and $f^j(b_j)$ belong to the same ball $B_j \in \mathcal{B}$, so $d(f^j(a_j), f^j(b_j)) < \delta$. Thus, $d(f^n(a_j), f^n(b_j)) < \epsilon$ for $-\infty < n \leq j$. Since W is finite, there are distinct $x_0, y_0 \in W$ such that

$$a_j = x_0 \quad \text{and} \quad b_j = y_0$$

for infinitely many positive j and hence $d(f^n(x_0), f^n(y_0)) < \epsilon$ for all $n \geq 0$. □

Proposition 2.4.1 is also true for non-invertible maps (Exercise 2.4.3).

COROLLARY 2.4.2. *Let f be an expansive homeomorphism of an infinite compact metric space X. Then there are $x_0, y_0 \in X$ such that $d(f^n(x_0), f^n(y_0)) \to 0$ as $n \to \infty$.*

Proof. Let $\delta > 0$ be an expansiveness constant for f. By Proposition 2.4.1, there are $x_0, y_0 \in X$ such that $d(f^n(x_0), f^n(y_0)) < \delta$ for all $n \in \mathbb{N}$. Suppose $d(f^n(x_0), f^n(y_0)) \not\to 0$. Then by compactness, there is a sequence $n_k \to \infty$ such that $f^{n_k}(x_0) \to x'$ and $f^{n_k}(y_0) \to y'$ with $x' \neq y'$. Then $f^{n_k+m}(x_0) \to f^m(x')$ and $f^{n_k+m}(y_0) \to f^m(y')$ for any $m \in \mathbb{Z}$. For k large, $n_k + m > 0$ and hence $d(f^m(x'), f^m(y')) \leq \delta$ for all $m \in \mathbb{Z}$, which contradicts expansiveness. □

Exercise 2.4.1. Prove that every isometry of a compact metric space to itself is surjective and therefore is a homeomorphism.

Exercise 2.4.2. Show that the expanding circle endomorphisms E_m, $|m| \geq 2$, the full one- and two-sided shifts, the hyperbolic toral automorphisms, the horseshoe, and the solenoid are expansive, and compute expansiveness constants for each.

Exercise 2.4.3. Show that Proposition 2.4.1 is true for non-invertible continuous maps of infinite metric spaces.

2.5 Topological Entropy

Topological entropy is the exponential growth rate of the number of essentially different orbit segments of length n. It is a topological invariant that measures the complexity of the orbit structure of a dynamical system. Topological entropy is analogous to measure-theoretic entropy, which we introduce in Chapter 9.

Let (X, d) be a compact metric space, and $f: X \to X$ a continuous map. For each $n \in \mathbb{N}$, the function

$$d_n(x, y) = \max_{0 \le k \le n-1} d(f^k(x), f^k(y))$$

measures the maximum distance between the first n iterates of x and y. Each d_n is a metric on X, $d_n \ge d_{n-1}$, and $d_1 = d$. Moreover, the d_i are all equivalent metrics in the sense that they induce the same topology on X (Exercise 2.5.1).

Fix $\epsilon > 0$. A subset $A \subset X$ is (n, ϵ)-*spanning* if for every $x \in X$ there is $y \in A$ such that $d_n(x, y) < \epsilon$. By compactness, there are finite (n, ϵ)-spanning sets. Let $\mathrm{span}(n, \epsilon, f)$ be the minimum cardinality of an (n, ϵ)-spanning set.

A subset $A \subset X$ is (n, ϵ)-*separated* if any two distinct points in A are at least ϵ apart in the metric d_n. Any (n, ϵ)-separated set is finite. Let $\mathrm{sep}(n, \epsilon, f)$ be the maximum cardinality of an (n, ϵ)-separated set.

Let $\mathrm{cov}(n, \epsilon, f)$ be the minimum cardinality of a covering of X by sets of d_n-diameter less than ϵ (the diameter of a set is the supremum of distances between pairs of points in the set). Again, by compactness, $\mathrm{cov}(n, \epsilon, f)$ is finite.

The quantities $\mathrm{span}(n, \epsilon, f)$, $\mathrm{sep}(n, \epsilon, f)$, and $\mathrm{cov}(n, \epsilon, f)$ count the number of orbit segments of length n that are distinguishable at scale ϵ. These quantities are related by the following lemma.

LEMMA 2.5.1. $\mathrm{cov}(n, 2\epsilon, f) \le \mathrm{span}(n, \epsilon, f) \le \mathrm{sep}(n, \epsilon, f) \le \mathrm{cov}(n, \epsilon, f)$.

Proof. Suppose A is an (n, ϵ)-spanning set of minimum cardinality. Then the open balls of radius ϵ centered at the points of A cover X. By compactness, there exists $\epsilon_1 < \epsilon$ such that the balls of radius ϵ_1 centered at the points of A also cover X. Their diameter is $2\epsilon_1 < 2\epsilon$, so $\mathrm{cov}(n, 2\epsilon, f) \le \mathrm{span}(n, \epsilon, f)$. The other inequalities are left as an exercise (Exercise 2.5.2). □

Let

$$h_\epsilon(f) = \overline{\lim_{n \to \infty}} \frac{1}{n} \log(\mathrm{cov}(n, \epsilon, f)). \tag{2.2}$$

The quantity $\mathrm{cov}(n, \epsilon, f)$ increases monotonically as ϵ decreases, so $h_\epsilon(f)$ does as well. Thus the limit

$$h_{\mathrm{top}} = h(f) = \lim_{\epsilon \to 0^+} h_\epsilon(f)$$

exists; it is called the *topological entropy* of f. The inequalities in Lemma 2.5.1 imply that equivalent definitions of $h(f)$ can be given using $\mathrm{span}(n, \epsilon, f)$ or sep(n, ϵ, f), i.e.,

$$h(f) = \lim_{\epsilon \to 0^+} \overline{\lim_{n\to\infty}} \frac{1}{n} \log(\text{span}(n, \epsilon, f)) \tag{2.3}$$

$$= \lim_{\epsilon \to 0^+} \overline{\lim_{n\to\infty}} \frac{1}{n} \log(\text{sep}(n, \epsilon, f)). \tag{2.4}$$

LEMMA 2.5.2. *The limit* $\lim_{n\to\infty} \frac{1}{n} \log(\text{cov}(n, \epsilon, f)) = h_\epsilon(f)$ *exists and is finite.*

Proof. Let U have d_m-diameter less than ϵ, and V have d_n-diameter less than ϵ. Then $U \cap f^{-m}(V)$ has d_{m+n}-diameter less than ϵ. Hence

$$\text{cov}(m+n, \epsilon, f) \le \text{cov}(m, \epsilon, f) \cdot \text{cov}(n, \epsilon, f),$$

so the sequence $a_n = \log(\text{cov}(n, \epsilon, f)) \ge 0$ is subadditive. A standard lemma from calculus implies that a_n/n converges to a finite limit as $n \to \infty$ (Exercise 2.5.3). □

It follows from Lemmas 2.5.1 and 2.5.2 that the lim sups in Formulas (2.2), (2.3), and (2.4) are finite. Moreover, the corresponding lim infs are finite, and

$$h(f) = \lim_{\epsilon \to 0^+} \underline{\lim}_{n\to\infty} \frac{1}{n} \log(\text{cov}(n, \epsilon, f)) \tag{2.5}$$

$$= \lim_{\epsilon \to 0^+} \underline{\lim}_{n\to\infty} \frac{1}{n} \log(\text{span}(n, \epsilon, f)) \tag{2.6}$$

$$= \lim_{\epsilon \to 0^+} \underline{\lim}_{n\to\infty} \frac{1}{n} \log(\text{sep}(n, \epsilon, f)). \tag{2.7}$$

The topological entropy is either $+\infty$ or a finite non-negative number. There are dramatic differences between dynamical systems with positive entropy and dynamical systems with zero entropy. Any isometry has zero topological entropy (Exercise 2.5.4). In the next section, we show that topological entropy is positive for several of the examples from Chapter 1.

PROPOSITION 2.5.3. *The topological entropy of a continuous map* $f: X \to X$ *does not depend on the choice of a particular metric generating the topology of* X.

Proof. Suppose d and d' are metrics generating the topology of X. For $\epsilon > 0$, let $\delta(\epsilon) = \sup\{d'(x, y): d(x, y) \le \epsilon\}$. By compactness, $\delta(\epsilon) \to 0$ as $\epsilon \to 0$. If U is a set of d_n-diameter less than ϵ, then U has d'_n-diameter at most $\delta(\epsilon)$. Thus $\text{cov}'(n, \delta(\epsilon), f) \le \text{cov}(n, \epsilon, f)$, where cov and cov$'$ correspond to the metrics d and d', respectively. Hence,

$$\lim_{\delta\to 0^+}\lim_{n\to\infty}\frac{1}{n}\log(\mathrm{cov}'(n,\delta,f)) \le \lim_{\epsilon\to 0^+}\lim_{n\to\infty}\frac{1}{n}\log(\mathrm{cov}(n,\epsilon,f)).$$

Interchanging d and d' gives the opposite inequality. □

COROLLARY 2.5.4. *Topological entropy is an invariant of topological conjugacy .*

Proof. Suppose $f\colon X\to X$ and $g\colon Y\to Y$ are topologically conjugate dynamical systems, with conjugacy $\phi\colon Y\to X$. Let d be a metric on X. Then $d'(y_1, y_2) = d(\phi(y_1), \phi(y_2))$ is a metric on Y generating the topology of Y. Since ϕ is an isometry of (X, d) and (Y, d'), and the entropy is independent of the metric by Proposition 2.5.3, it follows that $h(f) = h(g)$. □

PROPOSITION 2.5.5. *Let $f\colon X\to X$ be a continuous map of a compact metric space X.*

1. *$h(f^m) = m\cdot h(f)$ for $m\in\mathbb{N}$.*
2. *If f is invertible, then $h(f^{-1}) = h(f)$. Thus $h(f^m) = |m|\cdot h(f)$ for all $m\in\mathbb{Z}$.*
3. *If $A_i, i = 1,\ldots,k$ are closed (not necessarily disjoint) forward f-invariant subsets of X, whose union is X, then*

$$h(f) = \max_{1\le i\le k} h(f|A_i).$$

In particular, if A is a closed forward invariant subset of X, then $h(f|A)\le h(f)$.

Proof. 1: Note that

$$\max_{0\le i<n} d(f^{mi}(x), f^{mi}(y)) \le \max_{0\le j<mn} d(f^j(x), f^j(y)).$$

Thus, $\mathrm{span}(n,\epsilon, f^m)\le \mathrm{span}(mn,\epsilon, f)$, so $h(f^m)\le m\cdot h(f)$. Conversely, for $\epsilon>0$, there is $\delta(\epsilon)>0$ such that $d(x,y)<\delta(\epsilon)$ implies that $d(f^i(x), f^i(y))<\epsilon$ for $i=0,\ldots,m$. Then $\mathrm{span}(n,\delta(\epsilon), f^m)\ge \mathrm{span}(mn,\epsilon, f)$, so $h(f^m)\ge m\cdot h(f)$.

2: The nth image of an (n,ϵ)-separated set for f is (n,ϵ)-separated for f^{-1}, and vice versa.

3: Any (n,ϵ)-separated set in A_i is (n,ϵ)-separated in X, so $h(f|A_i)\le h(f)$. Conversely, the union of (n,ϵ)-spanning sets for the A_is is an (n,ϵ)-spanning set for X. Thus if $\mathrm{span}_i(n,\epsilon, f)$ is the minimum cardinality of an

(n, ϵ)-spanning subset of A_i, then

$$\operatorname{span}(n, \epsilon, f) \leq \sum_{i=1}^{k} \operatorname{span}_i(n, \epsilon, f) \leq k \cdot \max_{1\leq i\leq k} \operatorname{span}_i(n, \epsilon, f).$$

Therefore,

$$\begin{aligned}\varlimsup_{n\to\infty} \frac{1}{n}\log(\operatorname{span}(n, \epsilon, f)) &\leq \varlimsup_{n\to\infty} \frac{1}{n}\log k + \varlimsup_{n\to\infty} \frac{1}{n}\log\left(\max_{1\leq i\leq k} \operatorname{span}_i(n, \epsilon, f)\right)\\ &= 0 + \max_{1\leq i\leq k} \varlimsup_{n\to\infty} \frac{1}{n}\log(\operatorname{span}_i(n, \epsilon, f))\end{aligned}$$

The result follows by taking the limit as $\epsilon \to 0$. □

PROPOSITION 2.5.6. *Let (X, d^X) and (Y, d^Y) be compact metric spaces, and $f: X \to X$, $g: Y \to Y$ continuous maps. Then:*

1. *$h(f \times g) = h(f) + h(g)$; and*
2. *if g is a factor of f (or equivalently, f is an extension of g), then $h(f) \geq h(g)$.*

Proof. To prove part 1, note that the metric $d((x, y), (x', y')) = \max\{d^X(x, x'), d^Y(y, y')\}$ generates the product topology on $X \times Y$, and

$$d_n((x, y), (x', y')) = \max\left\{d_n^X(x, x'), d_n^Y(y, y')\right\}.$$

If $U \subset X$ and $V \subset Y$ have diameters less than ϵ, then $U \times V$ has d-diameters less than ϵ. Hence

$$\operatorname{cov}(n, \epsilon, f \times g) \leq \operatorname{cov}(n, \epsilon, f) \cdot \operatorname{cov}(n, \epsilon, g),$$

so $h(f \times g) \leq h(f) + h(g)$. On the other hand, if $A \subset X$ and $B \subset Y$ are (n, ϵ)-separated, then $A \times B$ is (n, ϵ)-separated for d. Hence

$$\operatorname{sep}(n, \epsilon, f \times g) \geq \operatorname{sep}(n, \epsilon, f) \cdot \operatorname{sep}(n, \epsilon, g),$$

so, by (2.7), $h(f \times g) \geq h(f) + h(g)$.

The proof of part 2 is left as an exercise (Exercise 2.5.5). □

PROPOSITION 2.5.7. *Let (X, d) be a compact metric space, and $f: X \to X$ an expansive homeomorphism with expansiveness constant δ. Then $h(f) = h_\epsilon(f)$ for any $\epsilon < \delta$.*

Proof. Fix γ and ϵ with $0 < \gamma < \epsilon < \delta$. We will show that $h_{2\gamma}(f) = h_\epsilon(f)$. By monotonicity, it suffices to show that $h_{2\gamma}(f) \leq h_\epsilon(f)$.

By expansiveness, for distinct points x and y, there is some $i \in \mathbb{Z}$ such that $d(f^i(x), f^i(y)) \geq \delta > \epsilon$. Since the set $\{(x, y) \in X \times X: d(x, y) \geq \gamma\}$ is compact, there is $k = k(\gamma, \epsilon) \in \mathbb{N}$ such that if $d(x, y) \geq \gamma$, then

$d(f^i(x), f^i(y)) > \epsilon$ for some $|i| \leq k$. Thus if A is an (n, γ)-separated set, then $f^{-k}(A)$ is $(n+2k, \epsilon)$-separated. Hence, by Lemma 2.5.1, $h_\epsilon(f) \geq h_{2\gamma}(f)$. □

REMARK 2.5.8. *The topological entropy of a continuous (semi)flow can be defined as the entropy of the time-1 map. Alternatively, it can be defined using the analog d_T, $T > 0$, of the metrics d_n. The two definitions are equivalent because of the equicontinuity of the family of time-t maps, $t \in [0, 1]$.*

Exercise 2.5.1. Let (X, d) be a compact metric space. Show that the metrics d_i all induce the same topology on X.

Exercise 2.5.2. Prove the remaining inequalities in Lemma 2.5.1.

Exercise 2.5.3. Let $\{a_n\}$ be a subadditive sequence of non-negative real numbers, i.e., $0 \leq a_{m+n} \leq a_m + a_n$ for all $m, n \geq 0$. Show that $\lim_{n\to\infty} a_n/n = \inf_{n\geq 0} a_n/n$.

Exercise 2.5.4. Show that the topological entropy of an isometry is zero.

Exercise 2.5.5. Let $g: Y \to Y$ be a factor of $f: X \to X$. Prove that $h(f) \geq h(g)$.

Exercise 2.5.6. Let Y and Z be compact metric spaces, $X = Y \times Z$, and π be the projection to Y. Suppose $f: X \to X$ is an isometric extension of a continuous map $g: Y \to Y$, i.e., $\pi \circ f = g \circ \pi$ and $d(f(x_1), f(x_2)) = d((x_1), (x_2))$ for all $x_1, x_2 \in Y$ with $\pi(x_1) = \pi(x_2)$. Prove that $h(f) = h(g)$.

Exercise 2.5.7. Prove that the topological entropy of a continuously differentiable map of a compact manifold is finite.

2.6 Topological Entropy for Some Examples

In this section, we compute the topological entropy for some of the examples from Chapter 1.

PROPOSITION 2.6.1. *Let $\tilde{A}$ be a 2×2 integer matrix with determinant 1 and eigenvalues λ, λ^{-1}, with $|\lambda| > 1$; and let $A: \mathbb{T}^2 \to \mathbb{T}^2$ be the associated hyperbolic toral automorphism. Then $h(A) = \log|\lambda|$.*

Proof. The natural projection $\pi: \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}^2$ is a local homeomorphism, and $\pi\tilde{A} = A\pi$. Any metric $\tilde{d}$ on $\mathbb{R}^2$ invariant under integer translations induces a metric d on $\mathbb{T}^2$, where $d(x, y)$ is the $\tilde{d}$-distance between the sets $\pi^{-1}(x)$ and $\pi^{-1}(y)$. For these metrics, π is a local isometry.

Let v_1, v_2 be eigenvectors of A with (Euclidean) length 1 corresponding to the eigenvalues λ, λ^{-1}. For $x, y \in \mathbb{R}^2$, write $x - y = a_1v_1 + a_2v_2$ and define $\tilde{d}(x, y) = \max(|a_1|, |a_2|)$. This is a translation-invariant metric on $\mathbb{R}^2$. A $\tilde{d}$-ball of radius ϵ is a parallelogram whose sides are of (Euclidean) length 2ϵ and parallel to v_1 and v_2. In the metric $\tilde{d}_n$ (defined for $\tilde{A}$), a ball of radius ϵ is a parallelogram with side length $2\epsilon|\lambda|^{-n}$ in the v_1-direction and 2ϵ in the v_2-direction. In particular, the Euclidean area of a $\tilde{d}_n$-ball of radius ϵ is not greater than $4\epsilon^2|\lambda|^{-n}$. Since the induced metric d on $\mathbb{T}^2$ is locally isometric to $\tilde{d}$, we conclude that for sufficiently small ϵ, the Euclidean area of a d_n-ball of radius ϵ in $\mathbb{T}^2$ is at most $4\epsilon^2|\lambda|^{-n}$. It follows that the minimal number of balls of d_n-radius ϵ needed to cover $\mathbb{T}^2$ is at least

$$\text{area}(\mathbb{T}^2)/4\epsilon^2|\lambda|^{-n} = |\lambda|^n/4\epsilon^2.$$

Since a set of diameter ϵ is contained in an open ball of radius ϵ, we conclude that $\text{cov}(n, \epsilon, A) \geq |\lambda|^n/4\epsilon^2$. Thus, $h(A) \geq \log|\lambda|$.

Conversely, since the closed $\tilde{d}_n$-balls are parallelograms, there is a tiling of the plane by ϵ-balls whose interiors are disjoint. The Euclidean area of such a ball is $C\epsilon^2|\lambda|^{-n}$, where C depends on the angle between v_1 and v_2. For small enough ϵ, any ϵ-ball that intersects the unit square $[0, 1] \times [0, 1]$ is entirely contained in the larger square $[-1, 2] \times [-1, 2]$. Therefore the number of the balls that intersect the unit square does not exceed the area of the larger square divided by the area of a $\tilde{d}_n$-ball of radius ϵ. Thus, the torus can be covered by $9\lambda^n/C\epsilon^2$ closed d_n-balls of radius ϵ. It follows that $\text{cov}(n, 2\epsilon, A) \leq 9\lambda^n/C\epsilon^2$, so $h(A) \leq \log|\lambda|$. □

To establish the corresponding result in higher dimensions, we need some results from linear algebra. Let B be a $k \times k$ complex matrix. If λ is an eigenvalue of B, let

$$V_\lambda = \{v \in \mathbb{C}^k : (B - \lambda I)^i v = 0 \text{ for some } i \in \mathbb{N}\}.$$

If B is real and γ is a real eigenvalue, let

$$V_\gamma^{\mathbb{R}} = \mathbb{R}^k \cap V_\gamma = \{v \in \mathbb{R}^k : (B - \gamma I)^i v = 0 \text{ for some } i \in \mathbb{N}\}.$$

If B is real and $\lambda, \bar{\lambda}$ is a pair of complex eigenvalues, let

$$V_{\lambda,\bar{\lambda}}^{\mathbb{R}} = \mathbb{R}^k \cap (V_\lambda \oplus V_{\bar{\lambda}}).$$

These spaces are called *generalized eigenspaces*.

LEMMA 2.6.2. *Let B be a $k \times k$ complex matrix, and λ be an eigenvalue of B. Then for every $\delta > 0$ there is $C(\delta) > 0$ such that*

$$C(\delta)^{-1}(|\lambda| - \delta)^n \|v\| \leq \|B^n v\| \leq C(\delta)(|\lambda| + \delta)^n \|v\|$$

for every $n \in \mathbb{N}$ and every $v \in V_\lambda$.

Proof. It suffices to prove the lemma for a Jordan block. Thus without loss of generality, we assume that B has λs on the diagonal, ones above and zeros elsewhere. In this setting, $V_\lambda = \mathbb{C}^k$ and in the standard basis $e_1, \dots, e_k$, we have $Be_1 = \lambda e_1$ and $Be_i = \lambda e_i + e_{i-1}$, for $i = 2, \dots, k$. For $\delta > 0$, consider the basis $e_1, \delta e_2, \delta^2 e_3, \dots, \delta^{k-1} e_k$. In this basis, the linear map B is represented by the matrix

$$B_\delta = \begin{pmatrix} \lambda & \delta & & & \\ & \lambda & \delta & & \\ & & \ddots & \ddots & \\ & & & \lambda & \delta \\ & & & & \lambda \end{pmatrix}.$$

Observe that $B_\delta = \lambda I + \delta A$ with $\|A\| \leq 1$, where $\|A\| = \sup_{v \neq 0} \|Av|/\|v\|$. Therefore

$$(|\lambda| - \delta)^n \|v\| \leq \left\| B_\delta^n v \right\| \leq (|\lambda| + \delta)^n \|v\|.$$

Since B_δ is conjugate to B, there is a constant $C(\delta) > 0$ that bounds the distortion of the change of basis. □

LEMMA 2.6.3. *Let B be a $k \times k$ real matrix and λ an eigenvalue of B. Then for every $\delta > 0$ there is $C(\delta) > 0$ such that*

$$C(\delta)^{-1}(|\lambda| - \delta)^n \|v\| \leq \|B^n v\| \leq C(\delta)(|\lambda| + \delta)^n \|v\|$$

for every $n \in \mathbb{N}$ and every $v \in V_\lambda$ (if $\lambda \in \mathbb{R}$) or every $v \in V_{\lambda,\bar{\lambda}}$ (if $\lambda \notin \mathbb{R}$).

Proof. If λ is real, then the result follows from Lemma 2.6.2. If λ is complex, then the estimates for V_λ and $V_{\bar{\lambda}}$ from Lemma 2.6.2 imply a similar estimate for $V_{\lambda,\bar{\lambda}}$, with a new constant $C(\delta)$ depending on the angle between V_λ and $V_{\bar{\lambda}}$ and the constants in the estimates for V_λ and $V_{\bar{\lambda}}$ (since $|\lambda| = |\bar{\lambda}|$). □

PROPOSITION 2.6.4. *Let $\tilde{A}$ be a $k \times k$ integer matrix with determinant 1 and with eigenvalues $\alpha_1, \dots, \alpha_k$, where*

$$|\alpha_1| \geq |\alpha_2| \geq \cdots \geq |\alpha_m| > 1 > |\alpha_{m+1}| \geq \cdots \geq |\alpha_k|.$$

Let $A\colon \mathbb{T}^k \to \mathbb{T}^k$ be the associated hyperbolic toral automorphism. Then

$$h(A) = \sum_{i=1}^{m} \log |\alpha_i|.$$

Proof. Let $\gamma_1, \dots, \gamma_j$ be the distinct real eigenvalues of $\tilde{A}$, and $\lambda_1, \overline{\lambda_1}, \dots, \lambda_m, \overline{\lambda_m}$ be the distinct complex eigenvalues of $\tilde{A}$. Then

$$\mathbb{R}^k = \bigoplus_{i=1}^{j} V_{\gamma_i} \oplus \bigoplus_{i=1}^{m} V_{\lambda_i,\overline{\lambda_i}},$$

any vector $v \in \mathbb{R}^k$ can be decomposed uniquely as $v = v_1 + \cdots + v_{j+m}$ with v_i in the corresponding generalized eigenspace. Given $x, y \in \mathbb{R}^k$, let $v = x - y$, and define $\tilde{d}(x, y) = \max(|v_1|, \ldots, |v_{j+m}|)$. This is a translation-invariant metric on $\mathbb{R}^k$, and therefore descends to a metric on $\mathbb{T}^k$. Now, using Lemma 2.6.3, the proposition follows by an argument similar to the one in the proof of Proposition 2.6.1 (Exercise 2.6.3). □

The next example we consider is the solenoid from §1.9.

PROPOSITION 2.6.5. *The topological entropy of the solenoid map* $F\colon S \to S$ *is* $\log 2$.

Proof. Recall from §1.9 that F is topologically conjugate to the automorphism $\alpha\colon \Phi \to \Phi$, where

$$\Phi = \left\{(\phi_i)_{i=0}^{\infty}\colon \phi_i \in [0, 1),\ \phi_i = 2\phi_{i+1} \bmod 1\right\},$$

and α is coordinatewise multiplication by 2 (mod 1). Thus, $h(F) = h(\alpha)$. Let $|x - y|$ denote the distance on $S^1 = [0, 1] \bmod 1$. The distance function

$$d(\phi, \phi') = \sum_{n=0}^{\infty} \frac{1}{2^n} |\phi_n - \phi'_n|$$

generates the topology in Φ introduced in §1.9.

The map $\pi\colon \Phi \to S^1$, $(\phi_i)_{i=0}^{\infty} \mapsto \phi_0$, is a semiconjugacy from α to E_2. Hence, $h(\alpha) \geq h(E_2) = \log 2$ (Exercise 2.6.1). We will establish the inequality $h(\alpha) \leq \log 2$ by constructing an (n, ϵ)-spanning set.

Fix $\epsilon > 0$ and choose $k \in \mathbb{N}$ such that $2^{-k} < \epsilon/2$. For $n \in \mathbb{N}$, let $A_n \subset \Phi$ consist of the 2^{n+2k} sequences $\psi^j = (\psi_i^j)$, where $\psi_i^j = j \cdot 2^{-(n+k+i)} \bmod 1$, $j = 0, \ldots, 2^{n+2k} - 1$. We claim that A_n is (n, ϵ)-spanning. Let $\phi = (\phi_i)$ be a point in Φ. Choose $j \in \{0, \ldots, 2^{n+2k} - 1\}$ so that $|\phi_k - j \cdot 2^{-(n+2k)}| \leq 2^{-(n+2k+1)}$. Then $|\phi_i - \psi_i^j| \leq 2^{k-i} 2^{-(n+2k+1)}$, for $0 \leq i \leq k$. It follows that for $0 \leq m \leq n$,

$$\begin{aligned} d(\alpha^m \phi, \alpha^m \psi^j) &= \sum_{i=0}^{\infty} \frac{\left|2^m \phi_i - 2^m \psi_i^j\right|}{2^i} < \sum_{i=0}^{k} \frac{2^m \left|\phi_i - \psi_i^j\right|}{2^i} + \frac{1}{2^k} \\ &< 2^m \sum_{i=0}^{k} \frac{2^{k-i} 2^{-(n+2k+1)}}{2^i} + \frac{1}{2^k} < \frac{1}{2^{k-1}} < \epsilon. \end{aligned}$$

Thus $d_n(\phi, \psi^j) < \epsilon$, so A_n is (n, ϵ)-spanning. Hence,

$$h(\alpha) \leq \lim_{n \to \infty} \frac{1}{n} \log \operatorname{card} A_n = \log 2.$$

□

Note that $\alpha: \Phi \to \Phi$ is expansive with expansiveness constant 1/3 (Exercise 2.6.4), so by Proposition 2.5.7, $h_\epsilon(\alpha) = h(\alpha)$ for any $\epsilon < 1/3$.

Exercise 2.6.1. Compute the topological entropy of an expanding endomorphism $E_m: S^1 \to S^1$.

Exercise 2.6.2. Compute the topological entropy of the full one- and two-sided m-shifts.

Exercise 2.6.3. Finish the proof of Proposition 2.6.4.

Exercise 2.6.4. Prove that the solenoid map (§1.9) is expansive.

2.7 Equicontinuity, Distality, and Proximality[1]

In this section, we describe a number of properties related to the asymptotic behavior of the distance between corresponding points on pairs of orbits.

Let $f: X \to X$ be a homeomorphism of a compact Hausdorff space. Points $x, y \in X$ are called *proximal* if the closure $\overline{\mathcal{O}((x, y))}$ of the orbit of (x, y) under $f \times f$ intersects the diagonal $\Delta = \{(z, z) \in X \times X: z \in X\}$. Every point is proximal to itself. If two points x and y are not proximal, i.e., if $\overline{\mathcal{O}((x, y))} \cap \Delta = \emptyset$, they are called *distal*. A homeomorphism $f: X \to X$ is *distal* if every pair of distinct points $x, y \in X$ is distal. If (X, d) is a compact metric space, then $x, y \in X$ are proximal if there is a sequence $n_k \in \mathbb{Z}$ such that $d(f^{n_k}(x), f^{n_k}(y)) \to 0$ as $k \to \infty$; points $x, y \in X$ are distal if there is $\epsilon > 0$ such that $d(f^n(x), f^n(y)) > \epsilon$ for all $n \in \mathbb{Z}$ (Exercise 2.7.2)

A homeomorphism f of a compact metric space (X, d) is said to be *equicontinuous* if the family of all iterates of f is an equicontinuous family, i.e., for any $\epsilon > 0$, there exists $\delta > 0$ such that $d(x, y) < \delta$ implies that $d(f^n(x), f^n(y)) < \epsilon$ for all $n \in \mathbb{Z}$. An isometry preserves distances and is therefore equicontinuous. Equicontinuous maps share many of the dynamical properties of isometries. The only examples from Chapter 1 that are equicontinuous are the group translations, including circle rotations.

We denote by $f \times f$ the induced action of f in $X \times X$, defined by $f \times f(x, y) = (f(x), f(y))$.

PROPOSITION 2.7.1. *An expansive homeomorphism of an infinite compact metric space is not distal.*

Proof. Exercise 2.7.1. □

[1] Several arguments in this section were conveyed to us by J. Auslander.