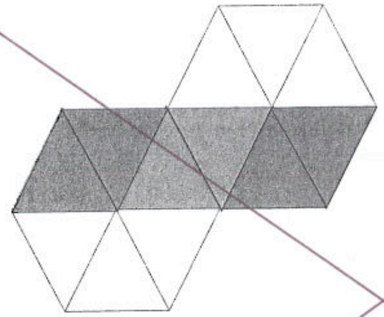

Figure 4.2.15. Parallelogram.

Energy conservation reduces this to three dimensions, and conservation of the size of angular momentum to two. These two dimensions are parametrized by a time parameter along an ellipse and a perihelion angle. This is therefore a system similar to the mathematical pendulum (Section 6.2.2) where one gets flows on circles, that is, one-dimensional invariant tori. However, the Kepler problem for several planets *without mutual interaction* gives higher-dimensional invariant tori with linear flows on them. This is the central feature of *complete integrability* in Hamiltonian dynamical systems.

■ EXERCISES

■ Exercise 4.2.1 Give a detailed proof of (4.2.1).

■ Exercise 4.2.2 Verify directly that for any fixed number m the sum of $\lg(p+1) - \lg p$ over all p with exactly m digits is 1, as it should be according to Proposition 4.2.7.

■ Exercise 4.2.3 Verify the calculation needed to deduce Proposition 4.2.7 from Proposition 4.1.7 or Theorem 4.2.3.

■ Exercise 4.2.4 Referring to Proposition 4.2.7, determine $\lim_{n\to\infty} F^2_{10}(n)/n$ and find the asymptotic frequencies of 0 and 9, respectively, as the *second* digit of powers of 2.

■ Exercise 4.2.5 Referring to the proof of Proposition 4.2.8, assume $\gamma \neq 0$ and replace the section C_1 by the section $C_2 := \{x_2 = 0\}$. Prove that the resulting return map is a rotation and determine the rotation angle in terms of γ.

■ Exercise 4.2.6 Verify by direct calculation of the time derivatives that the functions $x_1^2 + x_2^2$ and $x_3^2 + x_4^2$ are invariant under (4.2.5).

■ Exercise 4.2.7 Formulate the natural uniform distribution property referred to in Proposition 4.2.9 and proved in Section 4.2.5.4.

■ Exercise 4.2.8 Prove that any closed proper subgroup Γ of $\mathbb{R}$ is cyclic, that is, $\Gamma = \{na\}_{n\in\mathbb{Z}}$ for some $a \in \mathbb{R}$.

■ Exercise 4.2.9 Given an initial direction, how many slopes are there for the billiard flow in a square and in each of the two triangles, and what are they?

■ Exercise 4.2.10 Suppose a horizontal light beam enters a circular room with mirrored walls. Describe the possibilities for which areas of the room will be best lit.

■ Exercise 4.2.11 Prove that a complete unfolding of a regular pentagon covers every point of the plane infinitely many times.

■ Exercise 4.2.12 Obtain the continuation of orbits in the billiard description of the 2-particle system by interpreting double collisions as limits of a series of simple collisions.

4.3 INVERTIBLE CIRCLE MAPS

The success in analyzing circle rotations is due in large part to the fact that these come from linear dynamical systems, namely, from rotations of the plane (Section 3.1). This causes the great homogeneity of the orbit structure that gives uniform density of orbits and uniform distribution. However, another ingredient, perhaps less apparent, is the simple structure of the circle itself. Analogously to the study of interval homeomorphisms (Section 2.3.1) this makes it possible to give a fairly satisfactory analysis of the orbit structure of any invertible map of the circle. One-dimensionality of the circle provides two (related) features that make a fairly detailed analysis possible: the (cyclic) ordering of its points and the Intermediate-Value Theorem. These have the effect of tying together different orbits tightly enough to make the possible orbit structures relatively easy to describe. The importance of the order structure will become particularly apparent in Proposition 4.3.11 and Proposition 4.3.15.

For noninvertible maps of an interval or of the circle the order of points may not be preserved and hence use of this first property fails, while the Intermediate-Value Theorem can still be used so long as we have continuity. Accordingly, the structural features are much more complicated while still amenable to rather extensive analysis. Chapter 11 outlines this for some interval maps.

One principle that will manifest itself in various guises throughout this section is that while, unlike the situation with rotations, the orbit structure of invertible circle maps is not always entirely homogeneous, the asymptotic behavior is in various different ways about as homogeneous, or at least coherent, as the entire orbit structure of a rotation and, in fact, ultimately turns out to look much like a rotation.

In this section a fundamental dichotomy is central: A circle homeomorphism (Definition A.1.16) may or may not have periodic points. Every orbit has the same type of asymptotic behavior, and it corresponds in a precise sense to the behavior of an orbit of a rational or an irrational rotation, respectively. The tool that leads to this conclusion is a parameter that reflects asymptotic rotation rates and is rational or not according to whether there are periodic points.

4.3.1 Lift and Degree

Recall the relation between the circle $S^1 = \mathbb{R}/\mathbb{Z}$ and the line $\mathbb{R}$ (see Section 2.6.2). There is a projection $\pi : \mathbb{R} \to S^1$, $x \mapsto [x]$, where $[x]$ is the equivalence class of x in

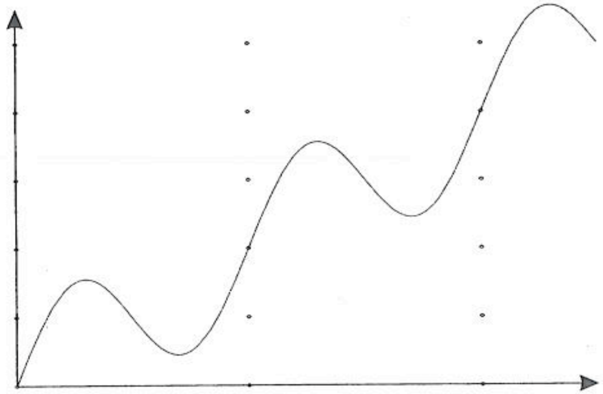

Figure 4.3.1. A lift and degree.

$\mathbb{R}/\mathbb{Z}$ as in Section 2.6.2. Here $[\cdot]$ denotes an equivalence class, whereas the integer part of a number is written $\lfloor\cdot\rfloor$. We use $\{\cdot\}$ for the fractional part.

Proposition 4.3.1 *If $f\colon S^1 \to S^1$ is continuous, then there exists a continuous $F\colon \mathbb{R} \to \mathbb{R}$, called a* lift *of f to $\mathbb{R}$, such that*

$$f \circ \pi = \pi \circ F, \tag{4.3.1}$$

that is, $f([z]) = [F(z)]$. Such a lift is unique up to an additive integer constant, and $\deg(f) := F(x+1) - F(x)$ is an integer independent of $x \in \mathbb{R}$ and the lift F. It is called the degree *of f. If f is a homeomorphism, then $|\deg(f)| = 1$.*

Proof *Existence:* Pick a point $p \in S^1$. Then $p = [x_0]$ for some $x_0 \in \mathbb{R}$ and $f(p) = [y_0]$ for some $y_0 \in \mathbb{R}$. From these choices of x_0 and y_0 define $F\colon \mathbb{R} \to \mathbb{R}$ by requiring that $F(x_0) = y_0$, F is continuous, and $f([z]) = [F(z)]$ for all $z \in \mathbb{R}$. One can construct such an F by varying the initial point p continuously, which causes $f(p)$ to vary continuously. Then there is no ambiguity of how to vary x and y continuously, and thus $F(x) = y$ defines a continuous map.[3]

Uniqueness: Suppose $\tilde{F}$ is another lift. Then $[\tilde{F}(x)] = f([x]) = [F(x)]$ for all x, meaning $\tilde{F} - F$ is always an integer. Because it is continuous it must be constant.

Degree: $F(x+1) - F(x)$ is an integer (now evidently independent of the choice of lift) because $[F(x+1)] = f([x+1]) = f([x]) = [F(x)]$. By continuity, $F(x+1) - F(x) =: \deg(f)$ must be a constant.

Invertibility: If $\deg(f) = 0$, then $F(x+1) = F(x)$ and thus F is not monotone. Then f is noninvertible because it cannot be monotone. If $|\deg(f)| > 1$, then $|F(x+1) - F(x)| > 1$ and, by the Intermediate-Value Theorem, there exists a

[3] To elaborate, take $\delta > 0$ such that $d([x], [x']) \le \delta$ implies $d(f([x]), f([x'])) < 1/2$. Then define F on $[x_0 - \delta, x_0 + \delta]$ as follows: If $|x - x_0| \le \delta$, then $d(f([x]), q) < 1/2$ and there is a unique $y \in (y_0 - 1/2, y_0 + 1/2)$ such that $[y] = f([x])$. Define $F(x) = y$. Analogous steps extend the domain by another δ at a time, until F is defined on an interval of unit length. Then $f([z]) = [F(z)]$ defines F on $\mathbb{R}$.

$y \in (x, x+1)$ with $|F(y) - F(x)| = 1$. Then $f([y]) = f([x])$ and $[y] \ne [x]$, so f is noninvertible. □

Definition 4.3.2 Suppose f is invertible. If $\deg(f) = 1$, then we say that f is orientation-preserving; if $\deg(f) = -1$, then f is said to reverse orientation.

Remark 4.3.3 The function $F(x) - x\deg(f)$ is periodic because

$$F(x+1) - (x+1)\deg(f) = F(x) + \deg(f) - (x+1)\deg(f) = F(x) + x\deg(f)$$

for all x. In particular, if f is an orientation-preserving homeomorphism, then $F(x) - x$ is periodic and so $F - \mathrm{Id}$ is bounded. A slightly stronger observation will come in handy soon.

Lemma 4.3.4 *If f is an orientation-preserving circle homeomorphism and F a lift, then $F(y) - y \le F(x) - x + 1$ for all $x, y \in \mathbb{R}$.*

Proof Let $k = \lfloor y - x \rfloor$. Then

$$\begin{aligned} F(y) - y &= F(y) + F(x+k) - F(x+k) + (x+k) - (x+k) - y \\ &= (F(x+k) - (x+k)) + (F(y) - F(x+k)) - (y - (x+k)). \end{aligned} \tag{4.3.2}$$

Now $F(x+k) - (x+k) = F(x) - x$ and $0 \le y - (x+k) < 1$ by choice of k, so $F(y) - F(x+k) \le 1$. Thus the right-hand side above is at most $F(x) - x + 1 - 0$. □

4.3.2 Rotation Number

The presence or absence of periodic points is determined by a single parameter called the rotation number. It also tells us which rotation to compare a circle homeomorphism to.

Proposition 4.3.5 *Let $f\colon S^1 \to S^1$ be an orientation-preserving homeomorphism and $F\colon \mathbb{R} \to \mathbb{R}$ a lift of f. Then*

$$\rho(F) := \lim_{|n|\to\infty} \frac{1}{n}(F^n(x) - x) \tag{4.3.3}$$

exists for all $x \in \mathbb{R}$. $\rho(F)$ is independent of x and well defined up to an integer; that is, if $\tilde{F}$ is another lift of f, then $\rho(F) - \rho(\tilde{F}) = F - \tilde{F} \in \mathbb{Z}$. $\rho(F)$ is rational if and only if f has a periodic point.

The fact that the rotation number is independent of the point is the first manifestation of the coherent asymptotic behavior of orbits that we will come to expect. This proposition justifies the following terminology:

Definition 4.3.6 $\rho(f) := [\rho(F)]$ is called the *rotation number* of f.