SOME RECENT DEVELOPMENTS AND OPEN QUESTIONS ON DYNAMICS OF PIECEWISE ISOMETRIES

Miguel Mendes

University of Surrey
England

Supervisor: *Matthew Nicol*

Financial Support: *Fundação para a Ciência e Tecnologia*

http://www.maths.surrey.ac.uk/personal/pg/M.Mendes

m.mendes@eim.surrey.ac.uk

Summary

✓ Definitions

✓ The Exceptional set

✓ Periodic Points

✓ Attractors

✓ Topological Entropy and Complexity

Definitions

Let $\mathbf{X} = \mathbb{R}^n$ or $\mathbf{X} \subset \mathbb{R}^n$ a compact subset admitting a convex partition.

◇ **Convex Partition**

Let $\mathcal{P} = \{P_0, ..., P_{r-1}\}$ be a collection of open and convex sets such that:

$$P_i \cap P_j = \emptyset, i \neq j \; ;$$
$$\mathbf{X} = \cup_{k=0}^{r-1} \overline{P}_k \; .$$

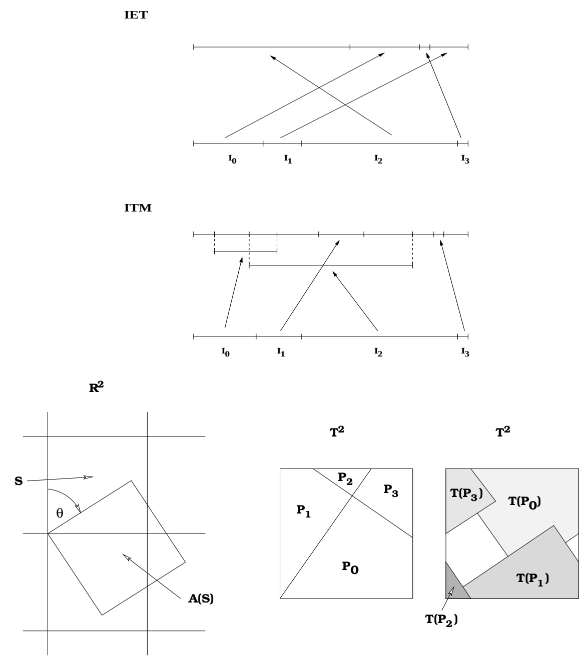
Each element P_i is called an *atom*.

◇ **Piecewise Isometries**

Given $\mathbf{X}$, $\mathcal{P}$ and a family of isometries $\{T_i\}_{i=0,\dots,r-1}$ we define,

$$T(x) = T_i(x), \text{ for } x \in P_i \; .$$

Examples:



◇ **Exceptional Set and Discontinuity**

A piecewise isometry is not defined along the *discontinuity set*,

$$\mathcal{D} = \cup_{k=0}^{r-1} \partial P_k$$

Consequently, the dynamics is defined for all points in M except,

$$\mathcal{E} = \cup_{k=0}^{\infty} T^{-k}(\mathcal{D})$$

denominated *exceptional set*.

◇ **The Coding Map and Cells**

To each $x \in \mathbf{X}^* = \mathbf{X} \backslash \mathcal{E}$ we assign an right-infinite word in the alphabet $\mathcal{A} = \{0, ..., r-1\}$ according to the following rule,

$$\chi(x) = \omega = \omega_0 ... \omega_n .... \text{ iff } T^n(x) \in P_{\omega_n} .$$

All points $x \in \mathbf{X}^*$ are classified as *rationals* or *irrationals* depending on whether $\chi(x) = ab...b...$ or not, where a and b are finite words in $\mathcal{A}$.

We can now build a *asymptotic refinement* $\Sigma \leq \mathcal{P}$ so that for all x, y belonging to the same *cell* ,

$$\chi(x) = \chi(y)$$

in the following manner,

$$\Sigma = \bigvee_{n=0}^{\infty} T^{-n}(\mathcal{P})$$

Comment:

For $x, y \in K(\in \Sigma), d(T^n(x), T^n(y)) = d(x, y), n \geq 0.$

Thm (A. Goetz [1996]): All cells are convex.

$$+$$

Thm (A. Goetz [1996]): If $\mathbf{X}$ is compact then all cells of positive measure are rational.

$$\Downarrow$$

Cor (A. Goetz [1996]): Irrational cells must have zero measure and therefore must be contained in hyperplanes of dimension $< n$.

Exceptional set

Thm (A. Goetz [1996]): $\overline{\mathcal{E}} = \{x \in \mathbb{X} : d(\mathcal{O}^+(x), \mathcal{D}) = 0\}$.

$$\Downarrow$$

For every point $x \in \overline{\mathcal{E}} \backslash \mathcal{E}$ and every $\varepsilon > 0$

the iterates of $B_\varepsilon(x)$ will intersect $\mathcal{D}$

infinitely often

Suppose $\overline{\mathcal{E}} \neq \mathbb{X}$. It then follows that:

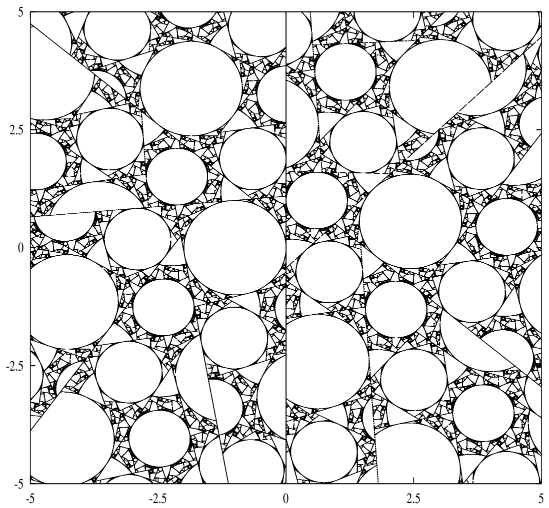
$\hookrightarrow$ $\mathbb{X}$ decomposes into a countable union of disjoint open connected components C_i ;

$\hookrightarrow$ For every component C_i there exist $j, k \in \mathbb{N}$ such that,

$$T(C_i) = T_k(C_i) \subset C_j$$

$\hookrightarrow$ The preimage of each component C_i is either empty or made of $n(\leq r)$ components $C_{j_1}, \ldots, C_{j_n} \in \mathcal{C}$, *i.e.*,

$$T^{-1}(C_i) = C_{j_1} \cup \ldots \cup C_{j_n}$$



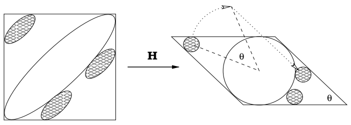
Thm (M. Mendes [2002?]): *For every rational cell K there exists a component $C \in \mathcal{C}$ such that*

$$C \subset K = \overline{C} \backslash \mathcal{E} \subsetneq \overline{C}$$

OQ: ¿Does $\overline{\mathcal{E}}$ have positive measure? (Note that the Lebesgue measure of $\mathcal{E}$ is zero)

OQ: It's not even known whether $\overline{\mathcal{E}}$ has empty interior or not!

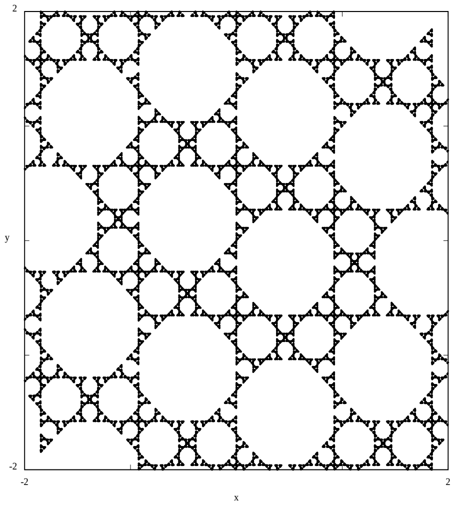
Conj: For the following piecewise rotation on the rhombus,



numerical investigations by Ashwin *et al.* [1997] lead to believe that the measure of $\overline{\mathcal{E}}$ is positive for all irrational values θ.

In some *rational* cases we can find self-similarities and hence compute the Hausdorff dimension.

Ex: For instance, in Roy Adler *et al.* [2000] for $\theta = \pi/4$,



the Hausdorff dimension is
$\frac{\log(3)}{\log(\sqrt{2})+1} = 1.2464774... < 2 \Longrightarrow \mu(\overline{\mathcal{E}}) = 0.$

Thm (A. Goetz [1996]): *Given a compact subspace of the plane* $\mathbf{X}$, *let* $T : \mathbf{X} \to \mathbf{X}$ *be an* ***irrational*** *piecewise rotation. Define* $f : \mathcal{T} \to \mathbb{R}_0^+$, $f(T) = \mu(\overline{\mathcal{E}}_T)$. *Then* f *is a upper semi-continuous, i.e.,*

$$\limsup_{T' \xrightarrow{d_{\mathcal{T}}} T} f(T') \leq f(T)$$

where $d_{\mathcal{T}}(T, T') = \max_{0 \leq i < r}\{|\rho_i - \rho_i'|\,, |z_i - z_i'|\,, d_H(\overline{P}_i, \overline{P}'_i)\}$ *and* $T_i(x) = \rho_i.x + z_i$.

Thm (P. Ashwin & X. Fu [2001]): *For a full measure set of values of* θ *there are no tangencies in the disc packing.*

NOTE: This does not imply that $\overline{\mathcal{E}}$ has positive measure. They present an example of a tangent-free disc packing with zero measure complement.

Periodic Points

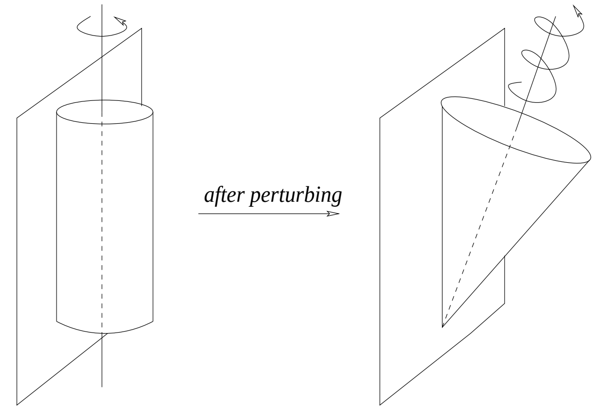
Thm (M. Mendes & M. Nicol [2002]) *There exist periodic points iff there exist recurrent points with rational coding.*

OQ: ¿How to prove that there are infinitely many periodic points in the invertible Goetz map?

REM: The invertible Goetz map is reversible. The Fixed Set Iteration Method (J. Lamb & G. Quispel [1994]) could be used but seems to be helpless in this case.

Thm (A. Goetz [1996]): *Every periodic point of an irrational piecewise rotation in the plane is stable under perturbation.*

NOTE: This does not generalise for odd dimensions.



Thm (M. Mendes & M. Nicol [2002]): *Almost every piecewise isometry in odd dimensions greater than 3 has no periodic points.*

Attractors

✓ Definition

PROB: Due to the discontinuities we cannot impose closedness and invariance at the same time.

$\Downarrow$

1. For topological reasons (on the boundary) closed sets are preferable according to examples available ;
2. ω-limit sets are closed independently of the continuity of the map .

$+$

Need an "attraction" condition!...$\Longrightarrow A \subset N, T(N) \subset N,$ $A = \cap_{n \geq 0} T^n(N) \cup Z.$

$+$

... and a "minimality" condition.

(Otherwise it will not be unique!)

$\Downarrow$

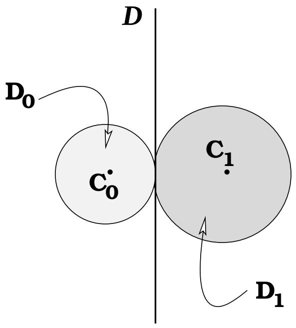
Definition of *piecewise isometric Attractor*

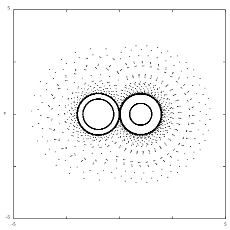
DEF: A set Q is said to be **quasi-invariant** if it is forward-invariant and $\mu(Q \backslash T(Q)) = 0$.

Prop (M. Mendes [2001]): *If $\mathcal{Q}$ is a compact quasi-invariant set then $T_{|\mathcal{Q}}$ is almost-invertible, measure-preserving and* $\mu(\mathcal{Q}) = \mu(\overline{\cup_{x \in \mathcal{Q}} \omega(x)})$.

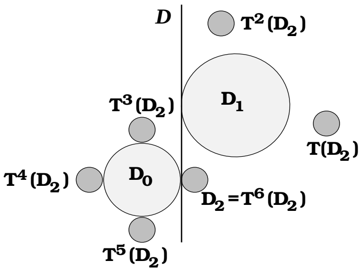
✓ Examples from the Goetz map

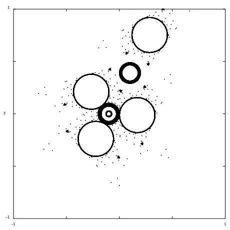
Thm (A. Goetz [1996]) 8-attractor: Let T be a piecewise rotation in the pane with two convex atoms. Suppose that the line containing both centres of rotation is orthogonal to the discontinuity line. Thus, al orbits will accumulate in the two maximal invariant discs.



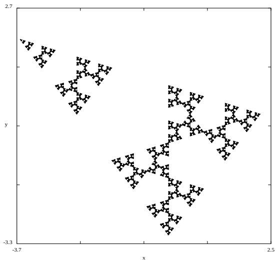


Thm (A. Goetz [1998]): Any small perturbation of a 8-attractor system possesses contains another 8-attractor system as a local map.

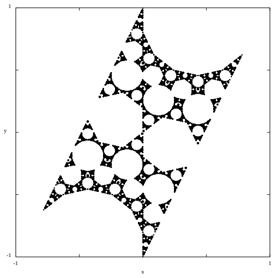


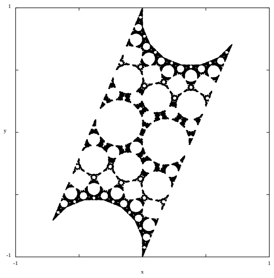


IMP: This result can be applied successively so that the resulting system has infinitely many periodic points. But in what setting does this happen? Is it an invertible map or is it a compact attractor like the following, *Example* (A. Goetz [1998]):



Other examples (M. Mendes [2001]):





Topological Entropy and Complexity

The *topological entropy* of a piecewise isometry T is defined as being

$$h(T) = \lim_{n\to\infty} \frac{\log(\#B_n)}{n}$$

where B_n is the set of blocks of length n generated by the coding map.

Thm (J. Buzzi [2000]): $h(T) = 0$ *for every piecewise isometry.*

OQ: What is the *polynomial growth* of B_n?

$$p = \lim_{n\to\infty} \frac{\log(\#B_n)}{\log n}$$

S. Vaienti & G. Poggiaspalla are carrying out some numerical investigations. Results might suggest that $\#B_n \approx n^r$ where r is the number of atoms.