M345PA46: Chaos and Fractals.
Class test 31-10-2014, 10-11am

1. Consider the following maps of the real line:

$$f(x) = \begin{cases} 3x & \text{if } x < \frac{1}{3}, \\ 2-3x & \text{if } \frac{1}{3} \le x \le \frac{2}{3}, \\ 3x-2 & \text{if } x > \frac{2}{3}. \end{cases}, \quad g(x) = \begin{cases} 3x & \text{if } x < \frac{5}{12}, \\ 5-9x & \text{if } \frac{5}{12} \le x \le \frac{7}{12}, \\ 3x-2 & \text{if } x > \frac{7}{12}. \end{cases}$$

Let the *non-wandering set* Λ_h of a map $h : \mathbb{R} \to \mathbb{R}$ be defined as the set

$$\Lambda_h := \{x \in \mathbb{R} \mid \limsup_{n\to\infty} |h^n(x)| < \infty\},$$

where as usual we denote the nth iterate of h as $h^n(x) := \underbrace{h \circ \ldots \circ h}_{n \text{ times}}(x)$.

(a) Show that $\Lambda_f = [0,1]$ and $\Lambda_g \subset [0,1]$ (but not equal to $[0,1]$).

(b) (i) Show that f on $[0,1]$ is topologically semi-conjugate to a full shift on three symbols. Discuss whether the semi-conjugacy that you have constructed is also a conjugacy.

(ii) Show that g on Λ_g is topologically conjugate to a full shift on three symbols.

2. Let Σ_3 denote the space of (half) infinite sequences of three symbols $\{0,1,2\}$, endowed with the metric $d(x,y) := \sum_{i=0}^{\infty} \delta_i(x,y)/4^i$ where $x = x_0x_1\ldots$ and $y = y_0y_1\ldots$ with $x_k, y_k \in \{0,1,2\}$ for all $k \in \mathbb{N}$ and $\delta_i(x,y) = 0$ if $x_i = y_i$, and $\delta_i(x,y) = 1$ otherwise. Consider the Markov chain defined by the (connectivity) matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

with the subshift of finite type $\sigma_A : \Sigma_{3,A} \to \Sigma_{3,A}$ where $\Sigma_{3,A} \subset \Sigma_3$ denotes the set of elements of Σ_3 that are compatible with A.

(i) Show that σ_A is <u>topologically mixing</u> and that <u>periodic orbits</u> of σ_A are dens in $\Sigma_{3,A}$.

(ii) Show that any positive number less than $\frac{16}{15}$, but no number larger than $\frac{16}{15}$, serves as a <u>sensitivity constant</u> for σ_A.