Liapunov-Schmidt reduction

(notes for M4A35)

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Introduction

We now consider an approach towards local bifurcation theory that is alternative to the center manifold reduction. It is called *Liapunov-Schmidt reduction.* The basic idea of this reduction, is to reduce a bifurcation problem to finding a solution $x(\lambda)$ of an equation taking the form

$$g(x, \lambda) = 0,$$

where $g : V \to V$ and $V = \ker(dg)_{0,0}$.

We will discuss the two classical type of codimension one bifurcations in this context: generic steady-state bifurcation (describing the local existence of equilibria) and generic Hopf bifurcation (describing the local existence of periodic solutions in the neighbourhood of an equilibrium point). The steady-state theory is most straightforward and illustrates well the essence of Liapunov-Schmidt reduction. The Hopf theory is more involved. Some of the details will be left to you as assessed homework.

Steady-state bifurcation

We consider a differential equation

$$\dot{y} = F(y.\alpha),$$

where $F : \mathbb{R}^n \times \mathbb{R}^\ell \to \mathbb{R}^n$, $F(0, 0) = 0$ and $(dF)_{0,0}$ is singular. We define

$$\mathcal{K} = \ker(dF)_{0,0} \neq \{0\}, \quad \text{and} \quad \mathcal{R} = \text{range}(dF)_{0,0}.$$

(Note that before we denoted the range of an operator also as the Im (image) of the operator.) We consider the following decompositions of $\mathbb{R}^n$:

$$\mathbb{R}^n = \mathcal{K} \oplus \widehat{\mathcal{K}}, \quad \mathbb{R}^n = \mathcal{R} \oplus \widehat{\mathcal{R}},$$

where $\widehat{\mathcal{K}}$ is called the *cokernel* and $\widehat{\mathcal{R}}$ is called the *corange.* Note that the latter complementary spaces are not uniquely defined by the above decompositions.

The Liapunov-Schmidt reduction entails the following observation: Let $x \in \mathcal{K}$ and $w \in \widehat{\mathcal{K}}$. Then

$$F(x, w, \alpha) = 0 \quad \Leftrightarrow \quad \begin{cases} EF(x, w, \alpha) = 0 \\ (I - E)F(x, w, \alpha) = 0, \end{cases}$$

where E and $(I - E)$ are the natural projections $E : \mathbb{R}^n \to \mathcal{R}$ and $(I - E) : \mathbb{R}^n \to \widehat{\mathcal{R}}$.

We start considering the first equation $EF(x,w,\alpha)=0$. When we differentiate with respect to w, we obtain

$$\frac{d}{dw}EF|_{0,0,0} = E\frac{d}{dw}F|_{0,0,0} = d_{\widehat{\mathcal{K}}}F|_{0,0,0}.$$

By definition, the latter derivative is not singular. Hence, by the Implicit Function Theorem, there exists $w(x,\alpha)$ with $w(0,0)=0$, locally uniquely solving this equation. In turn, we obtain locally near $(0,0,0)$,

$$F(x,w,\alpha)=0 \quad \Leftrightarrow \quad f(x,\alpha) := (I-E)F(x,w(x,\alpha),\alpha) = 0.$$

Now note that $\mathcal{R} = F(\widehat{\mathcal{K}})$ and $F|\widehat{\mathcal{K}}$ is nonsingular. Hence $\dim\mathcal{R} = \dim\widehat{\mathcal{K}}$, and $\dim\mathcal{K} = \dim\widehat{\mathcal{R}}$. Identifying $\mathcal{K}$ with $\mathbb{R}^m$, we have:

$$f:\mathbb{R}^m\times\mathbb{R}^\ell \to \mathbb{R}^m, \quad f(0,0)=0.$$

Example 1 (Generic steady-state bifurcation) In a one-parameter family of vector fields, with codimension one, we have may have a kernel $\mathcal{K}$ with $\dim\mathcal{K}=1$. Then we may reduce to $f:\mathbb{R}\times\mathbb{R}\times\mathbb{R}$, where

$$f(x,\alpha) = \alpha + a(\alpha)x + b(\alpha)x^2 + \dots,$$

with $a(0)=0$, and - as a generic assumption - $a'(0)\neq 0$ and $b(0)\neq 0$. The resulting bifurcation is a turning point (saddle-node).

Example 2 (Generic $\mathbb{Z}_2$-equivariant steady-state bifurcation) We now assume we have a one-parameter family of vector fields, each one of them commuting with $-I$. This implies that if $y(t)$ is a solution of the equations, then so is $-y(t)$.

It is not difficult to verify that in this case, with codimension one, we have may have a kernel $\mathcal{K}$ with $\dim\mathcal{K}=1$. Then, as our decompositions of $\mathbb{R}^n$ are invariant under $-I$, we may reduce to $f:\mathbb{R}\times\mathbb{R}\times\mathbb{R}$, satisfying $f(-x,\alpha) = -f(x,\alpha)$ [verify this!]. Hence, now f has the folliwing expansion

$$f(x,\alpha) = x(\alpha + a(\alpha)x^2 + \dots),$$

with $a(0)\neq 0$. The resulting bifurcation is a pitchfork:using the Implicit Function Theorem we find that locally near $(0,0)$

$$\{(x,\alpha)\in\mathbb{R}\times\mathbb{R} \mid f(x,\alpha)=0\} = \{(x,\alpha)\in\mathbb{R}\times\mathbb{R} \mid x=0\} \cup \{(x,\alpha)\in\mathbb{R}\times\mathbb{R} \mid x\simeq \pm\sqrt{\alpha/a(\alpha)}\}.$$

The Liapunov-Schmidt (LS) and Centre Manifold (CM) have in common the fact that both reduce to a system of dimension $\dim\mathcal{K}$. The main difference between the LS-reduction and the CM-reduction, is that the LS-reduction only provides information on steady-states. With CM-reduction, however, we retain information on all solutions that remain close to the origin.

Hopf bifurcation

We now will use LS-reduction to study Hopf bifurcation. We need to find a way to find periodic solutions close to the origin. We start from the differential equation

$$\frac{d}{dt}u + F(u,\alpha) = 0, \tag{1}$$

where $F : \mathbb{R}^n \times \mathbb{R}^\ell \to \mathbb{R}^n$. We define

$$A(\alpha) = (d_{\mathbb{R}^n} F)_{0,\alpha}.$$

For later use, we write the ℓ-dimensional parameter as $\alpha = (\lambda, \alpha_2, \ldots, \alpha_\ell)$.

We aim to prove the following:

Theorem 3 (Liapunov-Schmidt reduction at Hopf bifurcation point) *Assume that (1) is such that*

- $A(0)$ *has simple eigenvalues* $\pm i$,
- $A(0)$ *has no other eigenvalues lying on the imaginary axis.*

Then there exists a smooth function $g(z^2, \alpha) = r(z^2, \alpha)z$, *with* $r(0, 0) = 0$ *such that locally near* $(0, 0)$, *solutions to* $g(z, \alpha) = 0$ *with* $z \geq 0$ *are in one-to-one correspondence with orbits of small amplitude periodic solutions with period near* 2π.

Theorem 4 (Hopf Theorem) *Assume that (1) is such that with* $|\alpha| \leq \varepsilon$ *for some* $\varepsilon > 0$,

- $A(\alpha)$ *has simple eigenvalues* $\pm\omega(\alpha)i + \sigma(\alpha)$, *with* $\omega(0) = 1$, $\sigma(0) = 0$ *and* $\frac{d\sigma}{d\lambda}(0) \neq 0$.
- $A(\alpha)$ *has no other eigenvalues intersecting the imaginary axis.*

Then there is a ℓ*-dimensional parameter family of periodic solutions bifurcating from the equilibrium at* $u = 0$, $\alpha = 0$.

Theorem 5 (second Hopf Theorem) *Assume that (1) satisfies the conditions of Theorem 4 and*

$$r_{z^2}(0, 0) \neq 0.$$

Then, the bifurcation equation is upto leading order equivalent to

$$z(\varepsilon z^2 + \delta\lambda) = 0,$$

where $\varepsilon = \operatorname{sgn} r_{z^2}(0, 0)$ *and* $\delta = \operatorname{sgn} \frac{d\sigma}{d\lambda}(0)$.

Example 6 (Simplest Hopf bifurcation) Consider a one-parameter (λ) family of vector fields in $\mathbb{R}^2$ with equilibrium 0, whose linear part has eigenvalues $\pm i\omega$ at $\lambda = 0$. Then, generically, the conditions of Theorem 4 and Theorem 5 are satisfied, implying that there will be a branch of periodic solutions who period approaches $2\pi\omega$ when $\lambda \to 0$, and whose amplitude x (measured in $||\cdot||_1$-norm) goes to zero as $\lambda \to 0$ as $x = \sqrt{\pm\lambda}$, where the $\pm$ sign depends on whether the Hopf bifurcation is super- or sub-critical.

The Liapunov-Schmidt reduction

We aim to define an operator Φ whcih has the property that solutions to $\Phi = 0$ correspond to solutions of (1) with period near 2π. Application of the Liapunov-Schmidt reduction to Φ then will yield g.

In order to define Φ, we view (1) as an operator Φ on the space of periodic functions. The problem with this approach is that such a space is not a linear space, as the sum of two periodic functions is in general not periodic. In order to resolve this problem, we introduce a time-scaling parameter τ. Let $s = (1+\tau)t$, then (1) is equivalent to

$$(1+\tau)\frac{du}{ds} + F(u,\alpha) = 0. \tag{2}$$

We now aim to look for 2π-periodic functions of (2).

Let $C_{2\pi}$ denote the (Banach) space of continuous, 2π-periodic functions from $\mathbb{R} \to \mathbb{R}^n$ with norm

$$||u|| = \max_s |u(s)|,$$

and $C^1_{2\pi}$ be the (Banach) space of such functions that are continuously differentiable, with norm

$$||u||_1 = ||u|| + ||\frac{du}{ds}||.$$

We define

$$\Phi : C^1_{2\pi} \times \mathbb{R}^\ell \times \mathbb{R} \to C_{2\pi},$$

by

$$\Phi(u,\alpha,\tau) = (1+\tau)\frac{du}{ds} + F(u,\alpha).$$

Then $u \in C^1_{2\pi}$ is a 2π-periodic solution of (2) if and only if

$$\Phi(u,\alpha,\tau) = 0.$$

Importantly, the map Φ is S^1-equivariant. Let the circle group $S^1 \cong [0,2\pi\rangle$ act on $C_{2\pi}$in the following way:

$$(\theta \cdot u)(s) = u(s-\theta), \quad \theta \in [0,2\pi\rangle.$$

Then

$$\Phi(\theta \cdot u, \alpha, \tau) = \theta\Phi(u,\alpha,\tau).$$

Namely,

$$\frac{d(\theta \cdot u(s))}{ds} = \frac{du(s-\theta)}{ds} = \theta \cdot \frac{du(s)}{ds}, \quad \text{and} \quad F(\theta \cdot u, \alpha) = \theta \cdot F(u,\alpha),$$

the latter because F does not depend explicitly on s. The S^1-equivariance of Φ is an expression of that fact that if $u(s)$is a solutions, then so is $u(s-\theta)$ for all θ, which is in fact a consequence of the fact that (2) is autonomous, ie F does not depend explicitly on s.

Remark 7 Compare the occurrence of S^1-equivariance here with the formal normal form symmetry in the CM-reduction/normal form approach to Hopf bifurcation.

We now linearize Φ around $(u,\alpha,\tau) = (0,0,0)$:

$$L : C^1_{2\pi} \to C_{2\pi}, \quad Lu := \frac{du}{ds} + A(0)u, \quad \text{where} \quad A(\alpha) = (d_{\mathbb{R}^n}F)_{0,\alpha}.$$

We now summarize some important properties.

Proposition 8 *Let $A(0)$ satisfy the simple eigenvalue hypothesis of Theorem 3. Then*

(a) $\dim \ker L = 2$.

(b) *There exists a basis* v_1, v_2 *for* $\ker L$ *so that if we parameterize* $\ker L$ *by* $9x, y) = xv_1 + yv_2$, *then* S^1 *acts on* $\ker L$ *as*

$$\theta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

(c) *We have the following decompositions:*

$$C_{2\pi} = \mathrm{range}L \oplus \ker L, \quad C^1_{2\pi} = M \oplus \ker L, \quad \text{where} \quad M = \mathrm{range}L \cap C^1_{2\pi}.$$

Proof: We leave the details of the proof for the assessed work. In part (c), it becomes important that L is *Fredholm of index 0*, which has the important implication that

$$\text{codim range}L = \dim \ker L.$$

(This is not obvious, since $C_{2\pi}$ and $C^2_{2\pi}$ are infinite dimensional!) You may use this fact in the proof, without further explanation (although you are encouraged to find more about this!). ∎

We now proceed with the Liapunov-Schmidt reduction. We decompose:

$$\Phi = 0 \quad \Leftrightarrow \quad \begin{cases} E\Phi = 0, \\ (I-E)\Phi = 0, \end{cases}$$

where E and $(I-E)$ are the projections $E : C_{2\pi} \to \mathrm{range}L$ and $(I-E) : C_{2\pi} \to \ker L$. The equation $E\Phi = 0$ can be solved to yield a locally unique solution $w(v, \alpha, \tau)$ (with $w : \ker L \times \mathbb{R}^\ell \times \mathbb{R} \to M$), so that we are left with the equation $\phi(v, \alpha, \tau) = 0$, where

$$\phi : \ker L \times \mathbb{R}^\ell \times \mathbb{R} \to \ker L, \quad \phi(v, \alpha, \tau) = (I-E)\Phi(v, w(v, \alpha, \tau), \alpha, \tau).$$

Because the decompositions in Proposition 8(c) are S^1-invariant, it follows that (show in assessed work!)

$$\theta \cdot \phi(v, \alpha, \tau) = \phi(\theta \cdot v, \alpha, \tau).$$

This S^1-equivariance of the reduced equation $\phi = 0$, will help us a lot to solve it. Importantly, it enforced the form of ϕ (proof is left for assessed work):

Proposition 9 *In the coordinates of Proposition 8(b) on* $\ker L$, *the reduced mapping* ϕ *has the following form:*

$$\phi(x, y, \alpha, \tau) = p(x^2 + y^2, \alpha, \tau) \begin{pmatrix} x \\ y \end{pmatrix} + q(x^2 + y^2, \alpha, \tau) \begin{pmatrix} -y \\ x \end{pmatrix},$$

where $p(0,0,0) = q(0,0,0) = \frac{dp}{d\tau}(0,0,0) = 0$ *and* $\frac{dq}{d\tau}(0,0,0) = -1$.

Proof of Theorem 3

From the explicit form of ϕ in Proposition 9, we see that $\phi = 0$ if and only if either $x = y = 0$, or $p = q = 0$. The solution $x = y = 0$ refers to the equilibrium solution. The remaining solutions of $p = q = 0$ refer to nearby periodic solutions. Due to the S^1-equivariance, p and q are functions of $z^2 = x^2 + y^2$ (with $z \geq 0$). We thus have

$$\phi(x, y, \alpha, \tau) = 0 \ \wedge \ (x, y) \neq (0, 0) \quad \Leftrightarrow \quad p(z^2, \alpha, \tau) = q(z^2, \alpha . \tau) = 0.$$

Fortunately, because $\frac{dq}{d\tau}(0,0,0) \neq 0$, $q(z^2,\alpha,\tau) = 0$ may be solved for τ: there exists $\tau(z^2,\alpha)$ for (z^2,α) near $(0,0)$. We define

$$r(z^2,\tau) := p(z^2,\alpha,\tau(z^2,\alpha)).$$

We now have $x = y = 0$ ($\Leftrightarrow$ $z = 0$) or $r(z^2,\tau) = 0$. This implies:

$$\phi = 0 \quad \Leftrightarrow \quad g(z^2,\tau) = zr(z^2,\tau) = 0.$$

Remark 10 Compare the reduced equation with the $\dot{r}$-component of the Hopf bifurcation normal form.

Proof of Theorem 4

Suppose that $\frac{dr}{d\lambda}(0,0) \neq 0$, then - by the Implicit Function Theorem - $r = 0$ can be solve by $\lambda = \mu(z^2,\alpha')$, where $\alpha' = (\alpha_2,\ldots,\alpha_\ell)$. In other words, if $\frac{dr}{d\lambda}(0,0) \neq 0$, the differential equation (1) has an ℓ-parameter family of periodic solutions near 0, with period close to 2π.

The proof now relies on the fact that

$$\frac{dr}{d\lambda}(0,0) = \frac{d\sigma}{d\lambda}(0).$$

We leave the proof of the latter as homework.

Proof of Theorem 5

Theorem 5 deals with how the periodic solutions depend on the parameter λ. To address this issue we consider the power series expansion of $\lambda = \mu(z^2,\alpha')$:

$$\lambda = \mu(z^2,\alpha') = \mu_0(\alpha') + \mu_2(\alpha')z^2 + \mu_4(\alpha')z^4 + \ldots$$

where $\mu_0(0) = 0$. If $\mu_2(0) \neq 0$, then for each λ this equation has precisely one or no solutions for z^2 close to zero, as

$$z^2 \simeq \frac{\lambda - \mu_0(\alpha')}{\mu_2(\alpha')}.$$

By implicit differentiation of $r(z^2,(\mu(z^2,\alpha'),\alpha')) = 0$ one finds (show!) that

$$\mu_2(0) = -\frac{\frac{dr}{dz^2}(0,0)}{\frac{dr}{d\lambda}(0,0)}.$$

We know already that $\frac{dr}{d\lambda}(0,0) \neq 0$, hence

$$\frac{dr}{dz^2}(0,0) \quad \Rightarrow \quad \mu_2(0) \neq 0.$$

Assessed Homework

Write an essay detailing the proofs of the results in the Hopf bifurcation section, using Liapunov-Schmidt reduction. The handout from Golubitsky & Schaeffer might be sufficient, though you may use also any additional literature from the library (provide references if you do). You will be asked to discuss your assessed work with me later in person (so make sure you understand what you wrote down!).

0.0.1 Literature:

M. Golubitsky and D. Schaeffer. *Group theoretical methods in bifurcation theory. Vol 1* (1996), chaper VIII: Hopf Bifurcation.