \varnothing$. The *boundary* of S is the set ∂A of its boundary points.

For $A \subset X$ the set $\bar{A} := \{x \in X \mid B(x, r) \cap A \neq \varnothing \text{ for all } r > 0\}$ is called the *closure* of A. A is said to be *closed* if $\bar{A} = A$. A set $A \subset X$ is said to be *dense* if $\bar{A} = X$ and ϵ-dense if $X \subset \bigcup\{B(x, \epsilon) \mid x \in A\}$. A set is said to be *nowhere dense* if its closure has empty interior (that is, contains no nonempty open set). This is true for finite sets but fails for $\mathbb{Q}$ and intervals. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to *converge* to $x \in X$ if for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for every $n \geq N$ we have $d(x_n, x) < \epsilon$.

It is easy to see that a set is closed if and only if its complement is open. (Therefore, any intersection of closed sets is closed.) Another way to define a closed set is via accumulation points:

Definition A.1.5 An *accumulation point* of a set A is a point x for which every ball $B(x, \epsilon)$ intersects $A \smallsetminus \{x\}$. The set of accumulation points of A is called the *derived set* of A and denoted by A'. A set is closed if $A' \subset A$ and the closure $\bar{A}$ of a set A is $\bar{A} = A \cup A'$. A set A is said to be *perfect* if $A' = A$, that is, there are no points missing (all accumulation points are there) nor any extraneous (isolated) ones.

Note that $x \in A'$ if and only if there is a sequence of points in A that does not include x but converges to x.

Example A.1.6 Perfect sets are closed. $\mathbb{R}$ is perfect, as are $[0, 1]$, closed balls in $\mathbb{R}^n$, S^1, and the middle-third Cantor set (see Section A.1.7). But $\mathbb{Z}$ or finite subsets of $\mathbb{R}^n$ are not (they have no accumulation points) and nor are the rationals $\mathbb{Q}$ (they have irrational accumulation points).

On the real line finite sets are nowhere dense, but this fails for $\mathbb{Q}$ and intervals. The ternary Cantor set is nowhere dense, because it is closed and has empty interior (contains no interval).

Here is an interesting, pertinent special case of Theorem A.1.38:

Proposition A.1.7 *All sets in $\mathbb{R}$ that are bounded, perfect, and nowhere dense are homeomorphic to the ternary Cantor set.*

Definition A.1.8 A metric space X is said to be *connected* if it contains no two disjoint nonempty open sets. A *totally disconnected* space is a space X where for every two points $x_1, x_2 \in X$ there exist disjoint open sets $O_1, O_2 \subset X$ containing x_1, x_2, respectively, whose union is X.

$\mathbb{R}$ or any interval of $\mathbb{R}$, as well as $\mathbb{R}^n$ and open balls in $\mathbb{R}^n$, or the circle in $\mathbb{R}^2$ are connected. Examples of totally disconnected spaces are provided by finite subsets of $\mathbb{R}$ with at least two elements as well as the rationals and, in fact, any countable subset of $\mathbb{R}$. The ternary Cantor set is an uncountable totally disconnected set.

A.1.2 Completeness

One important property sets apart the real number system from that of rational numbers. This property is called *completeness*, and it reflects the fact that the real line "has no holes," like the rationals do. There are several equivalent ways of expressing this property precisely, and different versions may be useful in different circumstances.

(1) If a nondecreasing sequence of real numbers is bounded above, then it is convergent.
(2) If a subset of $\mathbb{R}$ has an upper bound, then it has a smallest upper bound.
(3) A Cauchy sequence of real numbers converges.

A Cauchy sequence is a sequence $(a_n)_{n\in\mathbb{N}}$ such that for any $\epsilon > 0$ there exists an $n \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon$ for any $n, m \geq N$.

The first two versions of completeness refer to the ordering of the real numbers (by using the notions of upper bound and nondecreasing). The last one does not, and it is used to define completeness of metric spaces.

Definition A.1.9 A sequence $(x_i)_{i\in\mathbb{N}}$ is said to be a *Cauchy sequence* if for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $d(x_i, x_j) < \epsilon$ whenever $i, j \geq \mathbb{N}$. A metric space X is said to be *complete* if every Cauchy sequence converges.

Example A.1.10 For example, $\mathbb{R}$ is complete, whereas an open interval is not, when one uses the usual metric $d(x, y) = |x - y|$ (the endpoints are "missing"). If, however, we define a metric on the open interval $(-\pi/2, \pi/2)$ by $d_*(x, y) = |\tan x - \tan y|$, then this unusual metric space is indeed complete. The endpoints are no longer perceived as "missing" because sequences that look like they converge to an endpoint are not Cauchy sequences with respect to this metric since it stretches distances near the endpoints.

Remark A.1.11 This is an example of the *pullback* of a metric. If (Y, d) is a metric space and $h\colon X \to Y$ is an injective map, then $d_*(x, y) := d(h(x), h(y))$ defines a metric on X. Here we took $X = (-\pi/2, \pi/2)$, $Y = \mathbb{R}$, and $h = \tan$.

Lemma A.1.12 *A closed subset Y of a complete metric space X is itself a complete metric space.*

Proof A Cauchy sequence in Y is a Cauchy sequence in X and hence converges to some $x \in X$. Then $x \in Y$ because Y is closed. □

An important example is the space of continuous functions (Definition A.1.16).

Theorem A.1.13 *The space*

$$\mathcal{C}([0,1], \mathbb{R}^n) := \{f\colon [0,1] \to \mathbb{R}^n \mid f \text{ is continuous}\}$$

is a complete metric space with the metric induced by the norm $\|f\| := \max_{x\in[0,1]} \|f(x)\|$ *(see Section A.1.5).*

Proof Suppose $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}([0,1],\mathbb{R}^n)$. Then it is easy to see that $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}^n$ for all $x\in[0,1]$. Therefore, $f(x) := \lim_{n\to\infty} f_n(x)$ is well defined by completeness of $\mathbb{R}^n$. To prove $f_n\to f$ uniformly fix any $\epsilon>0$ and find $N\in\mathbb{N}$ such that $\|f_k-f_l\|<\epsilon/2$ whenever $k,l\geq N$. Now fix $k\geq N$. For any $x\in[0,1]$ there is an N_x such that $l\geq N_x\Rightarrow \|f_l(x)-f(x)\|<\epsilon/2$. Taking $l\geq N$ gives $\|f_k(x)-f(x)\|\leq\|f_k(x)-f_l(x)\|+\|f_l(x)-f(x)\|<\epsilon$. This proves the claim because k was chosen independently of x. □

Likewise one proves completeness of the space of bounded sequences.

Theorem A.1.14 *The space l^∞ of bounded sequences $(x_n)_{n\in\mathbb{N}_0}$ with the sup-norm $\|(x_n)_{n\in\mathbb{N}_0}\|_\infty := \sup_{n\in\mathbb{N}_0}|x_n|$ is complete.*

Proof The proof is the same, except that the domain is $\mathbb{N}$ rather than $[0,1]$. (Boundedness is assumed to make the norm well defined; for continuous functions on $[0,1]$ it is automatic.) □

Lemma A.1.15 (Baire Category Theorem). *In a complete metric space any intersection of countably many open dense sets is dense.*

Proof If $\{O_i\}_{\mathrm{i}\in\mathrm{N}}$ are open and dense in X and $\varnothing\neq B_0\subset X$ is open, then inductively choose a ball B_{i+1} of radius at most ϵ/i such that $\bar{B}_{i+1}\subset O_{i+1}\cap B_i$. The centers form a Cauchy sequence and hence converge by completeness. Thus $\varnothing\neq\bigcap_i\bar{B}_i\subset B_0\cap\bigcap_i O_i$. □

A.1.3 Continuity

Definition A.1.16 Let $(X,d),(Y,d')$ be metric spaces. A map $f\colon X\to Y$ is said to be an *isometry* if $d'(f(x),f(y))=d(x,y)$ for all $x,y\in X$. It is said to be *continuous* at $x\in X$ if for every $\epsilon>0$ there exists a $\delta>0$ such that $f(B(x,\delta))\subset B(f(x),\epsilon)$ or, equivalently, if $d(x,y)<\delta$ implies $d'(f(x),f(y))<\epsilon$. f is said to be continuous if f is continuous at x for every $x\in X$. An equivalent characterization is that the preimage of each open set is open. f is said to be *uniformly continuous* if the choice of δ does not depend on x, that is, for all $\epsilon>0$ there is a $\delta>0$ such that for all $x,y\in X$ with $d(x,y)<\delta$ we have $d'(f(x),f(y))<\epsilon$. f is said to be an *open map* if it maps open sets to open sets.

A continuous bijection (one-to-one and onto map) with continuous inverse is said to be a *homeomorphism*. A map $f\colon X\to Y$ is said to be *Lipschitz-continuous* (or Lipschitz) with Lipschitz constant C, or C-Lipschitz, if $d'(f(x),f(y))\leq Cd(x,y)$. A map is said to be a *contraction* (or, more specifically, a *λ-contraction*) if it is Lipschitz-continuous with Lipschitz constant $\lambda<1$.

Continuity does not imply that the image of an open set is open. For example, the map x^2 sends $(-1,1)$ or $\mathbb{R}$ to sets that are not open.

There are various ways in which two metrics can be similar, or equivalent. The easiest way to describe these is to view the process of changing metrics as taking the identity map on X as a map between two different metric spaces.

Definition A.1.17 We say that two metrics are *isometric* if the identity establishes an isometry between them. Two metrics are said to be *uniformly equivalent* (sometimes just equivalent) if the identity and its inverse are Lipschitz maps between the two metric spaces. Finally, two metrics are said to be homeomorphic (sometimes also equivalent) if the identity is a homeomorphism between them.

A.1.4 Compactness

An important class of metric spaces is that of *compact* ones:

Definition A.1.18 A metric space (X, d) is said to be *compact* if any open cover of X has a finite subcover; that is, if, whenever $\{O_i \mid i \in I\}$ is a collection of open sets of X indexed by I such that $X \subset \bigcup_{i \in I} O_i$, there is a finite subcollection $\{O_{i_1}, O_{i_2}, \dots, O_{i_n}\}$ such that $X \subset \bigcup_{l=1}^{n} O_{i_l}$.

Proposition A.1.19 *Compact sets are closed and bounded.*

Proof Suppose X is a metric space and $C \subset X$ is compact. If $x \notin C$, then the sets $O_n := \{y \in X \mid d(x, y) > 1/n\}$ form an open cover of $X \smallsetminus \{x\}$ and hence of C. There is a finite subcover $\mathcal{O}$ of $\{O_n\}_{n\in\mathbb{N}}$. Let $n_0 := \max\{n \in \mathbb{N} \mid O_n \in \mathcal{O}\}$. Then $d(x, y) > 1/n_0$ for all $y \in C$, so $x \notin \bar{C}$. This proves $\bar{C} \subset C$.

C is bounded because the open cover $\{B(x, r) \mid r > 0\}$ has a finite subcover. □

The Heine–Borel Theorem tells us that in Euclidean space a set is compact if and only if it is closed and bounded. In some important metric spaces, closed bounded sets may fail to be compact, however, and this definition of compactness describes the property that is useful in a general metric space. Indeed, this definition uses the metric only to the extent that it involves open sets.

If a metric is given, compactness is equivalent to being both complete and totally bounded:

Definition A.1.20 A metric space is said to be totally bounded if for any $r > 0$ there is a finite set C such that the r-balls with center in C cover the space.

Proposition A.1.21 *Compact sets are totally bounded.*

Proof If C is compact and $r > 0$, then $\{B(x, r) \mid x \in C\}$ has a finite subcover. □

Proposition A.1.22 *If (X, d) and (Y, d') are metric spaces, X is compact, and $f\colon X \to Y$ is a continuous map, then f is uniformly continuous and $f(X) \subset Y$ is compact; hence it is closed and bounded. If $Y = \mathbb{R}$, this shows that f attains its minimum and maximum.*

Among the most used facts about compact spaces is this last observation that a continuous real-valued function on a compact set attains its minimum and maximum.

Proof For every $\epsilon > 0$ there is a $\delta = \delta(x, \epsilon) > 0$ such that $d'(f(x), f(y)) < \epsilon/2$ whenever $d(x, y) < \delta$. The balls $B(x, \delta(x, \epsilon)/2)$ cover X, so by compactness of X there is a finite subcover by balls $B(x_i, \delta(x_i, \epsilon)/2)$. Let $\delta_0 = (1/2)\min\{\delta(x_i, \epsilon)\}$.

If $x, y \in X$ with $d(x, y) < \delta_0$, then $d(x, x_i) < \delta_0 < \delta(x_i, \epsilon)$ for some x_i and, by the triangle inequality, $d(y, x_i) \le d(x, x_i) + d(x, y) < \delta_0 + \delta_0 \le \delta(x_i, \epsilon)$. These two facts imply $d'(f(x), f(y)) \le d'(f(x), f(x_i)) + d'(f(y), f(x_i)) < \epsilon/2 + \epsilon/2 = \epsilon$.

To see that $f(X) \subset Y$ is compact, consider any open cover $f(X) \subset \bigcup_{i \in I} O_i$ of $f(X)$. Then the sets $f^{-1}(O_i) = \{x \mid f(x) \in O_i\}$ cover X, and hence there is a finite subcover $X \subset \bigcup_{l=1}^{n} f^{-1}(O_{i_l})$. But then $f(X) \subset \bigcup_{l=1}^{n} O_{i_l}$. □

Proposition A.1.23 *Suppose* $\{C_i \mid i \in I\}$ *is a collection of compact sets in a metric space* X *such that* $\bigcap_{l=1}^{n} C_i \neq \varnothing$ *for any finite subcollection* $\{C_{i_l} \mid 1 \le l \le n\}$. *Then* $\bigcap_{i \in I} C_i \neq \varnothing$.

Proof We prove the contrapositive: Suppose $\{C_i \mid i \in I\}$ is a collection of compact sets with $\bigcap_{i \in I} C_i = \varnothing$. Let $O_i = C_1 \setminus C_i$ for $i \in I$. Then $\bigcap_{i \in I} C_i = \varnothing$ implies that $\bigcup_{i \in I} O_i = C_1$, that is, the O_i form an open cover of the compact set C_1. Thus there is a finite subcover $\bigcup_{l=1}^{n} O_{i_l} = C_1$. This means that $\bigcap_{l=1}^{n} C_{i_l} = \varnothing$. □

Proposition A.1.24

(1) *A closed subset of a compact set is compact.*
(2) *The intersection of compact sets is compact.*
(3) *A continuous bijection between compact spaces is a homeomorphism.*
(4) *A sequence in a compact set has a convergent subsequence.*

Proof (1) Suppose $C \in X$ is a closed subset of a compact space and $\bigcup_{i \in I} O_i$ is an open cover of C. If $O = X \setminus C$, then $X = O \cup C \subset O \cup \bigcup_{i \in I} O_i$ is an open cover of X and hence has a finite subcover $O \cup \bigcup_{l=1}^{n} O_{i_l}$. Since $O \cap C = \varnothing$, we get a finite subcover $\bigcup_{l=1}^{n} O_{i_l}$ of C.

(2) The intersection of compact sets is an intersection of closed subsets and hence a closed subset of any of these compact sets. Therefore it is compact by (1).

(3) We need to show that the image of an open set is open. Using bijectivity, note that the complement of the image of an open set O is the image of the complement O^c of O. O^c is a closed subset of a compact space, hence it is compact, and thus its image is compact, and hence closed. Its complement, the image of O, is then open, as required.

(4) Given a sequence $(a_n)_{n \in \mathbb{N}}$, let $A_n := \{a_i \mid i \ge n\}$ for $n \in \mathbb{N}$. Then the closures $\overline{A_n}$ satisfy the hypotheses of Proposition A.1.23 and there exists an $a_0 \in \bigcap_{n \in \mathbb{N}} \overline{A_n}$. This means that for every $k \in \mathbb{N}$ there exists an $n_k > n_{k-1}$ such that $a_{n_k} \in B(a_0, 1/k)$, that is, $a_{n_k} \to a_0$. □

An interesting example of a metric space is given by the Hausdorff metric:

Definition A.1.25 If (X, d) is a compact metric space and $K(X)$ denotes the collection of closed subsets of X, then the *Hausdorff metric* d_H on $K(X)$ is defined by

$$d_H(A, B) := \sup_{a \in A} d(a, B) + \sup_{b \in B} d(b, A),$$

where $d(x, Y) := \inf_{y \in Y} d(x, y)$ for $Y \subset X$.

Notice that d_H is symmetric by construction and is zero if and only if the two sets coincide (here we use that these sets are closed, and hence compact, so the "sup" are actually "max"). Checking the triangle inequality requires a little extra work. To show that $d_H(A, B) \le d_H(A, C) + d_H(C, B)$, note that $d(a, b) \le d(a, c) + d(c, b)$ for $a \in A$, $b \in B$, $c \in C$, so taking the infimum over b we get $d(a, B) \le d(a, c) + d(c, B)$ for $a \in A$, $c \in C$. Therefore, $d(a, B) \le d(a, C) + \sup_{c\in C} d(c, B)$ and $\sup_{a\in A} d(a, B) \le \sup_{a\in A} d(a, C) + \sup_{c\in C} d(c, B)$. Likewise, one gets $\sup_{b\in B} d(b, A) \le \sup_{b\in B} d(b, C) + \sup_{c\in C} d(c, A)$. Adding the last two inequalities gives the triangle inequality.

Lemma A.1.26 *The Hausdorff metric on the closed subsets of a compact metric space defines a compact topology.*

Proof We need to verify total boundedness and completeness. Pick a finite $\epsilon/2$-net N. Any closed set $A \subset X$ is covered by a union of ϵ-balls centered at points of N, and the closure of the union of these has Hausdorff distance at most ϵ from A. Since there are only finitely many such sets, we have shown that this metric is totally bounded. To show that it is complete, consider a Cauchy sequence (with respect to the Hausdorff metric) of closed sets $A_n \subset X$. If we let $A := \bigcap_{k\in\mathbb{N}} \overline{\bigcup_{n\ge k} A_n}$, then one can easily check that $d(A_n, A) \to 0$. □

Any homeomorphism of a compact metric space X induces a natural homeomorphism of the collection of closed subsets of X with the Hausdorff metric, so we have the following:

Lemma A.1.27 *The set of closed invariant sets of a homeomorphism f of a compact metric space is a closed set with respect to the Hausdorff metric.*

Proof This is just the set of fixed points of the induced homeomorphism; hence it is closed. □

Definition A.1.28 A metric space (X, d) is said to be *locally compact* if for every x and every neighborhood O of x there is a compact set K in O that contains x. It is said to be *separable* if it contains a countable dense subset (such as the rationals in $\mathbb{R}$).

A.1.5 Norms Define Metrics in $\mathbb{R}^n$

There is a particular class of metrics in the Euclidean space $\mathbb{R}^n$ that are invariant under translations.

Definition A.1.29 A function N on a linear space is said to be a *norm* if

(1) $N(\lambda x) = |\lambda| N(x)$ for $\lambda \in \mathbb{R}$ *(homogeneity)*,
(2) $N(x) \ge 0$ and $N(x) = 0 \Leftrightarrow x = 0$ *(positivity)*,
(3) $N(x + y) \le N(x) + N(y)$ *(convexity)*.

A linear space with a norm is said to be a normed linear space.

Any norm determines a metric by setting the distance function $d(x, y) = N(x - y)$. For the metric thus defined, positivity follows from the positivity of the norm, symmetry follows from homogeneity for $\lambda = -1$, and triangle inequality follows from convexity. For such a metric the translations $T_v \colon x \to x + v$ are isometries by definition. Furthermore, the central symmetry $x \to -x$ is an isometry, and any homothety $x \to \lambda x$ multiplies all distances by $|\lambda|$ (we call the last property homogeneity of the metric).

Example A.1.30 The maximum distance on $\mathbb{R}^n$ is given by

$$d(x, y) = \max_{1 \le i \le n} |x_i - y_i|. \tag{A.1.1}$$

Of course, the standard Euclidean metric is of that kind (it is also invariant under rotations, which we do not require), as is the maximum metric (A.1.1).

Example A.1.31 The linear space $C([0,1])$ of continuous functions on $[0,1]$ is a linear space and carries the norm $\|f\| := \max\{|f(x)| \mid x \in [0, 1]\}$.

The following proposition is the main reason why norms are useful devices in dynamics.

Proposition A.1.32 *All metrics in $\mathbb{R}^n$ determined by norms are uniformly equivalent.*

Proof First, since the property of uniform equivalence is transitive, it is sufficient to show that any metric determined by a norm is uniformly equivalent to the standard Euclidean metric.

Second, since translations are isometries, it is suffient to consider distances from the origin, that is, we can work with the norms directly.

Third, by homogeneity it is sufficient to consider norms of vectors whose Euclidean norm is equal to one, that is, the points on the unit sphere.

But then the other norm is a convex, and hence continuous, function with respect to Euclidean distance, so by compactness of the sphere it is bounded from above. It also achieves its minimum on the unit sphere. The minimum cannot be zero because this would imply the existence of a nonzero vector with zero norm. Thus the ratio of the norms is bounded between two positive constants. □

A.1.6 Product Spaces

The construction of the torus as a product of circles illustrates the usefulness of considering products of metric spaces in general. To define the product of two metric spaces (X, d_X) and (Y, d_Y) we need to define a metric on the cartesian product $X \times Y$, such as

$$d_{X\times Y}((x_1, y_1), (x_2, y_2)) := \sqrt{(d_X(x_1, x_2))^2 + (d_Y(y_1, y_2))^2}.$$

That this defines a metric is checked in the same way as checking that the Euclidean norm on $\mathbb{R}^2$ defines a metric.

There are other choices of equivalent metrics on the product. Two evident ones are

$$d'_{X\times Y}((x_1, y_1), (x_2, y_2)) := d_X(x_1, x_2) + d_Y(y_1, y_2)$$

and

$$d''_{X\times Y}((x_1, y_1), (x_2, y_2)) := \max(d_X(x_1, x_2), d_Y(y_1, y_2)).$$

Showing that these metrics are pairwise uniformly equivalent is done in the same way as showing that the Euclidean norm, the norm $\|(x, y)\|_1 := |x| + |y|$, and the maximum norm $\|(x, y)\|_\infty := \max(|x|, |y|)$ define pairwise equivalent metrics (Proposition A.1.32). Indeed, this follows from it.

For products of finitely many spaces (X_i, d_{X_i}) $(i = 1, \dots, n)$ one can define several uniformly equivalent metrics on the product as follows: Fix a norm $\|\cdot\|$ on $\mathbb{R}^n$, and for any two points $(x_1, x_2, \dots, x_n)$, and $(x'_1, x'_2, \dots, x'_n)$ define their distance to be the norm of the vector in $\mathbb{R}^n$ whose entries are $d_{X_i}(x_i, x'_i)$. That the resulting metrics are uniformly equivalent follows from the uniform equivalence of any two norms on $\mathbb{R}^n$ (Proposition A.1.32).

We also encounter products of infinitely many metric spaces (or, usually, a product of infinitely many copies of the same metric space). In an infinite cartesian product of a set X every element is specified by its components; that is, if the copies of the set X are indexed by a label i that ranges over an index set I, then an individual element of the product set is specified by assigning to each value of i an element of X, the ith coordinate. This leads to the formal definition of the infinite product $\prod_{i\in I} X =: X^I$ as the set of all functions from I to X.

Unlike in the case of finite products, we have to choose our product metric carefully. Not only do we have to keep in mind questions of convergence, but different choices may give metrics that are not equivalent, even up to homeomorphism. To define a product metric assume that I is countable. In case $I = \mathbb{N}$ and if the metric on X is bounded, that is, $d(x, y) \le 1$, say, for all $x, y \in X$, we can define several homeomorphic metrics by setting

$$d_\lambda(x, y) := \sum_{i=1}^{\infty} \frac{d(x_i, y_i)}{\lambda^{|i|}}. \tag{A.1.2}$$

This converges for any $\lambda > 1$ by comparison with the corresponding geometric series.

If $I = \mathbb{Z}$, we make the same definition with summation over $\mathbb{Z}$ [this is the reason for writing $|i|$ in (A.1.2)].

Theorem A.1.33 (Tychonoff). *The product of compact spaces is compact.*

As a particular case we can perform this construction with $X = [0, 1]$, the unit interval. The product thus obtained is called the *Hilbert cube*. This is a new way to think of the collection of all sequences whose entries are in the unit interval.

A.1.7 Sequence Spaces

Generalizing from the ternary Cantor set introduced in Section 2.7.1 we now define a more general class of metric spaces of which there are many important examples.

Definition A.1.34 A *Cantor set* is a metric space homeomorphic to the middle-third Cantor set.

A natural and important example is the space Ω_2^R of sequences $\omega = (\omega_i)_{i=0}^{\infty}$ whose entries are either 0 or 1. This set is the product $\{0, 1\}^{\mathbb{N}_0}$ of countably many copies of the set $\{0, 1\}$ of two elements, so it is natural to endow it with a product metric. Up to multiplication by a constant there is only one metric on $\{0, 1\}$, which we define by setting $d(0, 1) = 1$. Referring to (A.1.2), we can endow Ω_2^R with the product metric

$$d(\omega, \omega') := \sum_{i=0}^{\infty} \frac{d(\omega_i, \omega_i')}{3^{i+1}}.$$

Proposition A.1.35 *The space $\Omega_2^R = \{0, 1\}^{\mathbb{N}_0}$ equipped with the product metric $d(\omega, \omega') := \sum_{i=0}^{\infty} d(\omega_i, \omega_i')3^{-(i+1)}$ is a Cantor set.*

To prove this we need a homeomorphism between the ternary Cantor set C and Ω_2^R:

Lemma A.1.36 *The one-to-one correspondence between the ternary Cantor set C and Ω_2^R defined by mapping each point $x = 0.\alpha_1\alpha_2\alpha_3 \cdots = \sum_{i=1}^{\infty} (\alpha_i/3^i) \in C$ $(\alpha_i \neq 1)$ to the sequence $f(x) := \{\alpha_i/2\}_{i=0}^{\infty}$ is a homeomorphism.*

Proof If $x = 0.\alpha_0\alpha_1\alpha_2 \cdots = \sum_{i=0}^{\infty} (\alpha_i/3^{i+1})$ $(\alpha_i \neq 1)$ and $y = 0.\beta_0\beta_1\beta_2 \cdots = \sum_{i=0}^{\infty} (\beta_i/3^{i+1})$ $(\beta_i \neq 1)$ in C, then

$$d(x, y) = |x - y| = \left| \sum_{i=0}^{\infty} \frac{\alpha_i}{3^{i+1}} - \sum_{i=0}^{\infty} \frac{\beta_i}{3^{i+1}} \right|$$
$$= \left| \sum_{i=0}^{\infty} \frac{\alpha_i - \beta_i}{3^{i+1}} \right| \leq \sum_{i=0}^{\infty} \frac{|\alpha_i - \beta_i|}{3^{i+1}} = 2d(f(x), f(y)).$$

Now let $\alpha = f(x)$, $\beta = f(y)$. Then $d(f^{-1}(\alpha), f^{-1}(\beta)) = d(x, y) \leq 2d(\alpha, \beta)$, so f^{-1} is Lipschitz-continuous with Lipschitz constant 2.

If $\omega, \omega' \in \Omega_2^R$ are two sequences with $d(\omega, \omega') \geq 3^{-n}$, then $\omega_i \neq \omega_i'$ for some $i \leq n$, because otherwise

$$d(\omega, \omega') \leq \sum_{i=n+1}^{\infty} 3^{-i-1} = \frac{3^{-n-2}}{1 - \frac{1}{3}} = 3^{-n-1}/2 < 3^{-n}.$$

Consequently, $f^{-1}(\omega)$ and $f^{-1}(\omega')$ differ in the ith digit for some $i \leq n$. This implies $d(f^{-1}(\omega), f^{-1}(\omega')) \geq 3^{-(n+1)}$ because the two points are in different pieces of C_{n+1}. Taking $x = f^{-1}(\omega)$, $x' = f^{-1}(\omega')$, we get $d(x, x') < 3^{-(n+1)} \Rightarrow d(f(x), f(y)) < 3^{-n}$. This shows that f is Lipschitz-continuous as well. □

We have shown in particular that Ω_2^R is compact and totally disconnected. Let us note in addition that every sequence in Ω_2^R can be approximated arbitrarily well by different sequences in Ω_2^R by changing only very remote entries. Thus every point of Ω_2^R is an accumulation point and Ω_2^R is a perfect set.

Proposition A.1.37 *Cantor sets are compact, totally disconnected, and perfect.*

It is not hard to see that the space $\Omega_2 = \{0, 1\}^{\mathbb{Z}}$ with a product metric is in turn homeomorphic to Ω_2^R, and therefore it is also a Cantor set. To that end let

$$\alpha \colon \mathbb{Z} \to \mathbb{N}_0, \qquad n \mapsto \begin{cases} 2n & \text{if } n \geq 0 \\ 1 - 2n & \text{if } n < 0 \end{cases}$$

and $f \colon \Omega_2^R \to \Omega_2$, $\omega \mapsto \omega \circ a = (\ldots \omega_3\omega_1\omega_0\omega_2\omega_4 \ldots)$. Endowing Ω_2 and Ω_2^R with any two of the product metrics (A.1.2) makes f a homeomorphism because two sequences α, α' are close if and only if they agree on a large stretch of initial entries. Then the resulting sequences $\omega = f(\alpha)$ and $\omega' = f(\alpha')$ agree on a long stretch of entries around the 0th entry and hence are also close. Thus f is a continuous bijection between compact spaces and therefore a homeomorphism by Proposition A.1.24. (It is as easy to see directly that f^{-1} is continuous.)

A.1.8 General Properties of Cantor Sets

Theorem A.1.38 *Every perfect totally disconnected compact metric space is a Cantor set.*

We have seen that sequence spaces are perfect and compact; it is easy to see in general that they are totally disconnected: If $\alpha \neq \beta$ are sequences, then $\alpha_i \neq \beta_i$ for some index i. The set of sequences ω with $\omega_i = \alpha_i$ is open, and likewise the set of sequences with $\omega_i = \beta_i$. But these sets are disjoint and their union is the entire space.

Corollary A.1.39 *Every nonempty, perfect, bounded, nowhere dense set on the line is a Cantor set.*

Proof A perfect bounded set on the line is compact by the Heine–Borel Theorem (a closed bounded subset of $\mathbb{R}^n$ is compact). Being perfect, it also contains more than one point. If it is not totally disconnected, then it has a connected component with more than one point and hence contains a nontrivial interval, contrary to being nowhere dense. □

A.1.9 Dyadic Integers

Define the following metric d_2 on the group $\mathbb{Z}$ of all integers: $d(n, n) = 0$ and $d_2(m, n) = \|m - n\|_2$ for $n \neq M$, where

$$\|n\|_2 = 2^{-k} \qquad \text{if } n = 2^k l \text{ with an odd number } l.$$

The completion of $\mathbb{Z}$ with respect to that metric is called the group of *dyadic integers* and is usually denoted by $\mathbb{Z}_2$. It is a compact topological group.