M345PA46: 2nd progress test 18/11/2013

1. Let Ω_3 denote the set of bi-infinite sequences $\{\omega_i\}_{i\in\mathbb{Z}}$ whose entries ω_i are taken from a set of three symbols, for instance $\{0,1,2\}$.

   (i) Consider the cylinder

$$C_{\alpha_{1-n},\ldots,\alpha_{n-1}} := \{\omega \in \Omega_3 \mid \omega_i = \alpha_i,\ |i| < n\}.$$

   Let

$$d(\omega,\omega') := \sum_{m\in\mathbb{Z}} \frac{\delta(\omega_m,\omega'_m)}{4^m},$$

   where $\delta(a,b) = 0$ if $a = b$ and $\delta(a,b) = 1$ if $a \neq b$.

   (a) Show that d is a metric on Ω_3.
   Answer: All properties follow by comparing components in the sum: (i) $d(x,y) = d(y,x)$ follows from the fact that $\delta(a,b) = \delta(b,a)$. (ii) $d(x,y) = 0 \Leftrightarrow x = y$ follows again from the definition of δ: as soon as two sequences have one different symbol, the distance is positive, and the distance between two equal sequences is equal to zero. (iii) $d(x,y) + d(y,z) \geq d(x,z)$ follows from the fact that $\delta(a,b) + \delta(b,c) \geq \delta(a,c)$. This is obviously satisfied if $a = c$. If $a \neq c$ then $b \neq c$ or $a \neq b$ so that the inequality is also satsified.

   (b) Consider Ω_3 as a metric space with metric d. Show that the cylinder $C_{\alpha_{-1}\alpha_0\alpha_1}$ is a ball in Ω_3 around any point α of the form $\alpha = \ldots\alpha_{-1}\alpha_0\alpha_1\ldots$ and determine its radius.
   Answer: Let $\alpha \in C_{\alpha_{1-n},\ldots,\alpha_{n-1}}$. If $\omega \in C_{\alpha_{1-n},\ldots,\alpha_{n-1}}$ then

$$d(\omega,\omega') := \sum_{|m|\geq\ n} \frac{\delta(\omega_m,\omega'_m)}{4^m} \leq \sum_{|m|\geq\ n} \frac{1}{4^m} = \frac{1}{4^{n-1}}\frac{2}{3} < \frac{1}{4^{n-1}}.$$

   On the other hand if $\omega \notin C_{\alpha_{1-n},\ldots,\alpha_{n-1}}$,

$$d(\omega,\omega') \geq \frac{1}{4^{n-1}}.$$

   Thus $C_{\alpha_{1-n},\ldots,\alpha_{n-1}}$ is exactly equal to the ball around α of radius 4^{1-n}. In the case that $n = 2$, as asked, this yields a ball or radius $1/4$.

   (ii) Give for each of the following, an example of a topological Markov chain on Ω_3 (endowed with metric d), by means of its transition matrix or Markov graph, that has this property:

   1. a topological Markov chain that is not transitive
   2. a topological Markov chain that is topologically mixing

Answer: For instance, $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ is a transition matrix for a transitive and topologically mixing Markov chain (full shift) and and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is a transition matrix for a Markov chain (identity) that is neither transitive nor topologically mixing .

2. Consider the piecewise linear map $F : [0,1] \to [0,1]$ given by

$$F(x) = \begin{cases} 3x & \text{if } x \in \Delta_0 \\ 2 - 3x & \text{if } x \in \Delta_1 \\ x - \frac{2}{3} & \text{if } x \in \Delta_2 \end{cases}$$

where $\Delta_0 = [0, \frac{1}{3}]$, $\Delta_1 = [\frac{1}{3}, \frac{2}{3}]$, and $\Delta_2 = [\frac{2}{3}, 1]$. [It may be useful to draw the graph of F]

We consider a coding of orbits by means of (half-)inifinite sequences of the form $\omega = \omega_0\omega_1 \ldots \in \Omega_3^R$, where Ω_3^R denotes the metric space of (half-)infinite sequences with symbols $\omega_i \in \{0, 1, 2\}$ and distance

$$d(\omega, \omega') := \sum_{m \in \mathbb{N}} \frac{\delta(\omega_m, \omega'_m)}{4^m},$$

where $\delta(a, b) = 0$ if $a = b$ and $\delta(a, b) = 1$ if $a \neq b$.

We propose the coding $h : \Omega_3^R \to [0, 1]$ to be such that if $x = h(\omega)$, with $\omega = \omega_0\omega_1\omega_2 \ldots$ and $y := f^n(x)$ then $y \in \Delta_{\omega_n}$.

(i) Let $\Delta_{\omega_0 \ldots \omega_n} := \overline{\bigcap_{i=0}^n f^{-i}\left(\text{Int}(\Delta_{\omega_i})\right)}$. Determine Δ_{ab} for all $a, b \in \{0, 1, 2\}$.
Answer: $\Delta_{00} = [0, \frac{1}{9}]$, $\Delta_{01} = [\frac{1}{9}, \frac{2}{9}]$, $\Delta_{02} = [\frac{2}{9}, \frac{1}{3}]$, $\Delta_{12} = [\frac{1}{3}, \frac{4}{9}]$, $\Delta_{11} = [\frac{4}{9}, \frac{5}{9}]$, $\Delta_{10} = [\frac{5}{9}, \frac{2}{3}]$, $\Delta_{20} = [\frac{2}{3}, 1]$, $\Delta_{21} = \emptyset$, $\Delta_{22} = \emptyset$.

(ii) Determine the sequences in Ω_3^R that represent the points 0, $\frac{1}{3}$, $\frac{2}{3}$ and 1, and point out whether or not these sequences are unique.
Answer: $h^{-1}(0) = \overline{0}(= 00000\ldots)$, $h^{-1}(\frac{1}{3}) = \{0\overline{2}, 1\overline{20}\}$, $h^{-1}(\frac{2}{3}) = \{1\overline{0}, \overline{20}\}$ and $h^{-1}(1) = \overline{20}$.

(iii) Show that the map F is topologically semi-conjugate to a three-state topological Markov chain determined by the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.
Answer: A represents the admissible transitions between the labelling regions. Main item to prove is that h is surjective function, where $h(\omega) = \Delta_\omega$. We note that F is expanding on $[0, \frac{2}{3}]$ and non-expanding (translation) on $[\frac{2}{3}, 1]$. We note that F^2 is expanding so that $\lim_{n\to\infty} \Delta_{\omega_0 \ldots \omega_n}$ is indeed a single point. Taking into account the fact that F does not expand on Δ_3 (and that $F(\Delta_3) = \Delta_1$) we obtain uniform expansion with rate 3 for F^2, so that $|\Delta_{\omega_0 \ldots \omega_n}| \leq \frac{1}{3}\left(\frac{1}{\sqrt{3}}\right)^{n-1}$.