Nonlinear Normal Form Theory

(notes for M4A35)

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Introduction

We consider a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$, with $f(0) = 0$, and ask the question how we can transform into a standard *normal* form by a well-chosen coordinate transformation.

If f is linear, we have the option of chosing for the Jordan normal form. We now consider the case that f is nonlinear.

We write f as a Taylor expansion into homogeneous (polynomial) maps f_i a degree ^{i} satisfying $f_i(ax) = a^i f_i(x)$ for all $x \in \mathbb{R}^n$ and $a \in \mathbb{R}$:

$$f = (\sum_{i=1}^{k} f_i) + R_k,$$

where R_k is a rest term satisfying $\lim_{\varepsilon\to 0} |R_k(\varepsilon)\varepsilon^{-k-1}| < \infty$.

Our aim is to *normalize* f order-by-order, starting with the linear part f_1 upto any finite order k. The linear part may be chosen to be in Jordan normal form (obtainable by using a linear coordinate transformation). We seek a scheme of coordinate transformations that can be used to normalize f_j without changing f_i for all $i < j$. We discuss this procedure in detail below.

We will here first summarize our main results. We define $L := f_1 = (df)_0$. Our main theorem is the following:

Theorem 1 (Normal form) *Let f be as above and k be some natural number ($< \infty$). Then there exists a coordinate transformation transforming f to $\tilde{f}$, so that for all $i = 2, \ldots, k$*

$$\mathrm{ad}_{L^T}(\tilde{f}_i) = 0.$$

This theorem has the follwing important consequence. In order to state this, we first recall an elementary result from Linear Algebra:

Lemma 2 (Jordan-Chevalley decomposition) *Any matrix $L \in \mathrm{gl}(n, \mathbb{R})$ can be decomposed uniquely as*

$$L = S + N,$$

where

(a) $SN = NS$.

(b) S is semi-simple, ie TST^{-1} is diagonal for some $T \in \mathrm{gl}(n, \mathbb{C})$.

(c) N is nilpotent, ie $N^m = 0$ for some integer $m < \infty$.

Proof: L is conjugate to a matrix in Jordan normal form. The spliting is obvious for this matrix in Jordan normal form: S consists of the diagonal components and N is the remaining upper triangular matrix. Now, if a matrix is semi-simple or nilpotent, then so is any matrix conjugate to it. ∎

Now we can state the second theorem:

Theorem 3 (Normal form symmetry) *Let f be the vector field after centre manifold reduction, and k be some natural number ($< \infty$). Then there exists a coordinate transformation transforming f to $\tilde{f}$, so that for all $\tilde{f}^{(k)} = \sum_{i=1}^{k} \tilde{f}_i$ satisfies*

$$\tilde{f}^{(k)} g = g \tilde{f}^{(k)},$$

for all

$$g \in G = \overline{\{\exp(tS^T) \mid t \in \mathbb{R}\}},$$

where S is the semi-simple part of L.

*In other words, $\tilde{f}^{(k)}$ is G-*equivariant*. In this context, the Lie group G, generated by S, is called the (formal) symmetry group of the normal form.*

Example 4 Let f be a vector field in $\mathbb{R}^2$, with linear part

$$L = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then,

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \mid a \in \mathbb{R} \right\} \cong \mathbb{R}.$$

Let $x = (u, v)$ be the standard Carthesian coordinates for $\mathbb{R}^2$, Then, for any value of k, f an be put in the following normal form $\tilde{f}$:

$$\dot{x} = \tilde{f}^{(k)}(x) \quad \Leftrightarrow \quad \left\{ \begin{array}{lcl} \dot{u} & = & v + g(u), \\ \dot{v} & = & h(u) + vg(u)/u, \end{array} \right.$$

where g and h are polynomials of degree k, and $h(0) = g(0) = 0$.

Example 5 Let f be a vector field in $\mathbb{R}^2$, with linear part

$$L = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then,

$$G = \left\{ \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \mid \alpha \in [0, 2\pi\rangle \right\} = SO(2) \cong S^1.$$

Let $x = (u, v)$ be the standard Carthesian coordinates for $\mathbb{R}^2$. Because of the G-equivariance, it is advantageous to write the normal form equations in polar coordinates $(r, \phi)_p$, where $r^2 = u^2 + v^2$ and $\tan\phi = v/u$.

Because of the normal form symmetry, in polar coordinates the normal form vector field does not depend explicitly on ϕ, and takes the form:

$$\dot{x} = \tilde{f}^{(k)}(x) \quad \Leftrightarrow \quad \left\{ \begin{array}{lcl} \dot{r} & = & rg(r^2), \\ \dot{\phi} & = & h(r^2), \end{array} \right.$$

where g and h are polynomials, and $h(0) = 1$.

Homework 6 (a) Verify the claims in Example 4.

(b) Verify the claims in Example 5, and determine the degrees of the polynomials g and h as a function of k.

Proofs of Theorem 1 and Theorem 3

We seek coordinate transformations T transforming f to $\tilde{f}$, where $f_i = \tilde{f}_i$, for all $i < j$. We propose to consider an invertible transformation T, whose Taylor expansion has the form

$$T = I + P_j + R_j,$$

where $P_j : \mathbb{R}^n \to \mathbb{R}^n$ is homogeneous of degree j, and R_j is a higher order rest term. It is clear that invertibility of T does not put any restriction on P_j. Correspondingly, we have the following expansion of T^{-1}:

$$T = I - P_j + \tilde{R}_j,$$

where $\tilde{R}_j$ is some rest term.

We now need to examine how f transforms under a coordinate transformation T. Let $y = T(x)$. We have

$$\frac{d}{dt} x = f(x).$$

Then,

$$\begin{aligned} \frac{d}{dt} T^{-1}(y) &= f(T^{-1}(y)), \\ (dT^{-1})_y \frac{d}{dt} y &= f(T(y)), \\ \frac{d}{dt} y &= ((dT^{-1})_y)^{-1} \cdot f(T^{-1}(y)). \end{aligned}$$

Hence

$$\tilde{f} = (dT) \cdot f \circ T^{-1} = dT \cdot f \circ T^{-1},$$

where $\circ$ denotes - as usual - functional composition.

We now consider the Taylor expansion of $\tilde{f}$, if $T = I + P_j + \ldots$.

$$\tilde{f} = (I + dP_j + \ldots) \cdot f \circ (I - P_j + \ldots) = f + dP_j \cdot f - df \cdot P_j + \ldots = (\sum_{i=1}^{j-1} f_i) + f_j + dP_j \cdot L - L \cdot P_j + \ldots,$$

where in the last equations the remaining terms are of order $j + 1$. Hence,

$$\begin{array}{lcl} \tilde{f}_i & = & f_i, \;\; \text{if } i < j, \\ \tilde{f}_i & = & \ldots, \;\; \text{if } i > j, \end{array}$$

and

$$\tilde{f}_j = f_j - L \cdot P_j + dP_j \cdot L := f_j - \mathrm{ad}_L(P_j).$$

In order to understand how we can transform f_j, we thus need to study the image of the operator ad_L as defined above. We define $\mathcal{H}_j$ as the (finite dimensional!) linear vector space of homogeneous maps of degree j, of $\mathbb{R}^n$ to itself:

$$\mathrm{ad}_L : \mathcal{H}_j \to \mathcal{H}_j, \quad \mathrm{ad}_L(P_j) = L \cdot P_j - (dP_j) \cdot L.$$

We aim to write

$$\mathcal{H}_j = \mathrm{Im}_{\mathcal{H}_j}(\mathrm{ad}_L) \oplus \mathcal{C}_j.$$

Clearly, we can always choose $\tilde{f}_j$ in the complement $\mathcal{C}_j$. However, $\mathcal{C}_j$ is not unique, so we may choose a convenient complement. We will show that we may choose $\mathcal{C}_j$ to equal $\mathrm{Ker}_{\mathcal{H}_j}(\mathrm{ad}_{L^T})$.

Lemma 7 $\mathcal{H}_j = \mathrm{Im}_{\mathcal{H}_j}(\mathrm{ad}_L) \oplus \mathrm{Ker}_{\mathcal{H}_j}(\mathrm{ad}_{L^T})$.

Proof: We construct $\mathcal{C}_j$ as the orthogonal complement to $\mathrm{Im}_{\mathcal{H}_j}(\mathrm{ad}_L)$ in $\mathcal{H}_j$ with respect to the scalar product $\ll \cdot, \cdot \gg$:

$$\ll P(x), Q(x) \gg = \sum_{i=1}^{n} < p_i(x), q_i(x) >,$$

where $P, Q \in \mathcal{H}_j$, with $P = (p_1, \ldots, p_n)$ and $Q = (q_1, \ldots, q_n)$, and $< \cdot, \cdot >$ is a symmetric bilinear form, defined by

$$< x^\alpha, x^\beta > := \delta_{\alpha,\beta} \alpha!,$$

where $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ and $\alpha! := \alpha_1! \ldots \alpha_n!$. It is readily verified that

$$< p_i(x), q_i(x) > = \left[p_i(\frac{\partial}{\partial x}) q_i(x) \right]_{x=0}.$$

We will now obtain an expression for the orthogonal complement. Let $A, B \in \mathcal{H}_j$, then

$$\ll A, \mathrm{ad}_L(B) \gg := \ll \mathrm{ad}_L^*(A), B \gg,$$

where ad_L^* is (by definition!) the transpose of ad_L with respect to the chosen scalar product. If $\mathcal{C}_j$ is the orthogonal complement of $\mathrm{Im}_{\mathcal{H}_j}(\mathrm{ad}_L)$ in $\mathcal{H}_j$, then

$$\mathcal{C}_j = \{ A \in \mathcal{H}_j \mid \langle A, \mathrm{ad}_L(B) \rangle = 0, \quad \forall\ B \in \mathcal{H}_j \}.$$

Now,

$$0 = \langle A, \mathrm{ad}_L(B) \rangle = \langle \mathrm{ad}_L^*(A), B \rangle.$$

If the latter expression is equal to 0 for all $B \in \mathcal{H}_j$, then $\mathrm{ad}_L^*(A) = 0$, ie $A \in \mathrm{Ker}_{\mathcal{H}_j}(\mathrm{ad}_L^*)$, so that

$$\mathcal{C}_j = \mathrm{Ker}_{\mathcal{H}_j}(\mathrm{ad}_L^*).$$

This choice of complement is known as the *Fredholm alternative.*

Finally, in order to prove the lemma, we need to show that with our choice of scalar product in $\mathcal{H}_j$, $\mathrm{ad}_L^* = \mathrm{ad}_{L^T}$. We assert that, with $A \in \mathrm{Gl}(n, \mathbb{R})$,

$$(a) \quad \ll AP(x), Q(x) \gg = \ll P(x), A^T Q(x) \gg, \quad (b) \quad \ll P(Ax), Q(x) \gg = \ll P(x), Q(A^T x) \gg .$$

We first show that (a) and (b) indeed lead to the proof. Namely, if (a) and (b) hold, then

$$\ll \exp(sL)P(\exp(-sL)x), Q(x) \gg = \ll P(x), \exp(sL^T)Q(\exp(-sL^T)x) \gg,$$

for all s. Differentiating the left and right hand sides of this equation with respect to s and then setting $s = 0$ yields

$$\ll \mathrm{ad}_L P(x), Q(x) \gg = \ll P(x), \mathrm{ad}_{L^T} Q(x) \gg,$$

so that indeed $(\mathrm{ad}_L)^* = \mathrm{ad}_{L^T}$.

It remains to verify the assertions (a) and (b) above.

To verify (a), we write out the scalar product, using the bilinearity of $< \cdot, \cdot >$. We let $a_{i,m}$ denote the matrix-elements of A.

$$\begin{aligned}
\ll AP(x), Q(x) \gg &= \sum_{i=1}^{n} < \sum_{m=1}^{n} a_{i,m} p_m(x), q_i(x) >, \\
&= \sum_{i=1}^{n} \sum_{m=1}^{n} a_{i,m} < p_m(x), q_i(x) >, \\
&= \sum_{i=1}^{n} \sum_{m=1}^{n} < p_m(x), a_{i,m} q_i(x) >, \\
&= \sum_{m=1}^{n} < p_m(x), \sum_{i=1}^{n} a_{i,m} q_i(x) >, \\
&= \ll P(x), A^T Q(x) \gg .
\end{aligned}$$

Now we verify (b). It suffices to show that

$$< p_i(Ax), q_i(x) > = < p_i(x), q_i(A^T x) > .$$

We use the definition of $< \cdot, \cdot >$, and let $x = A^T y$:

$$\begin{aligned}
< p_i(Ax), q_i(x) > &= [p(A\frac{\partial}{\partial x})q(x)]_{x=0}, \\
&= [p(\frac{\partial}{\partial y})q(A^T y)]_{y=0}, \\
&= [p(\frac{\partial}{\partial x})q(A^T x)]_{x=0}, \\
&= < p_i(x), q_i(A^T x) > .
\end{aligned}$$

In the second step we use that if $x = A^T y$, $\frac{\partial}{\partial x} = A^{-1}\frac{\partial}{\partial y}$. The third step is nothing more than relabelling coordinates. ∎

Proof of Theorem 1 This follows directly from Lemma 7. ∎

It remains to prove Theorem 3. It relies on the following elementary lemma:

Lemma 8 *Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be linear with Jordan-Chevalley decomposition $L = S + N$. Then*

$$\mathrm{Ker}_{\mathcal{H}_j}(\mathrm{ad}_L) = \mathrm{Ker}_{\mathcal{H}_j}(\mathrm{ad}_S) \cap \mathrm{Ker}_{\mathcal{H}_j}(\mathrm{ad}_N).$$

Proof: We decompose the proof into two parts. First we show that if $L = S + N$ is the Jordan-Chevalley decomposition of L, then

$$\mathrm{ad}_L = \mathrm{ad}_S + \mathrm{ad}_N$$

is the Jordan-Chevalley decomposition of ad_L. From the definition of ad_N, it is easy to see that ad_N is nilpotent if N is. Also, since $NS = SN$, it is easily shown that $\mathrm{ad}_S\mathrm{ad}_N = \mathrm{ad}_N\mathrm{ad}_S$. It remains to be verified that ad_S is semi-simple. This follows by verifying that ad_S is diagonal if we choose an appropriate basis for $\mathcal{H}_j$.

We first choose S to be diagonal (for this, we need to allow possible complex coordinates), so that on an orthogonal basis $\{e_j\}$, we have $Se_j = \lambda_j e_j$, where λ_j is the corresponding eigenvalue of S. We first consider the case that $j = 1$. We may choose the basis of $\mathcal{H}_1 \cong \mathrm{gl}(n, \mathbb{R})$ as $\{m_{i,j} \mid i, j = 1, \ldots n\}$ where $m_{i,j}e_k = \delta_{j,k}e_i$ (with $\delta_{j,k} = 0$ if $j \neq k$ and $\delta_{j,j} = 1$). Then $\mathrm{ad}_L S(m_{i,j}) = (\lambda_i - \lambda_j)m_{i,j}$, so ad_L is diagonal on this basis and hence semi-simple.

If $j > 1$, the procedure is similar. Again, we take S to be diagonal, and choose the natural basis of $\mathcal{H}_j$ as being $E_i^\alpha(x) = x^\alpha e_i$, with e_i a unit basis vector of $\mathbb{R}^n$, and $\alpha = (\alpha_1, \ldots, \alpha_n)$ as above. Then it follows that $\mathrm{ad}_L(E_i^\alpha) = c_i^\alpha E_i^\alpha$. We thus establish that ad_S is semi-simple on$\mathcal{H}_j$.

To conclude the proof, we claim that for any linear operator L with Jordan-Chevalley decomposition $L = S + N$, we have

$$\mathrm{Ker}L = \mathrm{Ker}S \cap \mathrm{Ker}N.$$

Applying this claim to the Jordan-Chevalley decomposition of ad_L yields the lemma.

It remains to verify the validity of the last claim. Clearly, $\mathrm{Ker}L \supset \mathrm{Ker}S \cap \mathrm{Ker}N$. If $\mathrm{Ker}L \subset \mathrm{Ker}S$ then also $\mathrm{Ker}L \subset \mathrm{Ker}N$, since $N = L - S$, and hence $\mathrm{Ker}L \subset \mathrm{Ker}S \cap \mathrm{Ker}N$, proving the assertion. It remains to be shown that $\mathrm{Ker}L \subset \mathrm{Ker}S$. Let $v \in \mathrm{Ker}L$. Then $S^m v = 0$ for some m, since $S^m v = (L - N)^m v = N^m v$. Since S is semi-simple, $S^m v = 0$ if and only if $Sv = 0$. ∎

Proof of Theorem 3 First, we note that $L \in \mathrm{Ker}(\mathrm{ad}_{S^T})$.

We apply Lemma 8, to obtain that $\mathrm{Ker}_{\mathcal{H}_j}(\mathrm{ad}_{L^T}) \subset \mathrm{Ker}_{\mathcal{H}_j}(\mathrm{ad}_{S^T})$. Then we use the fact that

$$f \in \mathrm{Ker}_{\mathcal{H}_j}(\mathrm{ad}_{S^T}) \quad \Leftrightarrow \quad f \exp(tS^T) = \exp(tS^T) f, \ \forall t \in \mathbb{R}.$$

Taking the closure of this set of transformations, yields the symmetry group G in the Theorem. ∎

Homework 9 (a) Verify that $\ll \cdot, \cdot \gg$ is a scalar product on $\mathcal{H}_j$.

(b) Verify that if $x = A^T y$, with $x, y \in \mathbb{R}^n$ and $A \in \mathrm{Gl}(n, \mathbb{R})$, $\frac{\partial}{\partial x} = A^{-1}\frac{\partial}{\partial y}$.

(c) Verify the proof of Theorem 3 in detail. In particular, prove that

$$f \in \mathrm{Ker}_{\mathcal{H}_j}(\mathrm{ad}_{S^T}) \quad \Leftrightarrow \quad fg = gf, \ \forall g \in G,$$

where G as in the Theorem 3.

(d) Verify the details of the claim that ad_S acts diagonally on $\mathcal{H}_j$ when the proposed basis is used. Calculate the eigenvalues of ad_S as a function of the eigenvalues of S, and α.

Centre Manifold Reduction

In bifurcation theory, there exists an important reduction method for bifurcation problems of equilibria, known as *Centre Manifold Reduction.* We wil sketch the result here (the proof is beyond our reach, for the moment).

We can now state the Centre Manifold Theorem:

Theorem 10 (Centre Manifold) *Let f be a vector field in $\mathbb{R}^n$, with $f(0) = 0$. Let $L := (df)_0$, with Jordan-Chevalley decomposition $L = S + N$. Let $E^s, E^u, E^c \subset \mathbb{R}^n$ correspond to the union of eigenspaces of the semi-simple part S whose corresponding eigenvalues have respectively negative real part, positive real part, or lie on the imaginary axis $i\mathbb{R}$. (Note that the eigenspaces of S correspond to the* generalized *eigenspaces of L.)*

Then, locally, in a (small) neighbourhood of 0 the vector field is equivalent to

$$\begin{cases} \dot{x_s} &= L|_{E^s} x_s, \\ \dot{x_u} &= L|_{E^u} x_u, \\ \dot{x_c} &= f_c(x_c), \end{cases}$$

where $(x_s, x_u, x_c) \in E^s \oplus E^u \oplus E^c$. Here the notion of equivalence is that of relation by a local change of coordinates.

Hence, when studying local bifurcations, ie local changes of the flow near the equilibrium 0, it suffices to study the reduced vector field f_c, which is known as the *Centre Manifold Reduction.*

Theorem 11 (Normal Form Symmetry after Centre Manifold Reduction) *Let f be a vector field after centre manifold reduction, satisfying $f(0) = 0$, and k be some natural number ($< \infty$). Then there exists a coordinate transformation transforming f to $\tilde{f}$, so that $\tilde{f}^{(k)} := \sum_{i=1}^{k} \tilde{f}_i$ is G-equivariant, where*

$$g \in G = \overline{\{\exp(tS) \mid t \in \mathbb{R}\}} \cong \mathbb{T}^m,$$

for some m, where S is the semi-simple part of $L := (df_c)_0$, and $\mathbb{T}^m$ is an m-torus (the direct product of m circles $S^1 \times \ldots \times S^1$).

Proof: The proof is a special case of Theorem 3. After Centre Manifold Reduction, all the eigenvalues of the semi-simple part of $(df)_0$ lie on the imaginary axis. Hence, $S^T = -S$, and $\exp(tS)$ is a rotation, generating a torus. ∎

Local bifurcations: parametrized normal forms

Sofar, we have just changed coordinates to bring our vector field in a nice form. Now we want to study local bifurcations in a parametrized family of vector fields;

$$\frac{d}{dt} x = f_\mu(x).$$

Here $\mu \in \mathbb{R}^\ell$ is the parameter. That is, we want to study the local changes in the solutions near a nonhyperbolic equilibrium point when we vary parameters.

Our strategy is as follows:

(1) We fix a type of nonhyperbolic equilibrium (by defining some degeneracy conditions on the linear part of the vector field).

(2) We perform a Centre Manifold Reduction (as this will provide all the information we are after), so that from now on we only need to study a vector field with a linear part that has all its eigenvalues on the imaginary axis $i\mathbb{R}$.

(3) We consider the reduced vector field and assume that its nonlinear terms are nondegenerate ("typical"), and choose parameters so as to achieve a miniversal deformation of the linear part of the vector field.

(4) We put the reduced parametrized vector field in normal form, as in Theorem 11.

(5) We study local solutions of (a truncated form of) the parametrized normal form vector field.

(6) Study the effect of the higher-order remainder terms on the results of the study in the previous step.

In step (4), it is important that also in the case of a parametrized vector fields, we can transform into a normal form with the prescribed normal form symmetry, ie inside $\mathrm{Ker}(\mathrm{ad}_S)$, where S is the semi-simple part of $L = (df)_0$ at parameter value $\mu = 0$.

Theorem 12 (Parametrized Normal Form after Centre Manifold Reduction) *Let f_μ be a parametrized vector field after centre manifold reduction, satisfying $f_0(0) = 0$, and k be some natural number ($< \infty$). Then there exists a coordinate transformation transforming f_μ to $\tilde{f}_{\tilde{\mu}}$, so that $\tilde{f}_{\tilde{\mu}}^{(k)} := \sum_{i=1}^{k} (\tilde{f}_{\tilde{\mu}})_i$ is G-equivariant, where*

$$g \in G = \overline{\{\exp(tS) \mid t \in \mathbb{R}\}} \cong \mathbb{T}^m,$$

for some m, where S is the semi-simple part of $L := (df_0)_0$.

Proof: First we do a parameter transformation. Note that the centralizer miniversal deformation is already such that the parameters appear in the linear part in such a way that it remains inside $\mathrm{Ker}(\mathrm{ad}_S)$. Hence, after a parameter transformation we may assume that the linear part of f_μ lies inside $\mathrm{Ker}(\mathrm{ad}_S)$.

Consider f_μ as a vector field in $\mathbb{R}^{n+\ell}$, treating the parameter as a variable, by augmenting $\dot{x} = f_\mu(x)$ with the equations $\dot{\mu} = 0$. Then the rest of the proof follows the proofs of Theorem 1 and Theorem 11. ∎

The last step (6) is crucial in order to appreciate the (formal) normal form approach towards local bifurcation problems. We cannot go into detail here on much of the complicated theory discussion this aspect, but we quote only a (recent!) result by M. Field, assuring that all branches of hyperbolic equilibria, periodic solutions, and (normally) hyperbolic invariant tori, are finitely determined, ie these solution branches can be detected by the study of bifurcations in (truncated) normal forms.

Homework 13 (a) Work out the details of the proof of Theorem 12.

(b) Apply Theorem 12, to obtain normal forms for linear (versal) deformations of the vector fields in Example 4 and Example 5. Any idea how what the solutions close to $\mu = 0$ look like for the vector field? [open-ended question, see what you can get (may use some software to draw phase portraits, for instance].

Elementary local bifurcations: codimension one

We now can discuss the most elementary local bifurcations, those that persistently arise in one-parameter families of dynamical systems.

Saddle-node bifurcation (Turning point)

We first consider the case when there is only one eigenvalue on the imaginary axis. Consequently, this eigenvalue must be equal to zero.

After Centre Manifold Reduction we thus get the following normal form for f:

$$f(x) = bx^2 + cx^3 + \ldots.$$

Here, we need to be careful with the deformation. Namely, here it is not sufficient to consider a deformation of the linear part only, as it would imply that 0 remains an equilibrium. We have to allow for the possibility that the equilibrium disappears (we always need to when we have an eigenvalue equal to zero). Hence, our deformation will have the form

$$f_\mu(x) = \mu + a(\mu)x + b(\mu)x^2 + \ldots,$$

where $a(\mu) = 0$, since we assumed $(df_0)_0 = 0$. We also assume, as a nondegenracy condition, $b(\mu) \neq 0$.

We have $(\frac{\partial}{\partial\mu} f_\mu)_0 = 1$. Hence by the IFT, we have a locally unique branch $x(\mu)$, with $x(0) = 0$, so that $f_\mu(x(\mu)) = 0$. Moreover, at $x = \mu = 0$, this curve satisfies $d\mu/dx = 0$, but $d^2\mu/dx^2 = b$ which we may assume (as a nondegeneracy condition) to be nonzero. Hence $\mu(x)$ locally has the shape of a parabola, and $x = \mu = 0$ is a turning point.

This type of bifurcation is also known as a saddle-node bifurcation, as the one-dimensional vector field displays the merger of two equilibria one unstable (saddle) point, and one stable point (node).

Hopf bifurcation

We now consider the case that there are two complex conjugate eigenvalues on the imaginary axis. Hence, after Centre Manifold Reduction, we end up with a vector field in $\mathbb{R}^2$ with linear part as in Example 5 (in general, after an appropriate additional rescaling of time). Taking into account the two-dimensional miniversal deformation of the linear part, we obtain:

$$\begin{cases} \dot{r} &= r(\mu_2 + a(\mu)r^2 + \ldots), \\ \dot{\phi} &= \mu_1 + 1 + b(\mu)r^2 + \ldots. \end{cases}$$

Here $\mu = (\mu_1, \mu_2)$ is the parameter. Because the linear part has codimension two, it appears as if this situation only arises persistenly in two-parameter families of vector fields. However, one of the parameters (here μ_1) only slightly shifts the imaginary part of the eigenvalue of the linear part. This corresponds to the fact that the μ_1-part of the deformation merely shifts the position where the eigenvalues intersect the imaginary axis. The precise intersection point turns out to be irrelevant for the bifurcation (as it can be changed by a rescaling of the time variable).

We notice that the $\dot{r}$ equation is independent of ϕ. By the IFT, assuming the natural nondegeneracy condition that $a(0) \neq 0$, we find a one-sided branch $r^2(\mu)$ for which $\dot{r} = 0$. At the same time $\dot{\phi} \neq 0$ for μ small enough. Hence, the solution curve $r^2(\mu)$ corresponds to a branch of periodic solutions, with period close to 2π.

Persistence of this branch under higher order perturbations is guaranteed by the finite determinacy results of Field.

Homework 14 (a) Verify the details of the turning point bifurcation. Verify the claim that there is a change of stability of the equilibria on the branch $\mu(x)$, at $x = 0$.

(b) Verify the details of the Hopf bifurcation. Analyze what happens when the vector field has eigenvalues $\pm i\omega$ on the imaginary axis, and we do not rescale time to set $\omega = 1$.