Shift dynamics in a neighbourhood of a T-point containing two saddle-foci

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Abstract

We study dynamics near T-points containing two saddle foci in both general and reversible systems. By using Lin's method we prove for each N the existence of dynamics which is topologically conjugated to the full shift on N symbols. Moreover we describe the set of nearby k-homoclinic and k-heteroclinic orbits.

With that we extend results of Bykov [1, 2] for general systems and generalise them to higher dimensinsions. For reversible systems we show that the results of Lamb, Teixeira and Webster [10] can be carried over to higher dimensinsions.

1 Introduction

Webster et al. studied in $\mathbb{R}^3$ the dynamics near a reversible T-point containing two saddle-foci, [10]. A T-point is a heteroclinic cycle connecting two hyperbolic equilibria p_1 and p_2 which consists of a non-robust and of a robust heteroclinic orbit, see [6, 7]. So, in $\mathbb{R}^3$ one heteroclinic orbit, say Γ_1, lies in the intersection of one-dimensional stable and unstable manifolds of say p_1 and p_2, and the other heteroclinic orbit Γ_2 lies in the transversal intersection of the corresponding two-dimensional unstable and stable manifolds of these fixed points. We refer to Figure 1 for a visualisation. One of the main results in [10] is the verification of shift dynamics on N-symbols for every $N \geq 2$. In reversible systems with one-dimensional fixed point space of the underlying involution the orbit Γ_1 is of codimension one, see also Section 4, while in the general situation the orbit Γ_1 is a codimension two connection.

Bykov [1, 2] studied T-points in $\mathbb{R}^3$ as depicted in Figure 1 in general systems. There the considerations are mainly restricted to nearby 1-periodic, 1-heteroclinic and 1-

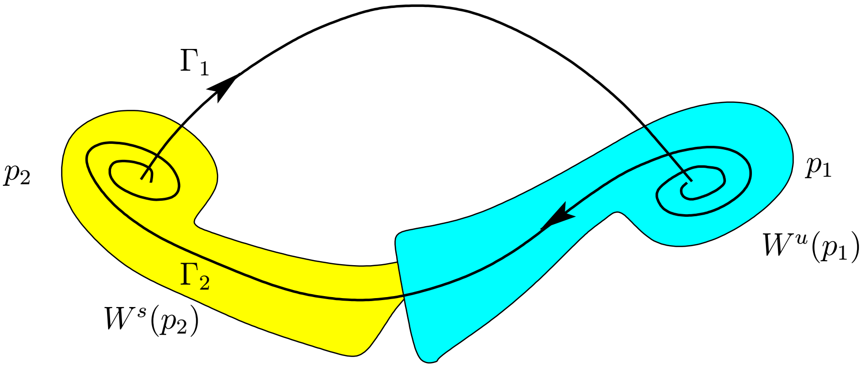

Figure 1: T-point in $\mathbb{R}^3$.

homoclinic orbits, see also Definition 1.1 below. Then by means of earlier results by Shilnikov and Newhouse one can conclude on more sophisticated dynamics.

Here our objectives are twofold: to generalise the results in [10] to higher dimensions, and to show that similar results hold true also in general systems. In particular we make this complicated dynamics more explicit. So, for the unperturbed system we proof (for both general and reversible systems) for each $N \in \mathbb{N}$ the existence of dynamics which is topologically conjugated to the full shift on N symbols. But under perturbation the shift dynamics survives only for finitely many N – although it is individually (for each N) structurally stable. We refer to Theorem 1.2 and Theorem 4.1 for a precise statement. The shift dynamics exists independently on the proportion of the real parts of the principal stable and unstable eigenvalues of both the equilibrium p_1 and the equilibrium p_2. This is a remarkable difference to the Shilnikov scenario for homoclinic orbits to a saddle-focus. Furthermore we consider k-homoclinic and k-heteroclinic orbits for $k > 1$.

In this paper we use Lin's method to gain the described results; this is an alternative way in comparsion with [10] or [1, 2] for the treatment of such T-points. We refer to Section 2 for a more detailed explanation of this method. Here we are content with mentioning that this method has been proved to be a powerful tool for detecting particular types of solutions such as periodic or connecting ones. This approach for searching orbits nearby a heteroclinic chain was introduced by Lin [11]. Later Sandstede [17] modified the procedure to get better estimates of the jump. We refer also to [9], where Lin's method has been extended to discrete systems and simultaneously it was allowed that the heteroclinic chain may consist of degenerate connecting orbits. In all these papers the equilibria which are involved in the heteroclinic chain have the same fixed point index. So, due to the different fixed point indices of p_1 and p_2 (see below) we have to make sure that the method can be applied to the problem under consideration. However, this method is also well situated to prove the existence of shift dynamics – and, what is more, it can be used to prove topological conjugacy (to the full shift on N symbols). Finally we want to remark that this approach allows an easy generalisation to higher dimensions as carried out in this paper.

Next we describe the exact setting of the problem and formulate our main theorem for general vector fields. The special features which arise in reversible vector fields we have devoted a separate section, see Section 4. We consider a two-parameter family of vector fields $f : \mathbb{R}^n \times \mathbb{R}^2 \to \mathbb{R}^n$, f smooth:

$$\dot{x} = f(x, \mu). \tag{1.1}$$

At $\mu = 0$ there are two hyperbolic fixed points p_1 and p_2, which are of saddle-focus type. The fixed points p_1, p_2 will persist and will remain hyperbolic for all sufficiently small $|\mu|$. Using C^∞ bump function transformations we can make $p_1(\mu), p_2(\mu)$ constant. Hence we shall assume this transformation has been carried out and write $(p_1(\mu), p_2(\mu)) = (p_1, p_2)$ for all sufficiently small $|\mu|$. Throughout this paper we denote the stable manifold of the equilibrium p_i by $W^s(p_i, \mu)$. For the sake of shortness we write just $W^s(p_i)$ for $W^s(p_i, 0)$. In the same manner we use $W^u(p_i, \mu)$ and $W^u(p_i)$ to denote the corresponding unstable manifolds.

Further we assume that at $\mu = 0$ there exits a T-point. That means that there exists a heteroclinic cycle Γ consisting of p_1, p_2, a heteroclinic orbit Γ_1 connecting p_2 and p_1 (in forward time) and another heteroclinic orbit Γ_2 between p_1 and p_2. The corresponding heteroclinic solutions we denote by $q_i(\cdot)$, $i = 1, 2$. The following assumptions characterise the T-point more closely. We assume that along Γ_2 the unstable manifold of p_1 and the stable manifold of p_2 intersect transversally

$$W^s(p_2) \pitchfork_{q_2(0)} W^u(p_1), \tag{1.2}$$

and what is more

$$\dim T_{q_2(0)} W^s(p_2) + \dim T_{q_2(0)} W^u(p_1) = n + 1. \tag{1.3}$$

Note that due to this assumption the fixed points p_1 and p_2 have different fixed point indices.

Also we assume that along Γ_1 the unstable manifold of p_2 and the stable manifold of p_1 intersect as clean as possible

$$T_{q_1(0)} W^s(p_1) \cap T_{q_1(0)} W^u(p_2) = \operatorname{span}\{f(q_1(0), 0)\}. \tag{1.4}$$

Finally we assume that the principal stable eigenvalue λ_1^s of $D_1 f(p_1, 0)$ is real and simple and that the principal unstable eigenvalues λ_1^u and $\overline{\lambda_1^u}$ of $D_1 f(p_1, 0)$ are non-real and simple; and *vice versa* the principal unstable eigenvalue λ_2^u of $D_1 f(p_2, 0)$ is real and simple and the principal stable eigenvalues λ_2^s and $\overline{\lambda_2^s}$ of $D_1 f(p_2, 0)$ are non-real and simple. For the corresponding unstable eigenvalues $\lambda_1^u(\mu)$ and $\overline{\lambda_1^u}(\mu)$ of $D_1 f(p_1, \mu)$ and the stable eigenvalues $\lambda_2^s(\mu)$ and $\overline{\lambda_2^s}(\mu)$ of $D_1 f(p_2, \mu)$ we write

$$\lambda_1^u(\mu) = \rho_1(\mu) + i\phi_1(\mu), \quad \lambda_2^s(\mu) = -\rho_2(\mu) + i\phi_2(\mu).$$

We refer again to Figure 1 for a visualisation of the situation in $\mathbb{R}^3$.

In order to exclude further degeneracies we assume the non-orbit flip condition

(H 1) $\quad \Gamma_i \not\subset W^{ss}(p_i), \quad \Gamma_i \not\subset W^{uu}(p_{i+1}), \quad i = 1, 2.$

Here $W^{ss}(p)$ and $W^{uu}(p)$ denotes the strong stable manifold and the strong unstable manifold, respectively, of the equilibrium p. Further we assume a non-inclination flip condition for Γ_1, which says that the stable manifold of p_1 intersects the local centre-unstable manifold of p_2 (this is an invariant manifold whose tangent space at p_2 comprises the unstable and weakest stable directions) intersect transversally:

(H 2) $$W^{cu}_{loc}(p_2) \pitchfork_{q_1(-t)} W^s(p_1), \quad W^{cs}_{loc}(p_1) \pitchfork_{q_1(t)} W^u(p_2), \quad t \gg 0.$$

Note that in $\mathbb{R}^3$ these conditions are automatically fulfilled, because of the saddle-focus structure of the equilibria.

Before we describe the nearby dynamics of the heteroclinic cycle we define some terminology. For that we introduce hyperplanes Σ_j which are transversal to Γ_j at $q_j(0)$ (which may be located "in the middle" of Γ_j), $j = 1, 2$.

Definition 1.1. *Let $\mathcal{U}$ be a sufficiently small neighbourhood of the primary heteroclinic cycle. All orbits which we shall describe in the following are assumed to be subsets of $\mathcal{U}$.*

- *A periodic orbit is called k-periodic, and a homoclinic orbit (to p_1 or p_2) is called k-homoclinic, if it passes through Σ_1 and Σ_2 k times in each case.*
- *A heteroclinic orbit connecting p_i to p_j (in forward time) is called a (i,j)-heteroclinic orbit. A (i,j)-heteroclinic orbit that passes k times through Σ_j is called a k-(i,j)-heteroclinic orbit.*

Let $\mathcal{H}$ be the space of vector fields containing a heteroclinic cycle under the assumptions stated above for $f(\cdot, 0)$. This space is endowed with the C^1 topology.

Our main result concerning shift dynamics is the following.

Theorem 1.2. *Consider the system* (1.1) *under the assumptions as stated above. There exists an open and dense set $\mathcal{D} \subset \mathcal{H}$ such that for each $f(\cdot, 0) \in \mathcal{D}$ the following is true. For each $N \geq 2$ there exists a set $\mathcal{S}^N_0 \subset \Sigma_1$ which is invariant under the first-return-map $\Pi : \Sigma_1 \to \Sigma_1$ (defined by the flow), and $(\mathcal{S}^N_0, \Pi)$ is topologically conjugated to the full shift on N symbols. Each set is furthermore hyperbolic, and therefore individually structurally stable. This means that for fixed $N \geq 2$ and sufficiently small μ there is a set $\mathcal{S}^N_\mu$ such that $(\mathcal{S}^N_\mu, \Pi(\mu))$ is toplogically conjugated to a full shift on N symbols, and $\mathcal{S}^N_\mu \to \mathcal{S}^N_0$ in the Hausdorff-metric as $\mu \to 0$.*

The proof of this theorem will be given in Section 3; here we only give some comments. The shift dynamics can be explained by the coexistence of several 1-periodic orbits. As symbols serve 1-periodic orbits $\mathcal{O}_1, \ldots, \mathcal{O}_N$, or more precisely their periods $\hat{\omega}_1, \ldots, \hat{\omega}_N$. Then a sequence $\mathbf{s} := (s_i)_{i \in \mathbb{Z}}$, $s_i \in \{1, \ldots, N\}$ is assigned to an orbit which is determined by an itinerary going in turns along $\mathcal{O}_1, \ldots, \mathcal{O}_N$ as prescribed by $\mathbf{s}$. These 1-periodic orbits correspond to transversal intersections of certain spirals, see Section 3.2. In order to prove the existence of shift dynamics we solve the bifurcation equation (2.1) in spaces of bounded sequences equipped with the supremum norm. For that we take advantage of the fact that there are even infinitely many transversal intersections of the addressed spirals, see Section 3.2.2. Indeed there exist infinitely many such intersections if the ratio of ρ_1 and ρ_2 is irrational, Lemma 3.6. If this ratio is rational and there is one transversal intersection then there are infinitely many ones, Lemma 3.7. A combination of these two facts yields that our theorem is true for an open and dense set of vector fields.

The verification of topological conjugacy is mainly based on the continuous dependence of Ξ (see (2.1)) on sequences $\boldsymbol{\omega}$ (of transition times) in spaces of sequences equipped with product topology. Those ideas trace back to similar considerations in [9, 17].

For fixed N there is not only one such set $\mathcal{S}_\mu^N$. For $\mu = 0$ there exist even countably many of them. This is again due to the existence of infinitely many intersections of the spirals we referred to already. These intersections correspond to an infinite set of 1-periodic orbits $\mathcal{O}_i$, $i \in \mathbb{N}$, which potentially can serve as symbols. For $\mu = 0$ the tips of the spirals coincide – a situation we exploit to verify infinitely intersections of them. For $\mu \neq 0$ the tips of the spirals are separated. Generically the tip of one spiral will not be located on the other spiral. Hence generically only for finitely many $N \in \mathbb{N}$ there exist corresponding sets $\mathcal{S}_\mu^N$, and for fixed N only finitely many different those sets do exist.

The shift dynamics does not involve any k-homoclinic or k-heteroclinic orbits of the original system (1.1). Our second result concerns the existence of such connecting orbits.

Lemma 1.3. *1. For each $k \geq 2$, $k \in \mathbb{N}$ there is a countable set $\mathcal{M}_k$ of parameter values, for which there exists a k-(2,1)-heteroclinic orbit.*

2. At $\mu = 0$ there exist countably many k-(1,2)-heteroclinic orbits for each $k \geq 2$. Moreover, for fixed k, $q_1(0)$ is an accumulation point of the intersections of the k-(1,2)-heteroclinic orbits with Σ_1. Each such k-(1,2)-heteroclinic orbit can be continued for $\mu \neq 0$. But for fixed $\mu \neq 0$ and fixed k there are only finitely many k-(1,2)-heteroclinic orbits.

3. For each $k \geq 2$, $k \in \mathbb{N}$ there is a countable set $\mathcal{M}_k^{hom}$ of parameter values, for which there exists a k-homoclinic orbit.

This lemma follows in the same way as corresponding statements about reversible systems given in Theorem 4.1 below. For that reason we do without presenting a separate proof and are just content with giving remarks how to adapt the corresponding proofs of the reversible situation. For that we refer to Sections 4.3.3 – 4.3.5.

We remark that our Hypothesis (H 3), see Section 3.1, which prescribes that the stable manifold of p_1 and the unstable manifold of p_2 split with positive speed, implies that there is just one 1-(2,1)-heteroclinic orbit, and this is the original one that exists at $\mu = 0$. In $\mathbb{R}^3$ it is clear that there can only be one k-(2,1)-heteroclinic orbit for each parameter value, because in this case the stable manifold of p_1 is one-dimensional.

1.1 Outline

We now outline the organisation of this paper. Our strategy for proving Theorem 1.2 is to use Lin's method, adapted to our problem. This is a method designed to calculate all bounded orbits within a neighbourhood of a given heteroclinic/homoclinic cycle configuration. Sections 2 and 3 will go through Lin's method in some detail for our case, but we would like first to give a general idea of the approach of the method.

We begin by introducing section hyperplanes Σ_k along the orbits Γ_j. We will make more precise how these sections are to be placed, but for now we may think of them as being roughly in "the middle" of Γ_j. Using Lin's method, we search for 'piecewise continuous orbits' of (1.1), which we call *Lin orbits*, a notion which is due to [8]. In the present context such orbits consist of pieces of actual orbits which stay forever close to the original heteroclinic cycle, but have certain discontinuities in the sections Σ_j. This discontinuity, or 'jump', is restricted to lie in a certain direction in the section planes, and will be made more precise in section 2. We refer to Figure 2 for a visualisation.

We are in fact able to prove existence and uniqueness of such orbits, depending only on the parameter μ, the times of passage between the sections Σ_j, and the jump discontinuity. Section 3.1 ends by deriving an equation for the ith jump of a Lin orbit, depending on μ and the times of passage $\omega_{1,i+1}$ and $\omega_{2,i}$. This equation is a two-dimensional equation, and is the bifurcation equation, see (**??**). Clearly, the solution set of the equation $\Xi_i(\omega, \mu) = 0$ (that is, we specify that each of the jumps should be zero) will be all real bounded orbits of the system (1.1) that lie close to the heteroclinic cycle. Therefore these orbits may be characterised only by the parameter μ and the times of passage in between sections.

It is by studying solutions to the equation $\Xi_i(\omega, \mu) = 0$ that we prove Theorem 1.2. This analysis is contained in section 3.2. As can be seen in equation (**??**), the bifurcation equation consists of a leading order term and remainder terms. The strategy for studying the bifurcation equation is to first study solutions to the truncated equation; that is, we set the leading order term of (**??**) equal to zero. It is proven in section 3.2 that nondegenerate solutions to this equation persist under addition of the higher order terms. Then our problem reduces to that of studying the truncated equation. A critical observation in our analysis is that the truncated equation is actually the equation to solve for the intersection points of two logarithmic spirals in the plane, see section 3.2.1. These spirals are parameterised by the times $\omega_{1,i+1}$ and $\omega_{2,i}$ respectively. Nondegenerate solutions of the truncated equation may therefore be thought of as transversal intersections of these two spirals.

In section 4 the method is adapted to study the reversible case. The main differences between the general case and reversible case are outlined. The bifurcation equations are modified, and the corresponding results for the reversible case are proved.

2 Lin's method

In this section we outline Lin's method, and explain the properties of Lin orbits corresponding to the system under consideration. The goal of Lin's method is to prove the existence of 'piecewise continuous orbits' X of (1.1) which we address as *Lin orbits*, a notion which is due to [8]. In the present context such orbits consist of pieces of actual orbits X_i, $X := (X_i)_{i\in\mathbb{Z}}$; the orbit X_i starts in Σ_1, follows Γ_1 until it reaches a neighbourhood of p_1 follows then Γ_2, meets Σ_2, stays further close to Γ_2 until it reaches a neighbourhood of p_2, follows then Γ_1 again, and terminates finally in Σ_1. Between two consecutive orbits X_{i-1} and X_i there may be a jump Ξ_i in a distinguished direction Z. We refer to Figure 2 for a visualisation. Note that different from the standard theory of Lin's method here there is no jump in Σ_2 – this is due to the transversal intersection of the unstable manifold of p_1 and the stable manifold of p_2 along Γ_2.

Let $2\omega_{1,i}$ and $2\omega_{2,i}$ be (prescribed) transition times of X_i from Σ_1 to Σ_2 and Σ_2 to Σ_1, respectively. It can be proved that for each μ which is sufficiently close to 0, and each sequence $\boldsymbol{\omega} := ((\omega_{1,i}, \omega_{2,i}))_{i\in\mathbb{Z}}$, where $\omega_{j,i}$ are sufficiently large, there exists a unique Lin orbit $X(\boldsymbol{\omega}, \mu)$, see Theorem 2.2. By making the jumps Ξ_i to zero one finds real orbits staying for all time close to the heteroclinic cycle Γ. Therefore the bifurcation equation for orbits staying close to Γ reads

$$\boldsymbol{\Xi} := (\Xi_i(\boldsymbol{\omega}, \mu))_{i\in\mathbb{Z}} = 0. \tag{2.1}$$

To begin the actual analysis, fix an inner product $\langle \cdot, \cdot \rangle$. Let, with respect to $\langle \cdot, \cdot \rangle$,

$$Y_i := \{f(q_i(0), 0)\}^{\perp}, \quad i = 1, 2.$$

With that we construct the cross-sections Σ_1 and Σ_2 as follows

$$\Sigma_i := q_i(0) + Y_i, \quad i = 1, 2. \tag{2.2}$$

Also, consistent with the standard theory, we define a subspaces $Z \subset Y_1$:

$$Z := (T_{q_1(0)}W^s(p_1) + T_{q_1(0)}W^u(p_2))^{\perp}.$$

By construction we have $\dim Z = 2$. Assigned to $q_1(0)$ and $q_2(0)$ we consider the following orthogonal direct sum decomposition of $\mathbb{R}^n$:

$$\begin{aligned} \mathbb{R}^n &= \operatorname{span}\{f(q_1(0), 0)\} \oplus W_1^+ \oplus W_1^- \oplus Z, \\ \mathbb{R}^n &= \operatorname{span}\{f(q_2(0), 0)\} \oplus W_2^+ \oplus W_2^-, \end{aligned} \tag{2.3}$$

where $W_i^+ = T_{q_i(0)}W^s(p_i) \cap Y_i$ and $W_i^- = T_{q_i(0)}W^u(p_j) \cap Y_i$, $i = 1, 2$, $j \neq i$.

A further assumption, that will simplify the analysis, says that the local stable/unstable manifolds of the fixed points p_1, p_2 are flat; that is:

$$W^s_{loc}(p_i, \mu) \subset T_{p_i}W^s(p_i), \quad W^u_{loc}(p_i, \mu) \subset T_{p_i}W^u(p_i). \tag{2.4}$$

We can bring the local stable/unstable manifolds into this form by means of local transformations based around each of the fixed points p_i.

2.1 Splitting of the stable and unstable Manifolds

The first step of Lin's method is to study the splitting of the stable and unstable manifolds in Σ_i with respect to the parameter μ. Because of (1.2) the situation in Σ_2 is clear. For each μ which is sufficiently close to 0 there is exactly one point $q_{2,\mu} \in \Sigma_2 \cap W^u(p_1, \mu) \cap W^s(p_2, \mu)$ such that the orbit $\Gamma_{2,\mu}$ through $q_{2,\mu}$ is a 1-(1,2)-heteroclinic orbit. The corresponding solution with $q_2(\mu)(0) = q_{2,\mu}$ we denote by $q_2(\mu)(\cdot)$; their restriction on $\mathbb{R}^{\pm}$ we denote by $q_2^{\pm}(\mu)(\cdot)$.

In $\mathbb{R}^3$ also the situation in Σ_1 is rather simple. In this case both the stable manifold of p_1 and the unstable manifold of p_2 are one-dimensional, and the intersection with the two-dimensional hyperplane Σ_1 consists of single points in each case. The heteroclinic connection Γ_1 will split up under perturbation. Let $q_{1,\mu}^+$ and $q_{1,\mu}^-$ be determined by the 'first hit' of the stable manifold of p_1 and the unstable manifold of p_2, respectively. Of course $q_{1,\mu}^+ - q_{1,\mu}^- \in Z$, recall that $Z = Y_1$ in this case. So, in a trivial way, for each μ we find a unique pair of orbits in the stable manifold of p_1 and the unstable manifold of p_2, respectively, such that the difference of their first hits in Σ_1 is in Z.

The main goal of this section is to show that this property remains in higher dimensions. More precisely we prove the following lemma:

Lemma 2.1. *For each μ which is sufficiently close to 0 there is a unique pair $(q_1^+(\mu)(\cdot),(q_1^-(\mu)(\cdot))$ of solutions of (1.1) such that:*

(i) $q_1^+(\mu)(0) \in \Sigma_1 \cap W^s(p_1,\mu)$, $\quad q_1^-(\mu)(0) \in \Sigma_1 \cap W^u(p_2,\mu)$,

(ii) $|q_1^+(\mu)(t) - q_1(t)|$ *small* $\forall t \in \mathbb{R}^+$ *and* $|q_1^-(\mu)(t) - q_1(t)|$ *small* $\forall t \in \mathbb{R}^-$,

(iii) $q_1^+(\mu)(0) - q_1^-(\mu)(0) \in Z$.

Proof. To investigate the splitting of Γ_1 we set

$$q_1^\pm(t) = q_1(t) + v^\pm(t). \tag{2.5}$$

This gives the following equations for $v^\pm$

$$\dot{v}^\pm = A(t)v^\pm + g(t, v^\pm, \mu), \tag{2.6}$$

which we consider on $\mathbb{R}^+$ or $\mathbb{R}^-$ respectively. Here $A(t) = D_x f(q_1(t), 0)$, and

$$g(t, v, \mu) = f(q_1(t) + v, \mu) - f(q_1(t), 0) - A(t)v. \tag{2.7}$$

With the above setting, we are looking for bounded solutions $v^\pm(\cdot)$ of (2.6). Hence $v^\pm(\cdot) \in C_b(\mathbb{R}^\pm)$, the space of continuous on $\mathbb{R}^\pm$ bounded functions equipped with the norm $\|v^\pm\|_\infty := \sup_{t\in\mathbb{R}^\pm} \|v^\pm(t)\|$. If $\|v^\pm\|_\infty$ is close to zero, then by the theory of stable and unstable manifolds, $q^\pm(t) := q_1(t)+v^\pm(t)$ is in the desired stable/unstable manifold. Further, if $v^\pm(0) \in Y_1$ and $v^+(0) - v^-(0) \in Z$, then (i) - (iii) of the lemma holds true.

In what follows we show that (2.6) has unique solutions $v^\pm$ with the outlined boundary conditions. An important role is played by the properties of the linear nonautonomous equations

$$\dot{v} = A(t)v, \tag{2.8}$$

which we consider either on $\mathbb{R}^+$ or on $\mathbb{R}^-$. Denote by $\Phi(t, s)$ the transition matrix for (2.8). We have that $\lim_{t\to\infty} q_1(t) = p_1$, so then $\lim_{t\to\infty} A(t) = D_x f(p_1, 0)$. Similarly $\lim_{t\to-\infty} A(t) = D_x f(p_2, 0)$. Also $D_x f(p_1, \mu) = -R \circ D_x f(p_2, \mu) \circ R$.

By the theory of exponential dichotomies [3], due to the fact that p_1, p_2 are hyperbolic, equation (2.8) has an exponential dichotomy on both $\mathbb{R}^+$ and $\mathbb{R}^-$. For $t \in \mathbb{R}_0^\pm$ we define projections $P^\pm(t)$ and $Q^\pm(t)$ as follows

$$\operatorname{im} P^+(0) = T_{q_1(0)} W^s(p_1), \quad \ker P^+(0) = W_1^- \oplus Z,$$

and

$$\ker P^-(0) = T_{q_1(0)} W^u(p_2) \quad \operatorname{im} P^-(0) = W_1^+ \oplus Z,$$

and for $t, s \geq 0$ or $t, s \leq 0$ let

$$P^\pm(t)\Phi(t, s) = \Phi(t, s)P^\pm(s).$$

Finally we define $Q^+ := I - P^+$ and $Q^- := I - P^-$.

Using the properties of exponential dichotomy we get that for each on $\mathbb{R}^+$ bounded function $g(\cdot)$

$$\int_0^t \Phi(t, s)P^+(s)g(s)ds - \int_t^\infty \Phi(t, s)Q^+(s)g(s)ds$$

is well defined on $\mathbb{R}^+$ and solves $\dot{v} = A(t)v + g(t)$. Therefore we see that the on $\mathbb{R}^+$ bounded solutions $v^+(\cdot)$ of (2.6) solve the following fixed point problem in $C_b^0(\mathbb{R}^+, \mathbb{R}^n)$ and vice versa

$$v^+(t) = \Phi(t,0)\nu + \int_0^t \Phi(t,s)P^+(s)g(s,v^+(s),\mu)ds - \int_t^\infty \Phi(t,s)Q^+(s)g(s,v^+(s),\mu)ds.$$

In a similar way it can be shown that the on $\mathbb{R}^-$ bounded solutions $v^-(\cdot)$ of (2.6) solve the following fixed point problem in $C_b^0(\mathbb{R}^-, \mathbb{R}^n)$ and vice versa

$$v^-(t) = \Phi(t,0)\eta + \int_{-\infty}^t \Phi(t,s)P^-(s)g(s,v^-(s),\mu)ds - \int_t^0 \Phi(t,s)Q^-(s)g(s,v^-(s),\mu)ds.$$

These both fixed point equations can be solved for $v^+ = v^+(\nu,\mu)$ and $v^- = v^-(\eta,\mu)$, near $(v^+,\nu,\mu) = (0,0,0)$ and $(v^-,\eta,\mu) = (0,0,0)$ in each case. We find that $v^\pm(0)$ can be written in the form

$$v^+(\nu,\mu)(0) = \nu + w^-(\nu,\mu) + z^+(\nu,\mu), \quad v^-(\eta,\mu)(0) = \eta + w^+(\eta,\mu) + z^-(\eta,\mu),$$

where $w^\pm \in W_1^\pm$ and $z^\pm \in Z$, and moreover $w^\pm(0,0) = 0$, $D_1 w^\pm(0,0) \neq 0$. In view of (iii) of the lemma we consider

$$\nu = w^+(\eta,\mu), \quad \eta = w^-(\nu,\mu).$$

This system can, near $(\nu,\eta,\mu) = (0,0,0)$, be solved for $\nu = \nu(\mu)$, $\eta = \eta(\mu)$.

The functions $q_1^+(\mu)(\cdot) := q_1(\cdot) + v^+(\nu(\mu),\mu)(\cdot)$ and $q_1^-(\mu)(\cdot) := q_1(\cdot) + v^-(\eta(\mu),\mu)(\cdot)$ are the wanted solutions of (1.1). ■

2.2 Construction of Lin orbits

The next step in the method is to search for orbits $X_{j,i}$, $j = 1,2$, $i \in \mathbb{Z}$, composing the Lin orbits $X = (X_i)_{i\in\mathbb{Z}}$ which we introduced in Section 1; more precisely $X_i = X_{1,i} \cup X_{2,i}$. Here $X_{j,i}$ is an orbit of the vector field starting in a point in Σ_j staying close to Γ_j until it reaches a neighbourhood of p_j, and continuing Γ_{j+1} until it reaches Σ_{j+1}; throughout this explanations the term "$j+1$" will be computed in $\mathbb{Z}_2$. By $x_{j,i}(\cdot)$ we denote solutions of (1.1) corresponding to the orbits $X_{j,i}$ with $x_{j,i}(0) \in \Sigma_j$ and $x_{j,i}(2\omega_{j,i}) \in \Sigma_{j+1}$. Actually $x_{1,i}(\cdot)$ will be composed of solutions $x_{1,i}^+(\cdot)$ and $x_{2,i}^-(\cdot)$ which are defined on $[0,\omega_{1,i}]$ and $[-\omega_{1,i},0]$, respectively. Similarly $x_{2,i}(\cdot)$ will be composed of solutions $x_{2,i}^+(\cdot)$ and $x_{1,i}^-(\cdot)$ which are defined on $[0,\omega_{2,i}]$ and $[-\omega_{2,i},0]$, respectively. This demands coupling conditions

$$x_{j,i}^+(\omega_{j,i}) = x_{j+1,i}^-(-\omega_{j,i}),\ j = 1,2, \tag{2.9}$$

and the jump conditions

$$\Xi_i := x_{1,i+1}^+(0) - x_{1,i}^-(0) \in Z, \quad x_{2,i}^+(0) = x_{2,i}^-(0), \tag{2.10}$$

for $i \in \mathbb{Z}$ in each case. Figure 2 visualises this situation. It is a specific feature of T-points that there is no jump in Σ_2, $x_{2,i}^+(0) - x_{2,i}^-(0) = 0$. However, large parts of the procedure run parallel to the explanations in [17], [9], [11] or [21]. In our

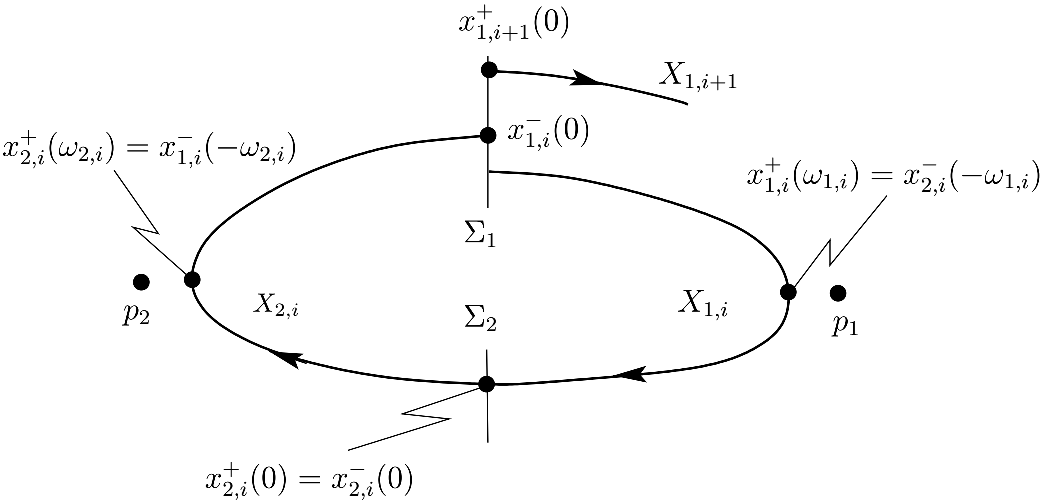

Figure 2: Ingredients of Lin orbits; $X_i = X_{1,i} \cup X_{2,i}$.

presentation we confine to explain only those parts of Lin's method more detailed which differ from the standard scheme.

To begin, we look for solutions of the form

$$x^{\pm}_{j,i}(t) = q^{\pm}_j(\mu)(t) + v^{\pm}_{j,i}(t). \tag{2.11}$$

Then

$$\dot v^{\pm}_{j,i} = A^{\pm}_j(t,\mu)v^{\pm}_{j,i} + g^{\pm}_j(t, v^{\pm}_{j,i}, \mu) \tag{2.12}$$

where $A^{\pm}_j(t,\mu) = D_x f(q^{\pm}_j(\mu)(t), \mu)$, and

$$g^{\pm}_j(t,v,\mu) = f(q^{\pm}_j(\mu)(t) + v, \mu) - f(q^{\pm}_j(\mu)(t), \mu) - A^{\pm}_j(t,\mu)v. \tag{2.13}$$

Let $\Phi^{\pm}_j(\mu,t,s)$ be the transition matrix for the equation

$$\dot v = A^{\pm}_j(t,\mu)v. \tag{2.14}$$

As before, these operators have an exponential dichotomy on $\mathbb{R}^+$ or $\mathbb{R}^-$, respectively, with corresponding projections $P^+_j(\mu,t)$, $Q^+_j(\mu,t) = I - P^+_j(\mu,t)$, and $Q^-_j(\mu,t)$, $P^-_j(\mu,t) = I - Q^-_j(\mu,t)$; therefore we have for $j = 1,2$

$$\operatorname{im} P^+_j(\mu,0) = T_{q^+_j(\mu)(0)}W^s(p_j), \quad \operatorname{im} Q^-_j(\mu,0) = T_{q^-_j(\mu)(0)}W^u(p_{j+1}).$$

Moreover we commit to

$$\ker P^+_1(\mu,0) = W^-_1 \oplus Z, \quad \ker P^+_2(\mu,0) = W^-_2$$

and

$$\ker Q^-_1(\mu,0) = W^+_1 \oplus Z, \quad \ker Q^-_2(\mu,0) = W^+_2.$$

Our main existence result in this respect is the following:

Theorem 2.2. *Consider the system (1.1) in the present setting. Then there are constants c, Ω such that for each μ with $|\mu| < c$ and each $\boldsymbol{\omega}$ with $\omega_{j,i} > \Omega$, $j = 1,2$, $i \in \mathbb{Z}$, there is a unique sequence of solutions $x^{\pm}_{j,i}(\boldsymbol{\omega},\mu)(\cdot)$, $j = 1,2$, $i \in \mathbb{Z}$, of (1.1) satisfying the coupling condition (2.9) and the jump condition (2.10).*

In other words this theorem says that for each μ and for each sequence $\boldsymbol{\omega}$ there exists a unique Lin orbit $X(\boldsymbol{\omega}, \mu)$.

Next we explain the steps leading to the proof of this theorem. We consider (instead of the original equation (1.1) with boundary conditions (2.9) and (2.10)) the differential equation (2.12) with corresponding boundary conditions (according to (2.11))

$$v^+_{j,i}(\omega_{j,i}) - v^-_{j+1,i}(-\omega_{j,i}) = q^-_{j+1}(-\omega_{j,i}) - q^+_j(\omega_{j,i}),\ j = 1, 2, \tag{2.15}$$

$$v^\pm_{j,i}(0) \in Y_j, \quad v^+_{1,i+1}(0) - v^-_{1,i}(0) \in Z, \quad v^+_{2,i}(0) = v^-_{2,i}(0), \tag{2.16}$$

for $i \in \mathbb{Z}$ in each case. We fix a sequence $\boldsymbol{\omega}$ with sufficiently large $\omega_{j,i}$ which we will keep throughout the following explanations. Assigned to this sequence we denote by $\mathcal{V}_{\boldsymbol{\omega}}$ the space of all sequences $\mathbf{v} := (v^+_{1,i}, v^-_{2,i}, v^+_{2,i}, v^-_{1,i})_{i\in\mathbb{Z}}$, where $v^+_{j,i} \in C([0, \omega_{j,i}], \mathbb{R}^n)$ and $v^-_{j,i} \in C([-\omega_{j+1,i}, 0], \mathbb{R}^n)$.

In a first step we consider a "linearised" equation

$$\dot{v}^\pm_{j,i} = A^\pm_j(t, \mu) v^\pm_{j,i} + h^\pm_{j,i}(t), \tag{2.17}$$

with continuous $h^\pm_{j,i}(\cdot)$, and with boundary conditions (2.16) and

$$Q^+_j(\mu, \omega_{j,i}) v^+_{j,i}(\omega_{j,i}) = a^+_{j,i}, \quad P^-_j(\mu, -\omega_{j+1,i}) v^-_{j,i}(-\omega_{j+1,i}) = a^-_{j+1,i}, \tag{2.18}$$

for any given $a^+_{j,i} \in \operatorname{Im} Q^+_j(\mu, \omega_{j,i})$, $a^-_{j+1,i} \in \operatorname{Im} P^-_j(\mu, -\omega_{j+1,i})$.

Lemma 2.3. *The boundary value problem ((2.17), (2.16), (2.18)) has a unique solution* $\mathbf{v}_{\boldsymbol{\omega}} \in \mathcal{V}_{\boldsymbol{\omega}}$.

Proof. Solutions of (2.17) can be written in the form

$$v^\pm_{j,i}(t) = \Phi^\pm_j(\mu, t, 0) v^\pm_{j,i}(0) + \int_0^t \Phi^\pm_j(\mu, t, s) h^\pm_{j,i}(s) ds. \tag{2.19}$$

Working in the boundary condition (2.18) gives for $i \in \mathbb{Z}$

$$\begin{aligned} Q^+_j(\mu, 0) v^+_{j,i}(0) &= \Phi^+_j(\mu, 0, \omega_{j,i}) a^+_{j,i} - \int\limits_0^{\omega_{j,i}} \Phi^+_j(\mu, 0, s) Q^+_j(\mu, s) h^+_{j,i}(s) ds, \\ P^-_j(\mu, 0) v^-_{j,i}(0) &= \Phi^-_j(\mu, 0, -\omega_{j+1,i}) a^-_{j,i} + \int\limits_{-\omega_{j+1,i}}^{0} \Phi^-_j(\mu, 0, s) P^-_j(\mu, s) h^-_{j,i}(s) ds, \end{aligned} \tag{2.20}$$

Because of (2.16) we have

$$v^+_{1,i+1}(0) = w^+_{1,i} + w^-_{1,i} + z^+_i,\ v^-_{1,i}(0) = w^+_{1,i} + w^-_{1,i} + z^-_i \quad \text{and} \quad v^\pm_{2,i}(0) = w^+_{2,i} + w^-_{2,i},$$

where $w^\pm_{j,i} \in W^\pm_j$ and $z^\pm_i \in Z$. For fixed j the left-hand sides of these equations are decoupled over i. Moreover, for each single $i \in \mathbb{Z}$ the left-hand side of (2.20) can be seen as a linear mapping

$$W^+_1 \times W^-_1 \times Z \times Z \to (W^-_1 \oplus Z) \times (W^+_1 \oplus Z) \quad \text{or} \quad W^+_2 \times W^-_2 \to W^+_2 \times W^-_2$$

depending on $j = 1$ or $j = 2$, respectively. These mappings are invertible. So the equations under consideration can be solved for

$$(w^\pm_{1,i}, z^\pm_i) = (w^\pm_{1,i}, z^\pm_i)(\mu, h^+_{1,i+1}, h^-_{2,i}, a^+_{1,i+1}, a^-_{2,i}), \quad w^\pm_{2,i} = w^\pm_{2,i}(\mu, h^-_{1,i}, h^+_{2,i}, a^-_{1,i}, a^+_{2,i}).$$

With (2.19) we get eventually the lemma. ■

In the next step we “replace” the boundary conditions (2.18) by

$$v^+_{j,i}(\omega_{j,i}) - v^-_{j+1,i}(-\omega_{j,i}) = d_{j,i}, \quad d_{j,i} \in \mathbb{R}^n, \quad j = 1, 2, \tag{2.21}$$

and consider the boundary value problem ((2.17), (2.16), (2.21)).

Lemma 2.4. *The boundary value problem ((2.17), (2.16), (2.21)) has a unique solution* $\hat{\mathbf{v}}_{\boldsymbol{\omega}} \in \mathcal{V}_{\boldsymbol{\omega}}$.

Proof. First we claim that for sufficiently large $\omega > 0$ and sufficiently small μ it holds $\operatorname{im} Q^+_j(\mu, \omega) \oplus \operatorname{im} P^-_{j+1}(\mu, -\omega) = \mathbb{R}^n$, $j = 1, 2$. This is due to asymptotical behaviour (as $\omega \to \infty$) of the involved projections and due to the hyperbolicity of the equilibria p_1 and p_2; we refer to [21] for more details.

The rest of the proof runs along the lines of the corresponding assertions in [17] or [9]: For each given sequence $(d_{j,i})$ one proves the existence of sequences $(a^+_{j,i})$ and $(a^-_{j,i})$ such that the corresponding solutions of the boundary value problem ((2.17), (2.16), (2.18)) solve the the boundary value problem ((2.17), (2.16), (2.21)). ■

Altogether we find $\hat{\mathbf{v}}_{\boldsymbol{\omega}} = \hat{\mathbf{v}}_{\boldsymbol{\omega}}(\mu, \mathbf{h}, \mathbf{d})$, where $\mathbf{h} := (h^+_{1,i}, h^-_{2,i}, h^+_{2,i}, h^-_{1,i})_{i\in\mathbb{Z}}$ and $\mathbf{d} := (d_{1,i}, d_{2,i})_{i\in\mathbb{Z}}$. It is worth to mention that actually each entry of $\hat{\mathbf{v}}_{\boldsymbol{\omega}}$ depends only on a finite part of the sequences $\mathbf{h}$ and $\mathbf{d}$.

A coupling of the gained solutions can be achieved by setting

$$d_{j,i} = d_{\boldsymbol{\omega}_{j,i}}(\mu) := q^-_{j+1}(\mu)(-\omega_{j,i}) - q^+_j(\mu)(\omega_{j,i}),\ j = 1, 2.$$

Now, the nonlinear boundary value problem ((2.12),(2.15),(2.16)) is equivalent to the following fixed point equation in $\mathcal{V}_{\boldsymbol{\omega}}$:

$$\mathbf{v} = \hat{\mathbf{v}}_{\boldsymbol{\omega}}(\mu, \mathcal{G}(\mathbf{v}, \mu), \mathbf{d}_{\boldsymbol{\omega}}(\mu)), \tag{2.22}$$

where

$$\begin{array}{rccl} \mathcal{G}: & \mathcal{V}_{\boldsymbol{\omega}} \times \mathbb{R} & \to & \mathcal{V}_{\boldsymbol{\omega}} \\ & (\mathbf{v}, \mu) & \mapsto & (h^+_{1,i}, h^-_{2,i}, h^+_{2,i}, h^-_{1,i})_{i\in\mathbb{Z}}, \quad h^\pm_{j,i}(\cdot) := g^\pm_j(\cdot, v^\pm_{j,i}(\cdot), \mu). \end{array}$$

The following Lemma provides solutions to the fixed point problem. (2.22)

Lemma 2.5. *For fixed* $\boldsymbol{\omega}$, $\omega_{j,i}$ *sufficiently large, and* $|\mu|$ *sufficiently small, the fixed point problem (2.22) has a unique solution* $\bar{\mathbf{v}}_{\boldsymbol{\omega}}$ *in a sufficiently small neighbourhood of* $0 \in \mathcal{V}_{\boldsymbol{\omega}}$. *Moreover, the mapping* $\mu \mapsto \bar{\mathbf{v}}_{\boldsymbol{\omega}}(\mu)$ *is smooth.*

Note that the necessary considerations have been done for fixed $\boldsymbol{\omega}$ in spaces $\mathcal{V}_{\boldsymbol{\omega}}$. Define

$$\bar{\mathbf{v}}(\boldsymbol{\omega}, \mu) = (\bar{v}^+_{1,i}, \bar{v}^-_{2,i}, \bar{v}^+_{2,i}, \bar{v}^-_{1,i})_{i\in\mathbb{Z}} := \bar{\mathbf{v}}_{\boldsymbol{\omega}}(\mu), \quad \bar{v}^\pm_{j,i} = \bar{v}^\pm_{j,i}(\boldsymbol{\omega}, \mu). \tag{2.23}$$

Lemma 2.6. *The mappings* $l^\infty_{\mathbb{R}^2} \times \mathbb{R} \to l^\infty_{\mathbb{R}^n}$, $(\boldsymbol{\omega}, \mu) \mapsto \bar{v}^\pm_{j,i}(\boldsymbol{\omega}, \mu)(0)$ *are smooth.*

For both Lemma 2.5 and Lemma 2.6 the proof of the corresponding statements in [17] or [9] can easily be adapted for the present situation.

3 Bifurcation Analysis – Proof of Theorem 1.2

In this section we shall study the nearby dynamics of the heteroclinic cycle Γ; our main concern is to prove Theorem 1.2. At first we develop the bifurcation equations.

3.1 The Bifurcation Equation

For given $\boldsymbol{\omega}$ and μ we find a unique Lin orbit $X(\boldsymbol{\omega},\mu)$, see Theorem 2.2, and a corresponding sequence $(\Xi_i(\boldsymbol{\omega},\mu))_{i\in\mathbb{Z}}$ of jumps, see (2.10). The Lin orbit becomes a real orbit if all jumps are equal to zero; this leads to the set of bifurcation equations

$$\Xi_i(\boldsymbol{\omega},\mu)=0,\quad i\in\mathbb{Z}, \tag{3.1}$$

where $\boldsymbol{\Xi}:=(\Xi_i)_{i\in\mathbb{Z}}$ can be read as a mapping

$$\boldsymbol{\Xi}:(l^\infty\times l^\infty)\times\mathbb{R}\to l^\infty\times l^\infty, \tag{3.2}$$

mind that $\boldsymbol{\omega}=(\omega_{1,i},\omega_{2,i})_{i\in\mathbb{Z}}$ and $\dim Z=2$. As in [17] or [9] we find (as a consequence of Lemma 2.6)

Lemma 3.1. *$\boldsymbol{\Xi}$ depends smoothly on $\boldsymbol{\omega}$ and μ.* ■

By solving the bifurcation equations (3.1) we find all kind of orbits staying close to the primary heteroclinic cycle Γ. For instance, a k-periodic orbit can be seen as a k-periodic Lin orbit where all jumps are zero. A Lin orbit is k-periodic if and only if $\boldsymbol{\omega}$ is k-periodic. So there arise only k different pairs $(\omega_{1,i},\omega_{2,i})$, $i\in\{1,\dots,k\}$. Therefore the bifurcation equation for detecting k-periodic orbits consists only of k equations

$$\Xi_i(\boldsymbol{\omega},\mu)=0,\quad i\in\{1,\dots,k\}, \tag{3.3}$$

and the corresponding $\boldsymbol{\Xi}$ can be read as a mapping $\mathbb{R}^{2k}\times\mathbb{R}^2\to\mathbb{R}^{2k}$.

Further it is worth to mention that our considerations remain true if some $\omega_{j,i}$ are formally put to infinity. If in addition the sequence $\boldsymbol{\omega}$ is periodic then this leads to the bifurcation equation for k-homoclinic orbits or heteroclinic cycles which are composed of k-heteroclinic connections. for instance a k-homoclinic orbit to p_1 corresponds to a k-periodic sequence $\boldsymbol{\omega}$ with $\omega_{1,1}=\infty$.

From (2.11) we see that the jumps Ξ_i have the form

$$\Xi_i(\boldsymbol{\omega},\mu):=\xi^\infty(\mu)+\xi_i(\boldsymbol{\omega},\mu), \tag{3.4}$$

where

$$\xi^\infty(\mu):=q_1^+(\mu)(0)-q_1^-(\mu)(0), \tag{3.5}$$

and

$$\xi_i(\boldsymbol{\omega},\mu)=\bar{v}_{1,i+1}^+(\boldsymbol{\omega},\mu)(0)-\bar{v}_{1,i}^-(\boldsymbol{\omega},\mu)(0). \tag{3.6}$$

First we consider $\xi^\infty(\mu)$. By introducing appropriate coordinates, $\xi^\infty(\cdot)$ can be seen as a mapping

$$\xi^\infty(\cdot):\mathbb{R}^2\to\mathbb{R}^2.$$

Of course $\xi^\infty(0)=0$, which represents that the unstable manifold of p_2 and the stable manifold of p_1 intersect (in the primary heteroclinic orbit Γ_1). The following hypothesis ensures that these manifolds split with positive speed by varying μ.

(H 3) $D\xi^{\infty}(0)$ is non-singular.

This justifies our setting $\mu \in \mathbb{R}^2$, $\mu := (\mu_1, \mu_2)$. Due to this hypothesis we may assume

$$\xi^{\infty}(\mu) = \mu. \tag{3.7}$$

We want to mention that Hypothesis (H 3) can be made more explicit by using a representation of ξ^{∞} by means of a Melnikov-type integral. For such a consideration we refer to Remark 4.10 below.

Next we turn to the representation of $\xi_i(\boldsymbol{\omega}, \mu)$:

$$\xi_i(\boldsymbol{\omega}, \mu) = \sum_{j=1}^{2} \langle \psi_j, \xi_i^{\omega}(\mu) \rangle \psi_j, \tag{3.8}$$

where $\{\psi_1, \psi_2\}$ is a orthonormal basis of Z. We can write

$$\xi_i(\boldsymbol{\omega}, \mu) = \sum_{j=1}^{2} \Big(\langle \psi_j, Q_1^{+}(\mu, 0) \bar{v}_{1,i+1}^{+}(\boldsymbol{\omega}, \mu)(0) \rangle - \langle \psi_j, P_1^{-}(\mu, 0) \bar{v}_{1,i}^{-}(\boldsymbol{\omega}, \mu)(0) \rangle \Big) \psi_j.$$

As in [17] or [9] we find that the leading order term (as $\boldsymbol{\omega} \to \infty$)) for the expression at the right-hand side can be extracted from

$$\sum_{j=1}^{2} \Big(S_1(\omega_{1,i+1}, \psi_j, \mu) - S_2(\omega_{2,i}, \psi_j, \mu) \Big) \psi_j,$$

where

$$S_1(t, \psi_j, \mu) := \langle \Psi_1^{+}(\mu, t, 0) Q_1^{+*}(\mu, 0) \psi_j, \tilde{Q}_1(\mu, t) q_2^{-}(\mu)(-t) \rangle$$

and

$$S_2(t, \psi_j, \mu) := \langle \Psi_1^{-}(\mu, -t, 0) P_1^{-*}(\mu, 0) \psi_j, (id - \tilde{Q}_2(\mu, t)) q_2^{+}(\mu)(t) \rangle.$$

Here $\Psi_j^{\pm}(\mu, \cdot, \cdot)$ is the transition matrix of the adjoint of (2.14), and $\tilde{Q}_j(\mu, \omega)$ is the projection which projects on $\operatorname{im} Q_j^{+}(\mu, \omega)$ along $\operatorname{im} P_{j+1}^{-}(\mu, -\omega)$, see also in the proof of Lemma 2.4. By P^* we denote the adjoint of the projection P.

Exemplarily we consider $S_1(t, \psi_1, \mu)$ somewhat closer. The following estimates can be found in [18, 19] or [9], respectively. As a consequence of Hypothesis (H 1) $q_2^{-}(\mu)(t)$ behaves asymptotically, as $t \to -\infty$, like $e^{D_1 f(p_1, \mu) t} \eta^u$, where $\eta^u = \eta^u(\mu)$ is in the generalised (real) eigenspace X^u of the principal unstable eigenvalues $\lambda_1^u(\mu)$, $\bar{\lambda}_1^u(\mu)$ of $D_1 f(p_1, \mu)$. The same holds true for the behaviour of $\tilde{Q}_1(\mu, t) q_2^{-}(\mu)(-t)$. Similarly, due to Hypothesis (H 2), $\Psi_1^{+}(\mu, t, 0) Q_1^{+*}(\mu, 0) \psi_1$ behaves asymptotically, as $t \to \infty$, like $e^{-(D_1 f(p_1, \mu))^* t} \eta^{+}$, where $\eta^{+} = \eta^{+}(\psi_1, \mu)$ belongs to the generalised (real) eigenspace of the principal stable eigenvalues $-\lambda_1^u(\mu)$, $-\bar{\lambda}_1^u(\mu)$ of $-(D_1 f(p_1, \mu))^*$. So, the leading term $L_{11}(t, \mu)$ of $S_1(t, \psi_1, \mu)$ reads

$$L_{11}(t, \mu) = \langle e^{-D_1 f(p_1, \mu) t} \eta^u,\ e^{-(D_1 f(p_1, \mu))^* t} \eta^{+} \rangle = \langle e^{-D_1 f(p_1, \mu) 2t} \eta^u,\ \eta^{+} \rangle.$$

We introduce an appropriate basis in $\mathbb{R}^n$ whose elements lie in X^u and $(X^u)^{\perp}$, respectively. Then η^u has a coordinate representation $(\eta_1^u, \eta_2^u, 0, \dots, 0)$ where $(\eta_1^u, \eta_2^u) \neq$

$(0,0)$, and $e^{-D_1 f(p_1,\mu)2t}\eta^u$ acts like

$$e^{-2\rho_1(\mu)t}\begin{pmatrix} \begin{pmatrix} \cos(2\phi_1(\mu)t) & \sin(2\phi_1(\mu)t) \\ -\sin(2\phi_1(\mu)t) & \cos(2\phi_1(\mu)t) \end{pmatrix}\begin{pmatrix} \eta_1^u \\ \eta_2^u \end{pmatrix} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Let $(\eta_1^+,\dots,\eta_n^+)$ be the coordinate representation of η^+ with respect to the chosen basis. Since $\eta^+ \in X^\perp$, where X denotes the generalised eigenspace of the other (the strong unstable and the stable) eigenvalues of $D_1 f(p_1,\mu)$, we have $(\eta_1^+,\eta_2^+) \neq (0,0)$. This gives finally that $L_{11}(t,\mu)$ takes the form

$$L_{11}(t,\mu) = e^{-2\rho_1(\mu)t}\Big((\eta_2^u\eta_1^+ - \eta_1^u\eta_2^+)\sin(2\phi_1(\mu)t) + (\eta_1^u\eta_1^+ + \eta_2^u\eta_2^+)\cos(2\phi_1(\mu)t)\Big).$$

Since both (η_1^u,η_2^u) and (η_1^+,η_2^+) are different from $(0,0)$ it holds

$$(\eta_1,\eta_2) := (\eta_2^u\eta_1^+ - \eta_1^u\eta_2^+,\ \eta_1^u\eta_1^+ + \eta_2^u\eta_2^+) \neq (0,0). \tag{3.9}$$

So there is an angle $\varphi_{11} = \varphi_{11}(\psi_1,\mu)$ such that

$$\sin\varphi_{11} = \eta_1\left(\eta_1^2+\eta_2^2\right)^{-1/2}, \quad \cos\varphi_{11} = \eta_2\left(\eta_1^2+\eta_2^2\right)^{-1/2}. \tag{3.10}$$

Hence $L_{11}(t,\mu)$ reads

$$L_{11}(t,\mu) = e^{-2\rho_1(\mu)t}c_{11}(\mu)\sin(2\phi_1(\mu)t+\varphi_{11}), \tag{3.11}$$

where $c_{11}(\mu) = (\eta_1^2+\eta_2^2)^{1/2}$. By construction $c_{11}(\cdot)$ is smooth and $c_{11}(0)\neq 0$.

In the same way we find constants $c_{ij}(\mu)$ and φ_{ij} such that the leading term $L_{ij}(t,\mu)$ of $S_i(t,\psi_j,\mu)$ reads

$$L_{ij}(t,\mu) = e^{-2\rho_i(\mu)t}c_{ij}(\mu)\sin(2\phi_i(\mu)t+\varphi_{ij}). \tag{3.12}$$

With that we obtain the following representation of $\xi_i(\boldsymbol{\omega},\mu)$:

Lemma 3.2. *For the system under consideration the jump $\xi_i(\boldsymbol{\omega},\mu)$ can be written in the form*

$$\begin{aligned}\xi_i(\boldsymbol{\omega},\mu) = \ & \psi_1\left(L_{11}(\omega_{1,i+1},\mu) - L_{21}(\omega_{2,i},\mu) + r_{1,i}(\boldsymbol{\omega},\mu)\right) \\ & +\psi_2\left(L_{12}(\omega_{1,i+1},\mu) - L_{22}(\omega_{2,i},\mu) + r_{2,i}(\boldsymbol{\omega},\mu)\right),\end{aligned}$$

where the for the quantities $r_{j,i}(\boldsymbol{\omega},\mu)$ it holds

$$r_{j,i}(\boldsymbol{\omega},\mu) = o(e^{-2\rho_1(\mu)\omega_{1,i+1}}) + o(e^{-2\rho_2(\mu)\omega_{2,i}}).$$

The functions $c_{ij}(\cdot)$ which arise in the definition of L_{ij} are smooth, and $c_{ij}(0)\neq 0$, $i,j=1,2$. ■

For the derivatives of the jumps, see also Lemma 3.1, there hold similar estimates:

Lemma 3.3. *The derivatives of the jumps ξ_i have the following form*

$$\begin{aligned} D_j\xi_i(\boldsymbol{\omega},\mu) = \;& \psi_1\Big(D_j\left(L_{11}(\omega_{1,i+1},\mu) - L_{21}(\omega_{2,i},\mu)\right) + \tilde{r}_{1,i}(\boldsymbol{\omega},\mu)\Big) \\ & +\psi_2\Big(D_j\left(L_{12}(\omega_{1,i+1},\mu) - L_{22}(\omega_{2,i},\mu)\right) + \tilde{r}_{2,i}(\boldsymbol{\omega},\mu)\Big), \end{aligned}$$

where the for the quantities $\tilde{r}_{j,i}(\boldsymbol{\omega},\mu)$ it holds

$$\tilde{r}_{j,i}(\boldsymbol{\omega},\mu) = o(e^{-2\rho_1(\mu)\omega_{1,i+1}}) + o(e^{-2\rho_2(\mu)\omega_{2,i}}).$$

■

For the proof we refer again to the corresponding statements in [17] or [9].

3.2 Proof of Theorem 1.2

This section is devoted the proof of Theorem 1.2. The bifurcation equation for orbits staying for all time close to the primary cycle reads:

$$\Xi_i(\boldsymbol{\omega},\mu) = \left\{ \begin{array}{ll} \mu_1 + L_{11}(\omega_{1,i+1},\mu) - L_{21}(\omega_{2,i},\mu) + r_{1,i}(\boldsymbol{\omega},\mu) & = 0 \\ \mu_2 + L_{12}(\omega_{1,i+1},\mu) - L_{22}(\omega_{2,i},\mu) + r_{2,i}(\boldsymbol{\omega},\mu) & = 0 \end{array} \right. \qquad (3.13)$$
$$i \in \mathbb{Z}.$$

An outstanding role in the analysis of the bifurcation equation $\Xi_i = 0$, $i \in \mathbb{Z}$, plays the discussion of

$$L((\omega_{1,i+1},\omega_{2,i}),\lambda) := \begin{pmatrix} \mu_1 + L_{11}(\omega_{1,i+1},\mu) - L_{21}(\omega_{2,i},\mu) \\ \mu_2 + L_{12}(\omega_{1,i+1},\mu) - L_{22}(\omega_{2,i},\mu) \end{pmatrix} = 0. \qquad (3.14)$$

For that it is worthwhile to observe that for $t \in \mathbb{R}^+$

$$\mathfrak{S}_1(t,\mu) := \begin{pmatrix} \mu_1 + L_{11}(t,\mu) \\ \mu_2 + L_{12}(t,\mu) \end{pmatrix} \quad \text{and} \quad \mathfrak{S}_2(t,\mu) := \begin{pmatrix} L_{21}(t,\mu) \\ L_{22}(t,\mu) \end{pmatrix} \qquad (3.15)$$

define spirals with the tip in $(\mu_1,\mu_2)^T$ and $(0,0)^T$, respectively. Note that $\mathfrak{S}_i(\cdot,\mu)$, are indeed spirals since $\varphi_{i1} - \varphi_{i2} \neq 0(\mod \pi)$. A proof of the latter fact will be presented in Lemma 3.4. Further we show, Lemma 3.6, that for $\mu = 0$ these spirals have infinitely many transversal intersections that accumulate at their coinciding tips. Roughly speaking these intersections serve as "starting solutions" for the Implicit Function Theorem which we use to tackle the full bifurcation equation, see Section 3.2.2.

3.2.1 Analysis of spirals

First we make clear that the curves $\mathfrak{S}_i(\cdot,\mu)$ which we defined in (3.15) are indeed spirals. For that, as already mentioned, we only have to prove

Lemma 3.4. $\varphi_{i1} - \varphi_{i2} \neq 0(\mod \pi)$*,* $i = 1, 2$*.*

Proof. We show that $\sin(\varphi_{i1} - \varphi_{i2}) \neq 0$; exemplarily we will do that for $i = 1$. The quantity φ_{11} is defined by (3.10). Similarly φ_{12} is associated to a pair $(\hat{\eta}_1, \hat{\eta}_2)$, which again, in accordance with (3.9) is defined by

$$(\hat{\eta}_1, \hat{\eta}_2) := (\eta_2^u \hat{\eta}_1^+ - \eta_1^u \hat{\eta}_2^+,\, \eta_1^u \hat{\eta}_1^+ + \eta_2^u \hat{\eta}_2^+) \neq (0, 0). \tag{3.16}$$

Note that the η_i^u appearing in (3.9) and (3.16) are the same in both equations. Exploiting $\sin(\varphi_{11} - \varphi_{12}) = \sin\varphi_{11}\cos\varphi_{12} - \sin\varphi_{12}\cos\varphi_{11}$ we find that $\sin(\varphi_{11} - \varphi_{12}) = 0$ if and only if (η_1, η_2) and $(\hat{\eta}_1, \hat{\eta}_2)$ are linearly dependent; and because of (3.9) and (3.16) this is the case if and only if (η_1^+, η_2^+) and $(\hat{\eta}_1^+, \hat{\eta}_2^+)$ are linearly dependent. But these vectors are linearly independent because ψ_1 and ψ_2 are linearly independent. ■

We continue with analysing the solutions of equations (3.14). We note again that these equations are equal to the real bifurcation equations (3.13) modulo the rest terms. For ease of notation in this section, we drop the μ-dependence of variables inside (3.14), and rename $\omega_{1,i+1}$ and $\omega_{2,i}$ as ω_1, ω_2 respectively. Then the equation reads

$$\begin{pmatrix} \mu_1 + e^{-2\rho_1\omega_1} c_{11} \sin(2\phi_1\omega_1 + \varphi_{11}) - e^{-2\rho_2\omega_2} c_{21} \sin(2\phi_2\omega_2 + \varphi_{21}) \\ \mu_2 + e^{-2\rho_1\omega_1} c_{12} \sin(2\phi_1\omega_1 + \varphi_{12}) - e^{-2\rho_2\omega_2} c_{22} \sin(2\phi_2\omega_2 + \varphi_{22}) \end{pmatrix} = 0 \tag{3.17}$$

The following Lemma defines a renormalisation of equations (3.17).

Lemma 3.5. *There exists an invertible change of coordinates such that equations* (3.17) *take the form*

$$\begin{pmatrix} \mu_1 + e^{-\rho_1\omega_1} \sin\omega_1 & -e^{-\rho_2\omega_2} c_{21} \sin\omega_2 \\ \mu_2 + e^{-\rho_1\omega_1} \sin(\omega_1 + \varphi_{12}) & -e^{-\rho_2\omega_2} c_{22} \sin(\omega_2 + \varphi_{22}) \end{pmatrix} = 0 \tag{3.18}$$

Proof. We define the variables

$$\begin{aligned} \hat{\omega}_1 &= 2\phi_1\omega_1 + \varphi_{11}, & \hat{c}_{11} &= e^{\rho_1\varphi_{11}/\phi_1} c_{11}, & \hat{c}_{21} &= e^{\rho_2\varphi_{21}/\phi_2} c_{21}, \\ \hat{\omega}_2 &= 2\phi_2\omega_2 + \varphi_{21}, & \hat{c}_{12} &= e^{\rho_1\varphi_{11}/\phi_1} c_{12}, & \hat{c}_{22} &= e^{\rho_2\varphi_{21}/\phi_2} c_{22}, \\ \hat{\varphi}_{12} &= \varphi_{12} - \varphi_{11}, & \hat{\varphi}_{22} &= \varphi_{22} - \varphi_{21}, & \hat{\rho}_1 &= \rho_1/\phi_1, \\ \hat{\rho}_2 &= \rho_2/\phi_2, \end{aligned}$$

under which (3.17) becomes

$$\begin{pmatrix} \mu_1 + e^{-\hat{\rho}_1\hat{\omega}_1} \hat{c}_{11} \sin\hat{\omega}_1 & -e^{-\hat{\rho}_2\hat{\omega}_2} \hat{c}_{21} \sin\hat{\omega}_2 \\ \mu_2 + e^{-\hat{\rho}_1\hat{\omega}_1} \hat{c}_{12} \sin(\hat{\omega}_1 + \hat{\varphi}_{12}) & -e^{-\hat{\rho}_2\hat{\omega}_2} \hat{c}_{22} \sin(\hat{\omega}_2 + \hat{\varphi}_{22}) \end{pmatrix} = 0$$

Now divide the first equation by $\hat{c}_{11}$ and the second equation by $\hat{c}_{12}$. Defining $\bar{c}_{21} = \hat{c}_{21}/\hat{c}_{11}$, $\bar{c}_{22} = \hat{c}_{22}/\hat{c}_{12}$, $\hat{\mu}_1 = \mu_1/\hat{c}_{11}$, $\hat{\mu}_2 = \mu_2/\hat{c}_{12}$ and dropping hats/bars provides equation (3.18).

Note that the new variables in equation (3.18) have the properties $c_{21}, c_{22} \neq 0$ and $\varphi_{12}, \varphi_{22} \neq 0 (\mod \pi)$. ■

Given the above change of variables, we define the spirals $\mathfrak{S}_1(t,\mu)$, $\mathfrak{S}_2(t,\mu)$ in the plane as before:

$$\mathfrak{S}_1(t,\mu) := \begin{pmatrix} \mu_1 + e^{-\rho_1 t}\sin t \\ \mu_2 + e^{-\rho_1 t}\sin(t+\varphi_{12}) \end{pmatrix}, \quad \mathfrak{S}_2(t,\mu) := \begin{pmatrix} e^{-\rho_2 t}c_{21}\sin t \\ e^{-\rho_2 t}c_{22}\sin(t+\varphi_{22}) \end{pmatrix}.$$

We will first consider solutions to the equations (3.18) where $\mu_1 = \mu_2 = 0$. We now prove the following Lemma, which proves the existence of infinitely many transversal intersections of $\mathfrak{S}_1(\omega_1,0)$ and $\mathfrak{S}_2(\omega_2,0)$ in the case where the ratio ρ_1/ρ_2 is irrational.

Lemma 3.6. *Assume that $\mu_1 = \mu_2 = 0$ and the ratio ρ_1/ρ_2 is irrational. Then equation* (3.18) *has infinitely many solutions for ω_1, ω_2 arbitrarily large. Moreover, at each of these solutions, the Jacobian of $(\omega_1,\omega_2) \mapsto \mathfrak{S}_1(\omega_1,0) - \mathfrak{S}_2(\omega_2,0)$ is nonsingular.*

Equivalently, given that the ratio ρ_1/ρ_2 is irrational, there are infinitely many transversal intersections of $\mathfrak{S}_1(\omega_1,0)$ and $\mathfrak{S}_2(\omega_2,0)$ for arbitrarily large ω_1, ω_2.

Proof. Central to the proof is the fact that numbers of the form $k_1\rho_1 - k_2\rho_2$ (where $k_1, k_2 \in \mathbb{N}$) are dense in the real line. This may be easily seen from the fact that numbers of the form

$$k_1\frac{\rho_1}{\rho_2} \pmod 1, \qquad k_1 \in \mathbb{N}$$

are dense in the unit interval (since ρ_1/ρ_2 is irrational). Therefore numbers of the form $k_1\frac{\rho_1}{\rho_2} - k_2$ (where besides k_1 also $k_2 \in \mathbb{N}$) are dense in the real line, and hence so are numbers of the form $k_1\rho_1 - k_2\rho_2$. We may make $k_1\rho_1 - k_2\rho_2$ arbitrarily close to any real number, by taking appropriate k_1 and k_2 (which, if necessary, have to be chosen sufficiently large).

Now consider the spirals $\mathfrak{S}_1(\omega_1,0)$ and $\mathfrak{S}_2(\omega_2,0)$. These spirals are centred on the origin in the plane. They also have the property that they are invariant under transformations $\omega \mapsto \omega - 2n_j\pi$ ($n_j \in \mathbb{N}$). Such a transformation has the effect of enlarging the spiral $\mathfrak{S}_i(\omega,0)$ by a factor of $e^{2\rho_i n_j\pi}$, which recovers the original spiral. Then it is clear that a straight line through the origin in the plane will intersect the spiral $\mathfrak{S}_i(\omega,0)$ in an infinite set of points $\mathfrak{S}_i(\omega_i^* + 2n_j\pi, 0)$ for some ω_i^*. Moreover, the tangents at $\mathfrak{S}_i(\cdot,0)$ at the points $\omega_i^* + 2n_j\pi$, $j \in \mathbb{N}$, are all parallel.

Now we would like to chose one such straight line such that the tangent directions to the spirals at $\mathfrak{S}_1(\omega_1^*,0)$ and $\mathfrak{S}_2(\omega_2^*,0)$ are different. Such a straight line exists because the exponential rates ρ_1 and ρ_2 are different. However note that $\mathfrak{S}_1(\omega_1^*,0)$ and $\mathfrak{S}_2(\omega_2^*,0)$ are not necessarily intersection points of the two spirals, see Figure 3.

Now let the constant M_0 be defined by

$$\mathfrak{S}_1(\omega_1^*,0) = M_0\mathfrak{S}_2(\omega_2^*,0).$$

Then (ω_1^*,ω_2^*) is a point of transversal intersection of the two spirals $\mathfrak{S}_1(\omega_1^*,0)$ and $M_0\mathfrak{S}_2(\omega_2^*,0)$. Equivalently, $((\omega_1^*,\omega_2^*),M_0)$ is a solution of the equation

$$\mathfrak{S}((\omega_1,\omega_2),M) := (\mathfrak{S}_1(\omega_1,0) - M\mathfrak{S}_2(\omega_2,0) = 0 \tag{3.19}$$

and that the Jacobian $D_1\mathfrak{S}((\omega_1^*,\omega_2^*),M_0)$ is nonsingular. Then by the Implicit Function Theorem, we may find functions $\omega_1^*(M)$, $\omega_2^*(M)$ for M close to M_0, such that

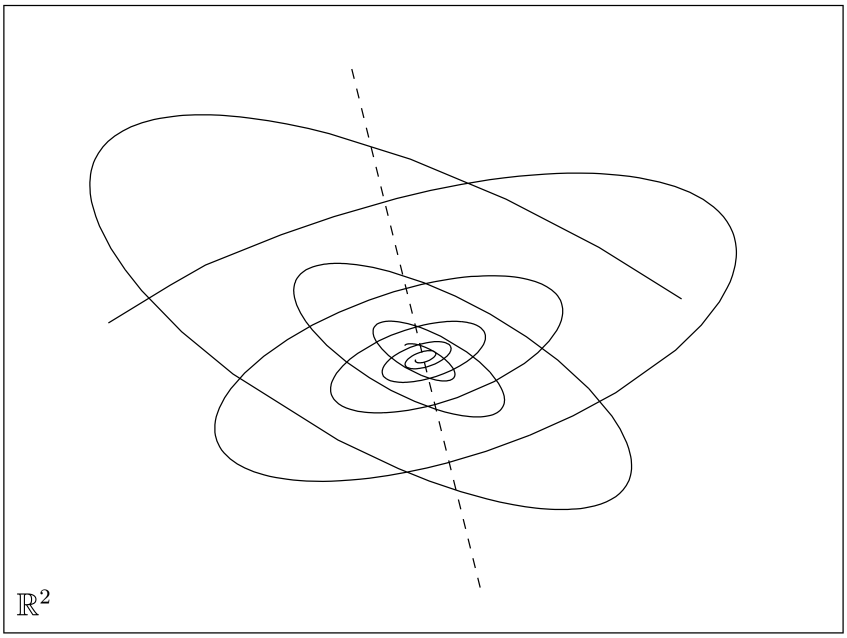

Figure 3: The spirals $\mathfrak{S}_1(\omega_1, 0)$ and $\mathfrak{S}_2(\omega_2, 0)$ together with a straight line through the origin that intersects both spirals at points that have different tangent direction.

$\omega_1^*(M_0) = \omega_1^*$, $\omega_2^*(M_0) = \omega_2^*$, and

$$\mathfrak{S}((\omega_1^*(M), \omega_2^*(M)), M) \equiv 0.$$

Now we can make $M := e^{(k_1\rho_1 - k_2\rho_2)2\pi}$ arbitrarily close to M_0 by choosing appropriate k_1, k_2 arbitrarily large. Hence we find for $\omega_i = \omega_i^*(M)$

$$\begin{pmatrix} e^{-2k_1\rho_1\pi} e^{-\rho_1\omega_1} \sin\omega_1 & -e^{-2k_2\rho_2\pi} e^{-\rho_2\omega_2} c_{21} \sin\omega_2 \\ e^{-2k_1\rho_1\pi} e^{-\rho_1\omega_1} \sin(\omega_1 + \varphi_{12}) & -e^{-2k_2\rho_2\pi} e^{-\rho_2\omega_2} c_{22} \sin(\omega_2 + \varphi_{22}) \end{pmatrix} = 0. \qquad (3.20)$$

The 2π-periodicity of $\sin(\cdot)$ gives

$$\mathfrak{S}_1(\omega_1^*(M) + 2k_1\pi, 0) = \mathfrak{S}_2(\omega_2^*(M) + 2k_2\pi, 0).$$

Now, we may choose sequences $(k_i(n))_{n\in\mathbb{N}}$ tending to infinity as n tends to infinity, such that the sequence (M_n), where $M_n := e^{(k_1(n)\rho_1 - k_2(n)\rho_2)2\pi}$ tends to M_0 (as $k_i \to \infty$). Therefore we find an infinite number of solutions to equation (3.20), and hence an infinite number of transversal intersections of the spirals $\mathfrak{S}_1(\omega_1, 0)$ and $\mathfrak{S}_2(\omega_2, 0)$. This proves the Lemma. ■

It is clear that under perturbation of μ, (3.18) will retain only finitely many transversal intersections, but this number of transversal may be arbitrarily large by taking μ sufficiently close to zero.

The following Lemma states that in the case where ρ_1/ρ_2 is rational, the spirals $\mathfrak{S}_1(\omega_1, 0)$ and $\mathfrak{S}_2(\omega_2, 0)$ are together self-similar.

Lemma 3.7. *Assume that that $\mu_1 = \mu_2 = 0$ and ρ_1/ρ_2 is rational, $\rho_1/\rho_2 = m/n$. Then equation* (3.18) *is periodic in* (ω_1, ω_2).

Proof. Under the transformations

$$\begin{aligned} \omega_1 &\to \omega_1 + 2n\pi \\ \omega_2 &\to \omega_2 + 2m\pi \end{aligned}$$

equation (3.18) becomes

$$\begin{pmatrix} e^{-2\rho_1 n\pi} e^{-\rho_1\omega_1} \sin\omega_1 & +e^{-2\rho_2 m\pi} e^{-\rho_2\omega_2} c_{21} \sin\omega_2 \\ e^{-2\rho_1 n\pi} e^{-\rho_1\omega_1} \sin(\omega_1 + \varphi_{12}) & +e^{-2\rho_2 m\pi} e^{-\rho_2\omega_2} c_{22} \sin(\omega_2 + \varphi_{22}) \end{pmatrix} = 0$$

Multiplying through both equations by $e^{2\rho_1 n\pi}$ and using $\rho_1 n - \rho_2 m = 0$ recovers the original equations (3.18). ■

Lemma 3.8. *Let $\mathcal{H}$ be defined as in the Introduction. Assume that $\mu_1 = \mu_2 = 0$. There exists an open and dense set $\mathcal{D} \subset \mathcal{H}$, such that for each $f(\cdot, 0) \in \mathcal{D}$, equation* (3.17) *has an infinite number of non-degenerate solutions, or equivalently the spirals $\mathfrak{S}_1(\omega_1, 0)$ and $\mathfrak{S}_2(\omega_2, 0)$ have an infinite number of transversal intersections.*

Proof. First we note that since $\mathcal{H}$ is endowed with the C^1 topology, the constants in equation (3.17) will vary continuously in this topology, meaning that they depend continuously on the vector field. Therefore Lemma 3.6 implies that there is a dense set $\tilde{\mathcal{D}} \subset \mathcal{H}$ such that equation (3.17) has an infinite number of transversal intersections.

We now consider a vector field f in $\tilde{\mathcal{D}}$, and take a small neighbourhood $B(f;\epsilon)$ around it. It is clear that if ϵ is sufficiently small, the constants in (3.17) will undergo an arbitrarily small perturbation, and at least one transversal intersection of $\mathfrak{S}_1(\omega_1, 0)$ and $\mathfrak{S}_2(\omega_2, 0)$ will persist. By Lemma 3.6, for every point in $B(f;\epsilon)$ where ρ_1/ρ_2 is irrational, there will be infinitely many transversal intersections. Also, since there is as least one transversal intersection at every point in $B(f;\epsilon)$, we may use Lemma 3.7 to show that where ρ_1/ρ_2 is rational, there must also be infinitely many transversal intersections.

This provides an open and dense set in $\mathcal{H}$ where equation (3.17) has infinitely many non-degenerate solutions, and completes the Lemma. ■

3.2.2 Symbolic dynamics

For our further analysis we assume that the coordinate transformation introduced in Lemma 3.5 has been carried out. Further we assume without loss of generality $0 < \rho_1 < \rho_2$. Finally we introduce an artificial parameter ϵ and define for $i \in \mathbb{Z}$:

$$\hat{\Xi}_i(\boldsymbol{\omega}, \mu, \epsilon) := e^{2\rho_1\omega_{1,i+1}} \begin{pmatrix} \mu_1 + L_{11}(\omega_{1,i+1}, \mu) - L_{21}(\omega_{2,i}, \mu) + \epsilon r_{1,i}(\boldsymbol{\omega}, \mu) \\ \mu_2 + L_{12}(\omega_{1,i+1}, \mu) - L_{22}(\omega_{2,i}, \mu) + \epsilon r_{2,i}(\boldsymbol{\omega}, \mu) \end{pmatrix}.$$

Then

$$\hat{\boldsymbol{\Xi}}(\boldsymbol{\omega}, \mu, 1) = 0,$$

where

$$\hat{\boldsymbol{\Xi}} : l^\infty_{\mathbb{R}^2} \times \mathbb{R} \times \mathbb{R} \to l^\infty_{\mathbb{R}^2} \quad (\boldsymbol{\omega}, \mu, \epsilon) \mapsto \left(\hat{\Xi}_i(\boldsymbol{\omega}, \mu, \epsilon)\right)_{i\in\mathbb{Z}},$$

is an equivalent formulation of the bifurcation equation (3.13). Here $l^\infty_{\mathbb{R}^2}$ denotes the space of bounded sequences in $\mathbb{R}^2$.

We define

$$\Omega_{n_0} := \{(\omega_1^*(M_n) + 2k_1(n)\pi, \omega_2^*(M_n) + 2k_2(n)\pi), n \geq n_0\}.$$

Further let $\Omega_{n_0,N} \subset \Omega_{n_0}$ comprise N elements, $|\Omega_{n_0,N}| = N$, for some $N \in \mathbb{N}$. With that we define the following set of sequences $\boldsymbol{\omega}$

$$\boldsymbol{\Omega}^{\mathbb{Z}}_{n_0,N} := \{\boldsymbol{\omega} : (\omega_{1,i+1}, \omega_{2,i}) \in \Omega_{n_0,N}\}.$$

Recall that the elements of $\Omega_{n_0,N}$ correspond to transversal intersections of the spirals $\mathfrak{S}_1(\cdot,0)$ and $\mathfrak{S}_2(\cdot,0)$.

Now we fix a sequence $\hat{\boldsymbol{\omega}} \in \boldsymbol{\Omega}^{\mathbb{Z}}_{n_0,N}$ and solve $\hat{\boldsymbol{\Xi}} = 0$ near $(\hat{\boldsymbol{\omega}}, 0, 0)$ for $\boldsymbol{\omega} = \boldsymbol{\omega}_{\hat{\boldsymbol{\omega}}}(\mu, \epsilon)$ by means of the Implicit Function Theorem. Indeed $\hat{\boldsymbol{\Xi}}(\hat{\boldsymbol{\omega}}, 0, 0) = 0$ and $D_1\hat{\boldsymbol{\Xi}}(\hat{\boldsymbol{\omega}}, 0, 0)$ is a invertible mapping $l^\infty_{\mathbb{R}^2} \to l^\infty_{\mathbb{R}^2}$; the Implicit Function Theorem can be applied and provides $\boldsymbol{\omega}_{\hat{\boldsymbol{\omega}}}(\cdot,\cdot)$.

We claim that for sufficiently large n_0 the solving function $\boldsymbol{\omega}_{\hat{\boldsymbol{\omega}}}(\cdot,\cdot)$ is defined for $\epsilon = 1$. This requires a closer look at the proof of the Implicit Function Theorem. The equation $\hat{\boldsymbol{\Xi}}(\boldsymbol{\omega}, \mu, \epsilon) = 0$ is equivalent to the fixed point equation

$$\boldsymbol{\omega} = \boldsymbol{\omega} - \Big[D_1\hat{\boldsymbol{\Xi}}(\hat{\boldsymbol{\omega}}, 0, 0)\Big]^{-1}\hat{\boldsymbol{\Xi}}(\boldsymbol{\omega}, \mu, \epsilon) =: \boldsymbol{\mathcal{A}}(\boldsymbol{\omega}, \mu, \epsilon).$$

The idea of the proof of the Implicit Function Theorem is to show that $\boldsymbol{\mathcal{A}}(\cdot, \mu, \epsilon)$ is, for (μ, ϵ) close to $(0,0)$, a contractive mapping of some closed neighbourhood of $\hat{\boldsymbol{\omega}}$ into itself. Here we want to make clear that this is even true for $\epsilon \in [0,1]$ only by requiring that n_0 is sufficiently large. For that we consider

$$D_1\boldsymbol{\mathcal{A}}(\boldsymbol{\omega}, \mu, \epsilon) = \Big[D_1\hat{\boldsymbol{\Xi}}(\hat{\boldsymbol{\omega}}, 0, 0)\Big]^{-1}\Big(D_1\hat{\boldsymbol{\Xi}}(\hat{\boldsymbol{\omega}}, 0, 0) - D_1\hat{\boldsymbol{\Xi}}(\boldsymbol{\omega}, \mu, \epsilon)\Big).$$

The main observation in this respect is that the norm of $D_1\hat{\boldsymbol{\Xi}}(\hat{\boldsymbol{\omega}}, 0, 0)$ is uniformely bounded for all n_o and that the norm of $D_1\hat{\boldsymbol{\Xi}}(\hat{\boldsymbol{\omega}}, 0, 0) - D_1\hat{\boldsymbol{\Xi}}(\boldsymbol{\omega}, \mu, \epsilon)$ can be made arbitrarily small independently of the choice of ϵ only by choosing n_o large enough and $\|\hat{\boldsymbol{\omega}} - \boldsymbol{\omega}\|$ sufficiently small. Moreover, the larger n_0 the closer $D_1\hat{\Xi}_i(\hat{\boldsymbol{\omega}}, 0, 0)$ is to the derivative $D_{(\omega_1,\omega_2)}(e^{2\rho_1\omega^*_{1,i+1}}\mathfrak{S}((\omega^*_{1,i+1}, \omega^*_{2,i}), M_0))$, see also Section 3.2.1, Equation (3.19). This gives the addressed boundedness of $D_1\hat{\boldsymbol{\Xi}}(\hat{\boldsymbol{\omega}}, 0, 0)$. We also refer to [16] where similar considerations (verification of shift dynamics in the Shilnikov scenario by using Lin's method) have been worked out in more detail.

Hence $\boldsymbol{\omega}_{\hat{\boldsymbol{\omega}}}(\cdot, 1)$ is well defined and we have

$$\Xi_i(\boldsymbol{\omega}_{\hat{\boldsymbol{\omega}}}(\mu, 1), \mu) = 0, \quad i \in \mathbb{Z}.$$

Note that moreover $\operatorname{dom}\boldsymbol{\omega}_{\hat{\boldsymbol{\omega}}}(\cdot,\cdot)$ does not depend on $\hat{\boldsymbol{\omega}} \in \boldsymbol{\Omega}^{\mathbb{Z}}_{n_0,N}$.

By $x_{\hat{\boldsymbol{\omega}}}(\mu)(\cdot)$ we denote the corresponding solution of (1.1), see also (2.11) and (2.23),

$$x_{\hat{\boldsymbol{\omega}}}(\mu)(0) = q_1^+(\mu)(0) + v^+_{1,1}(\boldsymbol{\omega}_{\hat{\boldsymbol{\omega}}}(\mu, 1), \mu)(0),$$

and define

$$\mathcal{S}^N_\mu := \{x_{\hat{\boldsymbol{\omega}}}(\mu)(0) : \hat{\boldsymbol{\omega}} \in \boldsymbol{\Omega}^{\mathbb{Z}}_{n_0,N}\}.$$

With these $\mathcal{S}_\mu^N$ we prove the theorem. For that purpose we we introduce a shift operator on $\boldsymbol{\Omega}_{n_0,N}^{\mathbb{Z}}$:

$$\zeta : \boldsymbol{\Omega}_{n_0,N}^{\mathbb{Z}} \to \boldsymbol{\Omega}_{n_0,N}^{\mathbb{Z}}, \quad \boldsymbol{\omega} \mapsto \boldsymbol{\tau}, \quad (\tau_{1,i}, \tau_{2,i}) := (\omega_{1,i+1}, \omega_{2,i+1}).$$

The system $(\boldsymbol{\Omega}_{n_0,N}^{\mathbb{Z}}, \zeta)$ is conjugated to the full shift on N symbols - even topologically conjugated by considering $\boldsymbol{\Omega}_{n_0,N}^{\mathbb{Z}}$ equipped with the product topology coming from $\boldsymbol{\Omega}_{n_0,N}^{\mathbb{Z}} \cong \{(\omega_1^k, \omega_2^k),\ k = 1, \ldots, N\}^{\mathbb{Z}}$. There is a canonical one-to-one mapping

$$h_\mu : \boldsymbol{\Omega}_{n_0,N}^{\mathbb{Z}} \to \mathcal{S}_\mu^N, \quad \hat{\boldsymbol{\omega}} \mapsto x_{\hat{\boldsymbol{\omega}}}(\mu)(0).$$

By Π_μ we denote the first-return map on Σ_1; $x \in \operatorname{dom} \Pi_\mu$, if there is a $t^* > 0$ such that $\phi_\mu^{t^*}(x) \in \Sigma_1$, ($\{\phi_\mu^{t^*}(\cdot)\}$ flow of (1.1)), and $\phi_\mu^t(x) \notin \Sigma_1$, $\phi_\mu^t(x) \in \mathcal{U}$, for all $t \in (0, t^*)$, see Definition 1.1 for the explanation of $\mathcal{U}$. We claim that $\mathcal{S}_\mu^N \subset \operatorname{dom} \Pi_\mu$, that $\mathcal{S}_\mu^N$ is Π_μ invariant, and moreover

$$\Pi_\mu \circ h_\mu = h_\mu \circ \zeta. \tag{3.21}$$

The latter equation can be concluded from the uniqueness part of Theorem 2.2 and the construction of $\boldsymbol{\omega}_{\hat{\boldsymbol{\omega}}}$ as follows. It holds

$$\Pi_\mu x_{\hat{\boldsymbol{\omega}}}(\mu)(0) = x_{\hat{\boldsymbol{\omega}}}(\mu)(2(\omega_{1,1}(\mu,1) + \omega_{2,1}(\mu,1))) = q_1^+(0) + \bar{v}_{1,2}^+(\boldsymbol{\omega}(\mu,1),\mu)(0),$$

and hence

$$\Pi_\mu x_{\hat{\boldsymbol{\omega}}}(\mu)(0) = x_{\zeta\hat{\boldsymbol{\omega}}}(\mu)(0).$$

This is just another representation of (3.21).

Equation (3.21) means that $(\mathcal{S}_\mu^N, \Pi_\mu)$ is conjugated to $(\boldsymbol{\Omega}_{n_0,N}^{\mathbb{Z}}, \zeta)$. So, in order to prove topological conjugacy, as claimed in the theorem, it remains to prove that h_μ is a homeomorphism. For that purpose we consider h_μ as a composition of mappings

$$\mathfrak{H} : \boldsymbol{\Omega}_{n_0,N}^{\mathbb{Z}} \to \mathfrak{O} := \{\boldsymbol{\omega}_{\hat{\boldsymbol{\omega}}}(\mu,1) : \hat{\boldsymbol{\omega}} \in \boldsymbol{\Omega}_{n_0,N}^{\mathbb{Z}}\}, \quad \hat{\boldsymbol{\omega}} \mapsto \boldsymbol{\omega}_{\hat{\boldsymbol{\omega}}}(\mu,1),$$

and

$$\mathfrak{h} : \mathfrak{O} \to \mathcal{S}_\mu^N, \quad \boldsymbol{\omega} \mapsto q_1^+(\mu)(0) + v_{1,1}^+(\boldsymbol{\omega}_{\hat{\boldsymbol{\omega}}}(\mu,1),\mu)(0).$$

First we show that $\mathfrak{H}$ is a homeomorphism, where both $\boldsymbol{\Omega}_{n_0,N}^{\mathbb{Z}}$ and $\mathfrak{O}$ are considered to be equipped with the product topology. We start with showing that $\mathfrak{O}$ is compact. Let ρ be that small that for any two (different) elements of $\boldsymbol{\Omega}_{n_0,N}$ the closed balls with radius ρ centred at these elements do not intersect:

$$B[(\omega_1^i, \omega_2^i), \rho] \cap B[(\omega_1^j, \omega_2^j), \rho] = \emptyset, \quad (\omega_1^i, \omega_2^i), (\omega_1^j, \omega_2^j) \in \boldsymbol{\Omega}_{n_0,N}, \quad i \neq j.$$

By construction $\mathfrak{O} \subset \left(\cup_{i=1}^N B[(\omega_1^i, \omega_2^i), \rho]\right)^{\mathbb{Z}}$. Since $\cup_{i=1}^N B[(\omega_1^i, \omega_2^i), \rho]$ is compact also $\left(\cup_{i=1}^N B[(\omega_1^i, \omega_2^i), \rho]\right)^{\mathbb{Z}}$ is compact by the Tychonoff theorem, see [5].

So, in order to verify the compactness of $\mathfrak{O}$ it remains to show that $\mathfrak{O}$ is closed. The set $\mathfrak{O}$ however is the set of zeros of the mapping $\hat{\boldsymbol{\Xi}}(\cdot, \mu, 1) : \left(\cup_{i=1}^N B[(\omega_1^i, \omega_2^i), \rho]\right)^{\mathbb{Z}} \to l_{\mathbb{R}^2}^\infty$. By [11, Lemma 3.4] this mapping is continuous, where the spaces are equipped with product topology. Altogether this proves that $\mathfrak{O}$ is compact.

Consider now the one-to-one map

$$\mathfrak{H}^{-1} : \mathfrak{O} \to \boldsymbol{\Omega}_{n_0,N}^{\mathbb{Z}}.$$

Obviously $\mathfrak{H}^{-1}$ is continuous. Due to the compactness of $\mathfrak{O}$ also $\mathfrak{H}$ is continuous – hence $\mathfrak{H}$ is a homeomorphism.

Next we consider $\mathfrak{h}$. Again by invoking [11, Lemma 3.4] or its proof respectively, we find that $\mathfrak{h}$ is continuous. (Again both $\mathfrak{O}$ and $\mathbf{\Omega}^{\mathbb{Z}}_{n_0,N}$ should be equipped with the product topology.) Once more the compactness of $\mathfrak{O}$ gives that $\mathfrak{h}$ is a homeomorphism.

All in all $h_\mu = \mathfrak{h} \circ \mathfrak{H}$ is a homeomorphism and therefore $(\mathcal{S}^N_\mu, \Pi_\mu)$ is topologically conjugated to the full shift on N symbols.

To conclude we remark that the sets $\mathcal{S}^N_\mu$ have the properties which we claimed in theorem. Further, for fixed $\mu \neq 0$ the spirals have only finitely many intersections. So there is an N_μ such that for $N > N_\mu$ there are neither N-periodic orbits nor shift dynamics in N symbols.

3.3 Hyperbolicity of the shift dynamics

Here we outline the argument provided in order to prove hyperbolicity of the sets found in section 3.2.2. The result is based on the following two theorems, the first of which was produced in a paper of Mañe [12]:

Theorem 3.9. *Every C^1 structurally stable diffeomorphism of a closed manifold satisfies Axiom A.*

We recall that an Axiom A diffeomorphism f is one for which the nonwandering set $\Omega(f)$ is hyperbolic, and the periodic points are dense in $\Omega(f)$. The following theorem is by Palis [14], and is concerned with the C^1 Ω-stability conjecture. Recall that a diffeomorphism f is Ω-stable if the nonwandering set $\Omega(f)$ is structurally stable.

Theorem 3.10. *If a C^1 diffeomorphism is Ω-stable then it satisfies Axiom A.*

It is clear from the proof of Theorem 1.2 that the solution set found which has dynamics that are topologically conjugate to a full shift on any finite number of symbols is C^1-persistent. Then we may apply Palis' theorem to this set, having defined a suitable diffeomorphism from Σ_1 to itself in a neighbourhood of this set. We then conclude that this set is indeed hyperbolic.

4 Reversible systems

In this section we consider the system (1.1) in $\mathbb{R}^{2n+1}$. We suppose that the family (1.1) is reversible with respect to a linear involution R, thus

$$R\, f(x, \mu) = -f(Rx, \mu),$$

and we assume that the fixed point space of R is n-dimensional, dim Fix $R = n$. Further the fixed points p_1 and p_2 are non-symmetric but lie in the same group orbit, that is $R(p_1) = p_2$. As well we assume that the heteroclinic cycle Γ is symmetric, that means

$$R\Gamma = \Gamma.$$

Altogether an orbit $\mathcal{O}$ of a reversible vector field is called *symmetric* if $R(\mathcal{O}) = \mathcal{O}$. The symmetry of Γ implies that both Γ_1 and Γ_2 are symmetric heteroclinic orbits. Besides these particularities we want, on the whole, to make the same assumptions as in the general situation. However, the reversibility which we imposed on the system enforces several particular properties and statements.

First of all we want to mention that such a symmetric heteroclinic cycle is a codimension-one configuration; so in the present case it is sufficient to consider a one-parameter family of vector fields, that means $\mu \in \mathbb{R}$ and $f : \mathbb{R}^{2n+1} \times \mathbb{R} \to \mathbb{R}^{2n+1}$.

The involution R maps the stable manifold of p_i onto the unstable manifold of Rp_i. We assume that the fixed point p_1 has a n-dimensional stable manifold, and a (n+1)-dimensional unstable (focus) manifold, and then due to the symmetry *vice versa* for p_2. This makes it possible to adopt the transversality assumption (1.2); then, by our assumptions, the second transversality assumption (1.3) is automatically fulfilled.

Due to the reversibility, the following relation for the principal stable and unstable eigenvalues holds true

$$-\lambda_2^s = \lambda_1^u =: \lambda^u \quad -\lambda_2^u = \lambda_1^s =: \lambda^s.$$

So $-\lambda^u$, $-\overline{\lambda^u}$ and $-\lambda^s$ are the principal stable and unstable eigenvalues respectively of $D_1 f(p_2, 0)$. In particular for the real and imaginary parts of the leading complex eigenvalues that means

$$\rho_1 = \rho_2 =: \rho, \quad \phi_1 = \phi_2 =: \phi.$$

We adopt the Hypotheses (H 1) and (H 2) as they are. We only want to remark that in the present case there is some redundancy in their formulation – the second assumption follows in each case from the first one. We perpetuate also Hypothesis (H 3). But in the case under consideration it turns out that $\xi^\infty(\cdot)$ can be seen as a mapping $\mathbb{R} \to \mathbb{R}$, see Lemma 4.3.

In the discussion of general systems transversal intersections of certain spirals play a key role. In order to ensure transversal intersections of the corresponding spirals in the reversible context we assume:

(H 4) $\quad \frac{\phi(0)}{\rho(0)} \neq \tan(\varphi_2 - \varphi_1)$

The main results regarding reversible systems are summarised in the following Theorem, which generalises results of [10] to higher dimensions.

Theorem 4.1. *Consider a reversible system as described above. Then it holds*

1. *At* $\mu = 0$, *there exists for each* $N \geq 2$ *a set* $\mathcal{S}_0^N \subset \Sigma_1$ *which is invariant under the first-return-map* Π *(defined by the flow), and* $(\mathcal{S}_0^N, \Pi)$ *is topologically conjugated to the full shift on* N *symbols. Each set* $\mathcal{S}_0^N$ *is setwise symmetric, that is* $R(\mathcal{S}_0^N) = \mathcal{S}_0^N$. *The collection of all these sets exists only for* $\mu = 0$, *although each set is individually structurally stable meaning that for* μ *close to* 0 *there is a set* $\mathcal{S}_\mu^N$ *which converges to* $\mathcal{S}_0^N$ *in the Hausdorff-metric as* $\mu \to 0$, *and* $(\mathcal{S}_\mu^N, \Pi(\mu))$ *is topologically conjugated to a full shift on* N *symbols.*
2. *At* $\mu = 0$, *there is a countably infinite set of symmetric 1-periodic orbits, with difference in period asymptotically tending to* $\pi/2\phi(\mu)$. *For* $\mu \neq 0$ *there are*

finitely many symmetric 1-periodic orbits. With the addition of the parameter to the phase space these periodic orbits form a 1 parameter family parameterised by period, see Figure 4. The periodic orbits converge to the heteroclinic cycle as the period tends to infinity.

3. *For each $k \geq 2$, $k \in \mathbb{N}$ there is a countable set $\mathcal{M}_k$ of parameter values, for which there exists a symmetric k-(2,1)-heteroclinic orbit. In particular there are sequences $(\mu_n^+(k))_{n\in\mathbb{N}}, (\mu_n^-(k))_{n\in\mathbb{N}} \subset \mathcal{M}_k$ with $\mu_n^+(k) \searrow 0$, $\mu_n^-(k) \nearrow 0$ as $n \to \infty$. There are no non-symmetric k-(2,1)-heteroclinic orbits.*

4. *At $\mu = 0$ there exist countably many k-(1,2)-heteroclinic orbits for each $k \geq 2$. Moreover, for fixed k, $q_1(0)$ is an accumulation point of the intersections of the k-(1,2)-heteroclinic orbits with Σ_1. Each such k-(1,2)-heteroclinic orbit can be continued for $\mu \neq 0$. But for fixed $\mu \neq 0$ and fixed k there are only finitely many k-(1,2)-heteroclinic orbits.*

5. *For each $k \geq 2$, $k \in \mathbb{N}$ there is a countable set $\mathcal{M}_k^{hom}$ of parameter values, for which there exists a k-homoclinic orbit. In particular there are sequences $(\mu_n^+(k))_{n\in\mathbb{N}}, (\mu_n^-(k))_{n\in\mathbb{N}} \subset \mathcal{M}_k^{hom}$ with $\mu_n^+(k) \searrow 0$, $\mu_n^-(k) \nearrow 0$ as $n \to \infty$.*

Before starting our analysis we want to comment the discovered dynamics. The symmetry property of the sets $\mathcal{S}_0^N$ makes sense if we introduce cross-sections Σ_j of Γ_j, $j = 1, 2$, containing $\operatorname{Fix} R$. The existence of shift dynamics can be explained as in the general case. Here we have the additional feature that the corresponding 1-periodic orbits are symmetric. However, the sets $\mathcal{S}_0^N$ contain also nonsymmetric orbits.

Figure 4 depicts the solutions of the bifurcation equation for 1-periodic orbits, see Section 4.3.2. The oscillatory behaviour exhibited in Figure 4 is characteristic of the

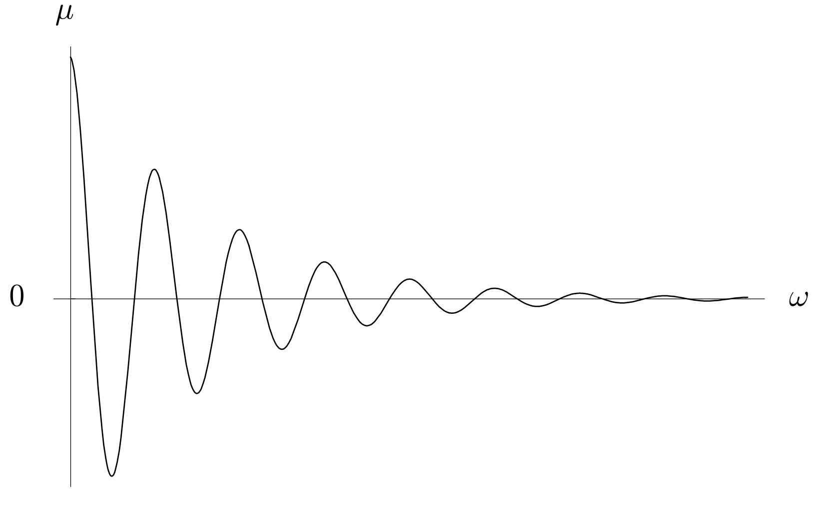

Figure 4: Bifurcation diagram for 1-periodic orbits. Here 2ω is the period of the corresponding 1-periodic orbits.

Shilnikov homoclinic orbit to a saddle-focus under certain eigenvalue conditions (the non-real eigenvalues are closer to the imaginary axis than the real one), where there exists chaotic dynamics. This represents a marked difference in our analysis: we do not require any eigenvalue condition to have this oscillatory behaviour, which is due to the fact that only the complex eigenvalues of p_1 and p_2 appear in the bifurcation equations.

Note that the cylinder orbit theorem that states that symmetric periodic orbits generically come in 1-parameter families, see Devaney [4] or Vanderbauwhede [20], does not work in the present context – $\dim \operatorname{Fix} R \neq \dim \operatorname{Fix}(-R)$. So there may exist isolated symmetric periodic orbits as stated in the theorem.

We also note that the bifurcation equation for 1-periodic orbits may be solved to find non-symmetric 1-periodic orbits. Those do not exist at $\mu = 0$. They are associated to the families of homoclinic orbits to each fixed point, see 5.) in the above theorem and Section 4.3.5. Of course the homoclinic orbits are nonsymmetric – the R-image of a homoclinic orbit to p_i is a homoclinic orbit to Rp_i.

We remark that our Hypothesis (H 3), see Section 3.1, which prescribes that the stable manifold of p_1 and the unstable manifold of p_2 split with positive speed, implies that there is just one 1-(2,1)-heteroclinic orbit, and this is the original one that exists at $\mu = 0$. In $\mathbb{R}^3$, $n = 1$, the statement on k-(2,1)-heteroclinic orbits is partly obvious. It is clear that there can only be at most one k-(2,1)-heteroclinic orbit for each parameter value, because in this case the stable manifold of p_1 is one-dimensional. Consequently such a connection must be symmetric, because the R-image is also a k-(2,1)-heteroclinic orbit.

4.1 Special features of Lin's method for reversible systems

First we want to remark that there are R-invariant C^∞ bump function transformations that make $p_1(\mu), p_2(\mu)$ constant, and that there are R-invariant C^∞ bump function transformations, with disjoint supports, based around each of the fixed points p_i, that bring the local stable/unstable manifolds into the form (2.4).

In order to establish the direct sum decomposition (2.3) we use here an R-invariant inner product $\langle \cdot, \cdot \rangle$. Such an inner product exists because $\{R, id\}$ forms a finite group. Further fix the heteroclinic solutions so that $q_1(0), q_2(0) \in \operatorname{Fix} R$. The reversibility of the vector field f effects that for $x \in \operatorname{Fix} R$ it holds $f(x) \in \operatorname{Fix}(-R)$. Therefore Y_i and hence also Σ_i, $i = 1, 2$, are R-invariant, see also (2.2). Moreover, Y_i, $i = 1, 2$, contain $\operatorname{Fix} R$ and an n-dimensional subspace of $\operatorname{Fix}(-R)$.

Lemma 4.2. *The space Z has a direct sum decomposition into one-dimensional subspaces of* $\operatorname{Fix} R$ *and* $\operatorname{Fix}(-R)$.

Proof. The vector field direction $f(q_1(0), 0)$ belongs to $\operatorname{Fix}(-R)$. The spaces W_1^+ and W_1^- are R-images of each other. So $W_1^+ \oplus W_1^-$ has a direct sum decomposition into $(n-1)$-dimensional subspaces of $\operatorname{Fix} R$ and $\operatorname{Fix}(-R)$, mind $\dim W_1^+ = \dim W_1^- = n - 1$. Since Z is R-invariant it has a direct sum decomposition into subspaces of $\operatorname{Fix} R$ and $\operatorname{Fix}(-R)$. Now, counting dimensions gives the lemma. ∎

Next we turn to the splitting of the stable and unstable manifolds which we considered in Section 2.1 for the general case. Additionally to our explanations there we observe that also $Rq_{2,\mu} \in \Sigma_2 \cap W^u(p_1, \mu) \cap W^s(p_2, \mu)$. So, due to uniqueness we have $Rq_{2,\mu} = q_{2,\mu}$, hence $q_{2,\mu} \in \operatorname{Fix} R$; the 1-(1,2)-heteroclinic orbit through $q_{2,\mu}$ is symmetric.

Further Lemma 2.1 remains true in the present case. There even holds the following "reversible supplementation".

Lemma 4.3. *Moreover, due to the reversibility, it holds*

$$q_1^+(\mu)(0) - q_1^-(\mu)(0) \in \operatorname{Fix}(-R).$$

Proof. Also the pair $(Rq_1^-(\mu)(\cdot), Rq_1^+(\mu)(\cdot))$ fulfils (i)-(iii) of Lemma 2.1. Therefore the uniqueness part of Lemma 2.1 provides $Rq_1^-(\mu)(0) = q_1^+(\mu)(0)$. ∎

In order to detect symmetric (periodic) orbits we use symmetric Lin orbits to propose the bifurcation equations. We set work on the definition of equality and periodicity of Lin orbits.

Definition 4.4. *1. Two Lin orbits $X(\boldsymbol{\omega}) = (X_i)_{i\in\mathbb{Z}}$ and $\hat{X}(\hat{\boldsymbol{\omega}}) = (\hat{X}_i)_{i\in\mathbb{Z}}$ are equal, $X(\boldsymbol{\omega}) = \hat{X}(\hat{\boldsymbol{\omega}})$, if there is an $i_0 \in \mathbb{Z}$ such that for all $i \in \mathbb{Z}$ it holds $X_i = \hat{X}_{i+i_0}$.*

2. A Lin orbit $X(\boldsymbol{\omega}) = (X_i)_{i\in\mathbb{Z}}$ is k-periodic, $k \in \mathbb{N}$, if for all $i \in \mathbb{Z}$ it holds $X_i = X_{i+k}$.

If $X(\boldsymbol{\omega}) = \hat{X}(\hat{\boldsymbol{\omega}})$ then

- (i) $\bigcup\limits_{i\in\mathbb{Z}} X_i = \bigcup\limits_{i\in\mathbb{Z}} \hat{X}_i$;
- (ii) for all $i \in \mathbb{Z}$ and $j \in \{1, 2\}$ it holds $\omega_{j,i} = \hat{\omega}_{j,i+i_0}$; this condition is also sufficient for the equality of X and $\hat{X}$;
- (iii) for all $i \in \mathbb{Z}$ it holds $\Xi_i = \hat{\Xi}_{i+i_0}$.

If $X(\boldsymbol{\omega})$ is an k-periodic Lin orbit then for all $i \in \mathbb{Z}$ we have $\Xi_i = \Xi_{i+k}$.

Now fix μ. Let $X := X(\boldsymbol{\omega}, \mu) = (X_i(\boldsymbol{\omega}, \mu))_{i\in\mathbb{Z}}$ be a Lin orbit associated to a given sequence $\boldsymbol{\omega} = (\omega_{1,i}, \omega_{2,i})_{i\in\mathbb{Z}}$.

Lemma 4.5. *The R-image RX of X is a Lin orbit $\hat{X} := \hat{X}(\hat{\boldsymbol{\omega}}, \mu) = (\hat{X}_i(\hat{\boldsymbol{\omega}}, \mu))_{i\in\mathbb{Z}}$ associated to the sequence $\hat{\boldsymbol{\omega}} = (\hat{\omega}_{1,i}, \hat{\omega}_{2,i})_{i\in\mathbb{Z}}$, with $\hat{\omega}_{j,i} = \omega_{j+1,-i}$.*

Proof. Let $x_{j,i}^{\pm}(\cdot)$ be the solutions of (1.1) which we introduced at the beginning of Section 2.2. With that we define solutions $\hat{x}_{j,i}^{\pm}(\cdot)$ of (1.1) by

$$\hat{x}_{j,i}^{+}(t) := Rx_{j,-i}^{-}(-t), \quad \hat{x}_{j,i}^{-}(t) := Rx_{j,-i}^{+}(-t).$$

By construction $\hat{x}_{1,i}^{+}$, $\hat{x}_{2,i}^{+}$, $\hat{x}_{1,i}^{-}$ and $\hat{x}_{2,i}^{-}$ are defined on $[0, \omega_{2,-i}]$, $[0, \omega_{1,-i}]$, $[-\omega_{1,-i}, 0]$ and $[-\omega_{2,-i}, 0]$, respectively. With

$$\hat{\omega}_{j,i} := \omega_{j+1,-i}$$

the functions $x_{j,i}^{\pm}(\cdot)$ satisfy the coupling conditions (2.9) and also the jump conditions (2.10). This shows that $\hat{X} = (\hat{X}_i)$ with $\hat{X}_i = RX_{-i}$ is indeed a Lin orbit associated to $\hat{\boldsymbol{\omega}}$; by construction $RX = \hat{X}$. ∎

Corollary 4.6. *Let X be a Lin orbit and let $(\Xi_i)_{i\in\mathbb{Z}}$ be the corresponding sequence of jumps (see (2.10)). For the sequence $(\hat{\Xi}_i)_{i\in\mathbb{Z}}$ associated to RX it holds $R\Xi_i = -\hat{\Xi}_i$.* ■

Definition 4.7. *A Lin orbit $X(\boldsymbol{\omega})$ is symmetric if $X = RX$.*

Lemma 4.8. *A Lin orbit $X(\boldsymbol{\omega})$ is symmetric if and only if there is $i_0 \in \mathbb{Z}$ such that for all $i \in \mathbb{Z}$ and $j \in \{1, 2\}$ it holds $\omega_{j+1,-i} = \omega_{j,i-i_0}$.* ■

From Corollary 4.6 and Lemma 4.8 we get

Corollary 4.9. *Let $X(\boldsymbol{\omega})$ be a symmetric Lin orbit, and let $i_0 \in \mathbb{Z}$ be such $\omega_{j+1,-i} = \omega_{j,i-i_0}$, for all $i \in \mathbb{Z}$, $j \in \{1, 2\}$. Then $R\,\Xi_i = -\Xi_{-i-(i_0+1)}$.* ■

4.2 The bifurcation equation

Lemma 4.3 says that $\xi^\infty(\mu) \in Z \cap \mathrm{Fix}\,(-R)$, and according to Lemma 4.2 this space is one-dimensional. So, by introducing appropriate coordinates, $\xi^\infty(\cdot)$ can be seen as a mapping

$$\xi^\infty(\cdot) : \mathbb{R} \to \mathbb{R}.$$

Hypothesis (H 3) says that the Jacobian $D\xi^\infty(0)$ is non-singular. Because the image of ξ^∞ is one-dimensional it suffices a one-dimensional preimage space of ξ^∞ to guarantee the non-singularity of $D\xi^\infty(0)$. This justifies our setting $\mu \in \mathbb{R}$.

Remark 4.10. Hypothesis (H 3) can be made more explicit by using a representation of ξ^∞ by means of a Melnikov-type integral.

Let ψ_1, ψ_2 be an orthonormal basis for Z, such that $Z \cap \mathrm{Fix}\ R = \mathrm{span}\{\psi_1\}$ and $Z \cap \mathrm{Fix}\ (-R) = \mathrm{span}\{\psi_2\}$. Set

$$\psi_j(t) := \Psi(t, 0)\psi_j \qquad \forall t \in \mathbb{R},\ j = 1, 2$$

where $\Psi(\cdot, \cdot)$ is the transition matrix of the adjoint (with respect to $\langle\cdot, \cdot\rangle$) of (2.8).

Representations (3.5) of ξ^∞ and (2.5) of $q_1^\pm$ give

$$\xi^\infty(\mu) = v^+(\mu)(0) - v^-(\mu)(0).$$

On the other hand we have, recall that $\xi^\infty \in \mathrm{Fix}\,(-R)$,

$$\xi^\infty(\mu) = \langle\psi_2, \xi^\infty(\mu)\rangle\psi_2.$$

Combining these two representations of ξ^∞, using further representations of $v^+(\mu)(0)$ or $v^-(\mu)(0)$ and exploit that the involved direct sum decompositions are orthogonal with respect to the scalar product $\langle\cdot, \cdot\rangle$ we end up with

$$\xi^\infty(\mu) = -\psi_2 \int_{-\infty}^{\infty} \langle\psi_2(s), g_1(s, v(\mu)(s), \mu)\rangle ds,$$

where $v(\mu)(t) := v^-(\mu)(t)$, $t \le 0$, and $v(\mu)(t) := v^+(\mu)(t)$, $t > 0$. From this we find, see also (2.7),

$$D\xi^\infty(0) = -\psi_2 \int_{-\infty}^{\infty} \langle\psi_2(s), D_2 f(q_1(s), 0)\rangle ds.$$

□

Throughout let $\{\psi_1, \psi_2\}$ be the basis for Z which has been introduced in Remark 4.10.

Lemma 4.11. *$S_1(t, \psi_1, \mu) = S_2(t, \psi_1, \mu), \quad S_1(t, \psi_2, \mu) = -S_2(t, \psi_2, \mu)$.*

Proof. The scalar product $\langle \cdot, \cdot \rangle$ is R-invariant. Therefore

$$S_1(t, \psi_j, \mu) := \langle R\, \Psi_1^+(\mu, t, 0) Q_1^{+*}(\mu, 0)\psi_j, R\, \tilde{Q}_1(\mu, t) q_2^-(\mu)(-t)\rangle.$$

Now the lemma follows immediately from the following symmetry properties

(i) $R\, \Psi_1^+(\mu, t, 0) = \Psi_1^-(\mu, -t, 0)\, R$;
(ii) $R\, Q_1^{+*}(\mu, 0) = P_1^{-*}(\mu, 0)\, R$;
(iii) $R\psi_1 = \psi_1, \quad R\psi_2 = -\psi_2$;
(iv) $R\, \tilde{Q}_1(\mu, t) = (id - \tilde{Q}_{2,}(\mu, t))\, R$;
(v) $R\, q_2^-(\mu)(-t) = q_2^+(\mu)(t), t \in \mathbb{R}^+$.

Item (i) follows from the reversibility of (1.1) and $Rq_1^+(\mu)(0) = q_1^-(\mu)(0)$, see Lemma 2.1 and its proof.

The relation $Rq_1^+(\mu)(0) = q_1^-(\mu)(0)$ also implies $T_{q_1^+(\mu)(0)} W^s(p_1) = T_{q_1^-(\mu)(0)} W^u(p_2)$. This again implies

$$RQ_1^+(\mu, 0) = P_1^-(\mu, 0)R. \tag{4.1}$$

Since R is selfadjoint with respect to the R-invariant scalar product $\langle \cdot, \cdot \rangle$ that means $R = R^*$, item (ii) holds true.

The symmetry properties of ψ_1 and ψ_2 which are stated in (iii) follow immediately from their definition.

From (4.1) we can conclude $RQ_1^+(\mu, \omega) = P_1^-(\mu, -\omega)R$. In the same way we find $RQ_2^+(\mu, \omega) = P_2^-(\mu, -\omega)R$. For the latter equality we used that $q_2(\mu)(0) \in \operatorname{Fix} R$, see at the beginning of Section 2.1. So $R\left(\operatorname{im} Q_j^+(\omega)\right) = \operatorname{im} P_{j+1}^-(-\omega)$, $j = 1, 2$. This provides item (iv).

Finally item (v) is a simple consequence of $q_2(\mu)(0) \in \operatorname{Fix} R$. ∎

As a consequence of the previous lemma we get

$$\varphi_{11} = \varphi_{21} =: \varphi_1, \quad \varphi_{12} = \varphi_{22} =: \varphi_2,$$

From Lemma 4.11 we see that the leading terms of S_2 read

$$\begin{aligned} L_{21}(t, \mu) &= e^{-2\rho(\mu)t} c_{11}(\mu) \sin(2\phi(\mu)t + \varphi_1) = -L_{11}(t, \mu), \\ L_{22}(t, \mu) &= -e^{-2\rho(\mu)t} c_{12}(\mu) \sin(2\phi(\mu)t + \varphi_2) = L_{12}(t, \mu). \end{aligned}$$

Altogether our considerations show that the bifurcation equation for orbits staying for all time close to the primary cycle reads:

$$\Xi_i(\boldsymbol{\omega}, \mu) = \left\{ \begin{array}{rl} L_{11}(\omega_{1,i+1}, \mu) - L_{11}(\omega_{2,i}, \mu) + r_{1,i}(\boldsymbol{\omega}, \mu) & = 0 \\ \mu + L_{12}(\omega_{1,i+1}, \mu) + L_{12}(\omega_{2,i}, \mu) + r_{2,i}(\boldsymbol{\omega}, \mu) & = 0 \end{array} \right. \qquad i \in \mathbb{Z}. \tag{4.2}$$

4.3 Proof of Theorem 4.1

In order to prove Theorem 4.1 we proceed in principle as in Section 3. But we have to take the particular features of reversible systems into account.

Our discussion in Section 3 was mainly based on the dicussion of the intersections of certain spirals. In particular we used that the spirals under consideration have infinitely many transversal intersections if the ratio of the real parts of the leading complex eigenvalues of the fixed points p_1 and p_2 is irrational, see Lemma 3.6. Here, due to the reversibility, this ratio is one. But in the current case we are able to state the necessary transversal intersections of the corresponding spirals explicitly. The remainder indeed runs parallel to our considerations in Section 3 – we have only to take care about additional symmetry statements.

As in Section 3.2.2 we introduce a parameter ϵ and define for $i \in \mathbb{Z}$:

$$\hat{\Xi}_i(\boldsymbol{\omega}, \mu, \epsilon) = e^{2\rho\omega_{1,i+1}} \begin{pmatrix} L_{11}(\omega_{1,i+1}, \mu) - L_{11}(\omega_{2,i}, \mu) + \epsilon r_{1,i}(\boldsymbol{\omega}, \mu) \\ \mu + L_{12}(\omega_{1,i+1}, \mu) + L_{12}(\omega_{2,i}, \mu) + \epsilon r_{2,i}(\boldsymbol{\omega}, \mu) \end{pmatrix}.$$

With that we consider the equation

$$\hat{\boldsymbol{\Xi}}(\boldsymbol{\omega}, \mu, \epsilon) = 0,$$

where again

$$\hat{\boldsymbol{\Xi}} : l^\infty_{\mathbb{R}^2} \times \mathbb{R} \times \mathbb{R} \to l^\infty_{\mathbb{R}^2} \quad (\boldsymbol{\omega}, \mu, \epsilon) \mapsto \left(\hat{\Xi}_i(\boldsymbol{\omega}, \mu, \epsilon)\right)_{i\in\mathbb{Z}}.$$

4.3.1 Symbolic Dynamics

We want to solve $\hat{\boldsymbol{\Xi}}(\boldsymbol{\omega}, \mu, \epsilon) = 0$ near $(\boldsymbol{\omega}, \mu, \epsilon) = (\hat{\boldsymbol{\omega}}, 0, 0)$ by means of the Implicit Function Theorem. As in Section 3.2.2 the "starting sequence" $\hat{\boldsymbol{\omega}}$ belongs to certain set $\boldsymbol{\Omega}^{\mathbb{Z}}_{n_0,N}$ of sequences which we define this time as follows:

$$\Omega_{n_0} := \{(\omega, \omega) : \omega = (n\pi - \varphi_2)/2\phi(0),\, n \geq n_0\}.$$

Again $\Omega_{n_0,N}$ is a subset of Ω_{n_0} with N elements and

$$\boldsymbol{\Omega}^{\mathbb{Z}}_{n_0,N} := \{\boldsymbol{\omega} : (\omega_{1,i+1}, \omega_{2,i}) \in \Omega_{n_0,N}\}.$$

Indeed, for $\omega_{1,i+1} = \omega_{2,i}$ we have $L_{11}(\omega_{1,i+1}, \mu) - L_{11}(\omega_{2,i}, \mu) = 0$, and it holds $L_{12}((n\pi - \varphi_2)/2\phi(0), 0) = 0$. Hypothesis (H 4) implies that $\rho(0)\sin(n\pi - \varphi_2 + \varphi_1) + \phi(0)\cos(n\pi - \varphi_2 + \varphi_1) \neq 0$, and this ensures that $D_1\hat{\boldsymbol{\Xi}}(\hat{\boldsymbol{\omega}}, 0, 0)$ is invertible, see also Lemma 3.3. So we can proceed as in Section 3.2.2. This includes the construction of the sets $\mathcal{S}^N_\mu$ and the first return map Π_μ as well as the topological conjugacy of $(\mathcal{S}^N_\mu, \Pi_\mu)$ to the full shift on N symbols.

It remains to show that $\mathcal{S}^N_\mu$ is setwise symmetric, that is $R\mathcal{S}^N_\mu = \mathcal{S}^N_\mu$. It can easily be checked that

$$\mathcal{R} : \boldsymbol{\Omega}^{\mathbb{Z}}_{n_0,N} \to \boldsymbol{\Omega}^{\mathbb{Z}}_{n_0,N}, \quad \boldsymbol{\omega} \mapsto \boldsymbol{\tau}, \quad \tau_{j,i} = \omega_{j+1,-i}$$

is an involution on $\boldsymbol{\Omega}^{\mathbb{Z}}_{n_0,N}$, see also Lemma 4.5. Again from the uniqueness part of Theorem 2.2 and from the construction of $\boldsymbol{\omega}_{\hat{\boldsymbol{\omega}}}(\mu, 1)$ it follows

$$R\, x_{\hat{\boldsymbol{\omega}}}(\mu)(0) = x_{\mathcal{R}\hat{\boldsymbol{\omega}}}(\mu)(-2(\omega_{1,0} + \omega_{2,0})) = x_{\zeta^{-1}(\mathcal{R}\hat{\boldsymbol{\omega}})}(\mu)(0).$$

This eventually proves the first statement of the theorem.

Completing we want to remark that in the addressed spirals (which can be regarded as spirals in Z) are, for $\mu = 0$, R-images of each other, and the above introduced set Ω_{n_0} corresponds to intersections of these spirals in $\operatorname{Fix} R$; the first components (corresponding to $\psi_1 \in \operatorname{Fix} R$) coincide, $L_{11}(\omega_{1,i+1}, 0) = L_{11}(\omega_{2,i}, 0)$, and the second one vanishes in each case, $L_{12}((n\pi - \varphi_2)/2\phi(0), 0) = 0$.

4.3.2 1-periodic orbits

Let $X(\boldsymbol{\omega})$ be a symmetric 1-periodic Lin orbit. In accordance with our results concerning symmetric Lin orbits we find

Corollary 4.12. *1. A 1-periodic Lin orbit $X(\boldsymbol{\omega})$ is symmetric if and only if for all $i \in \mathbb{Z}$ it holds $\omega_{1,i} = \omega_{2,i} := \omega$.*

2. For the jumps of a symmetric 1-periodic Lin orbit it holds $\Xi_i \equiv \Xi$, $i \in \mathbb{Z}$, and moreover $\Xi \in \operatorname{Fix}(-R)$. ■

Then the bifurcation equation for symmetric 1-periodic orbits reduces down to a 1-dimensional equation:

$$\mu + 2c_{12}(\mu)e^{-2\rho(\mu)\omega}\sin(2\phi(\mu)\omega + \varphi^2) + o(e^{-2\rho(\mu)\omega}) = 0. \tag{4.3}$$

Equation (4.3) has at $\mu = 0$ a countably infinite set of solutions. Therefore we have at $\mu = 0$ a countably infinite set of symmetric 1-periodic orbits, with difference in period asymptotically tending to $\pi/2\phi(\mu)$. Using the Implicit Function Theorem each of these solutions can be continued for $\mu \neq 0$. However, for fixed $\mu \neq 0$ there are only finitely many solutions of (4.3) and hence only finitely many 1-periodic orbits. With the addition of the parameter to the phase space these periodic orbits form a 1 parameter family parameterised by period, see Figure 4.

We note that one can also find non-symmetric 1-periodic orbits. These are associated to sequences $\boldsymbol{\omega}$ with $\omega_{1,i} \equiv \omega_1$, $\omega_{2,i} \equiv \omega_2$, $i \in \mathbb{Z}$, and $\omega_1 \neq \omega_2$. The corresponding bifurcation equation is two-dimensional (the $\operatorname{Fix} R$-component does not vanish as in the above situation). Such 1-periodic orbits are associated to the families of homoclinic orbits to each fixed point.

4.3.3 k-(2,1)-heteroclinic orbits

In this section we search for k-(2,1)-heteroclinic orbits for $k \geq 2$. A k-(2,1)-heteroclinic connection can be seen as part of a heteroclinic cycle consisting of this heteroclinic orbit and a 1-(1,2)-heteroclinic orbit. Such a cycle corresponds to a k-periodic sequence $\boldsymbol{\omega}$ with $\omega_{1,1} = \omega_{2,1} = \infty$. Hence the bifurcation equation for these orbits reads, see also Section 3.1, Equation (3.3).

$$
\begin{aligned}
\Xi_1(\boldsymbol{\omega},\mu) &= \left\{\begin{array}{rl} L_{11}(\omega_{1,2},\mu)+r_{1,1}(\boldsymbol{\omega},\mu) &= 0\\ \mu+L_{12}(\omega_{1,2},\mu)+r_{2,1}(\boldsymbol{\omega},\mu) &= 0\end{array}\right.\\
\Xi_i(\boldsymbol{\omega},\mu) &= \left\{\begin{array}{rl} L_{11}(\omega_{1,i+1},\mu)-L_{11}(\omega_{2,i},\mu)+r_{1,i}(\boldsymbol{\omega},\mu) &= 0\\ \mu+L_{12}(\omega_{1,i+1},\mu)+L_{12}(\omega_{2,i},\mu)+r_{2,i}(\boldsymbol{\omega},\mu) &= 0\end{array}\right.\\
& \qquad i=2,\dots,k-1\\
\Xi_k(\boldsymbol{\omega},\mu) &= \left\{\begin{array}{rl} -L_{11}(\omega_{2,k},\mu)+r_{1,k}(\boldsymbol{\omega},\mu) &= 0\\ \mu+L_{12}(\omega_{2,k},\mu)+r_{2,k}(\boldsymbol{\omega},\mu) &= 0.\end{array}\right.
\end{aligned}
\tag{4.4}
$$

In order to search for symmetric k-(2,1)-heteroclinic orbits we use sequences $\boldsymbol{\omega}$

$$\omega_{1,i}=\omega_{2,-i+2},\quad i=1,\dots,k. \tag{4.5}$$

This follows from Lemma 4.8, the k-periodicity of the corresponding sequence $\boldsymbol{\omega}$, and $\omega_{1,1}=\omega_{2,1}$. The i_0 of Lemma 4.8 is $i_0=-2$. Then from Corollary 4.9 we get

$$R\,\Xi_i=-\Xi_{-i+1},\quad i=1,\dots,k. \tag{4.6}$$

So the set of bifurcation equations (4.4) reduces down to

$$
\begin{aligned}
\Xi_1(\boldsymbol{\omega},\mu) &= \left\{\begin{array}{rl} L_{11}(\omega_{1,2},\mu)+r_{1,1}(\boldsymbol{\omega},\mu) &= 0\\ \mu+L_{12}(\omega_{1,2},\mu)+r_{2,1}(\boldsymbol{\omega},\mu) &= 0\end{array}\right.\\
\Xi_i(\boldsymbol{\omega},\mu) &= \left\{\begin{array}{rl} L_{11}(\omega_{1,i+1},\mu)-L_{11}(\omega_{2,i},\mu)+r_{1,i}(\boldsymbol{\omega},\mu) &= 0\\ \mu+L_{12}(\omega_{1,i+1},\mu)+L_{12}(\omega_{2,i},\mu)+r_{2,i}(\boldsymbol{\omega},\mu) &= 0\end{array}\right.\\
& \qquad i=2,\dots,\lfloor (k+1)/2\rfloor,
\end{aligned}
\tag{4.7}
$$

where $\lfloor x\rfloor$ denotes the integer part of x. Note that in these bifurcation equations $\boldsymbol{\omega}$ has the form (4.5). Further, if k is odd, then due to (4.6), $\Xi_{\lfloor (k+1)/2\rfloor}\in \mathrm{Fix}\,(-R)$.

First we consider a subsystem of (4.7) consisting of the first equation of $\Xi_1=0$, and the equations $\Xi_i=0$, $i=2,\dots,\lfloor (k+1)/2\rfloor$. There we introduce again an artificial parameter ϵ in the following manner:

$$
\hat{\Xi}(\boldsymbol{\omega},\mu,\epsilon) := \left\{\begin{array}{rl}
L_{11}(\omega_{1,2},\mu)+\epsilon r_{1,1}(\boldsymbol{\omega},\mu) &= 0\\
L_{11}(\omega_{1,i+1},\mu)-L_{11}(\omega_{2,i},\mu)+\epsilon r_{1,i}(\boldsymbol{\omega},\mu) &= 0\\
\mu+L_{12}(\omega_{1,i+1},\mu)+L_{12}(\omega_{2,i},\mu)+\epsilon r_{2,i}(\boldsymbol{\omega},\mu) &= 0\\
i=2,\dots,\lfloor (k+1)/2\rfloor. &
\end{array}\right.
\tag{4.8}
$$

In these equations appear $\lfloor (k+1)/2\rfloor$ variables $\omega_{j,i}$. Next we consider $\hat{\Xi}(\boldsymbol{\omega},0,0)=0$, where $\hat{\Xi}:\mathbb{R}^{\lfloor (k+1)/2\rfloor}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}^{\lfloor (k+1)/2\rfloor}$. For $\hat{\boldsymbol{\omega}}$ with $(\hat{\omega}_{1,i+1},\hat{\omega}_{2,i})\in\Omega_{n_0}$, $i=2,\dots,\lfloor (k+1)/2\rfloor$ and $\hat{\omega}_{1,2}=(n\pi-\varphi_1)/2\phi(0)$, $n\ge n_0$, we find that

$$\hat{\Xi}(\hat{\boldsymbol{\omega}},0,0)=0,\quad D_1\hat{\Xi}(\hat{\boldsymbol{\omega}},0,0)\text{ is invertible.}$$

So we can use the Implicit Function Theorem to solve $\hat{\Xi}(\boldsymbol{\omega},\mu,\epsilon)=0$ for $\boldsymbol{\omega}=\boldsymbol{\omega}(\mu,\epsilon)$ near $(\boldsymbol{\omega},\mu,\epsilon)=(\hat{\boldsymbol{\omega}},0,0)$. For sufficiently large n_0 one also $\epsilon=1$ makes sense, see also

Section 3.2.2 and Section 4.3.1. For the sake of brevity we write $\boldsymbol{\omega}(\mu)$ for $\boldsymbol{\omega}(\mu, 1)$. In this way we get

$$\hat{\Xi}(\boldsymbol{\omega}(\mu), \mu, 1) \equiv 0.$$

It remains to consider the second equation in $\Xi_1(\boldsymbol{\omega}, \mu) = 0$, which we write as fixed point equation

$$\mu = -L_{12}(\omega_{1,2}(\mu), \mu) - r_{2,1}(\boldsymbol{\omega}(\mu), \mu). \tag{4.9}$$

The right-hand side of this equation is a contraction. This finally allows to apply the Banach fixed point theorem which gives a fixed point μ of (4.9).

Finally, there are infinitely many $\hat{\boldsymbol{\omega}}^n$ we can start with. This eventually provides infinitely many $(\boldsymbol{\omega}^n(\mu_n), \mu_n)$ which solve the bifurcation equation (4.7). This proves the existence of a countable set of parameter values for which k-(2,1)-heteroclinic orbits exist. From the construction it follows that $\mu = 0$ is a accumulation point of this set.

The leading term of the right-hand side of (4.9) has the form

$$c(\mu) e^{-\left(\frac{\rho(\mu)}{\phi(\mu)}(n\pi - \varphi_1)\right)} \sin(n\pi + \varphi_2 - \varphi_1) = \tilde{c}(\mu) e^{-\left(\frac{\rho(\mu)}{\phi(\mu)}(n\pi - \varphi_1)\right)} (-1)^n,$$

see Lemma 3.4. This proves the existence of sequences (μ_n^+) and (μ_n^-) as stated in the theorem.

Completing we comment the non-existence of non-symmetric k-(2,1)-heteroclinic orbits. For that we emphasise that the bifurcation equations for detecting those orbits do not depend on the dimension of the phase space. In other words the existence or non-existence of non-symmetric k-(2,1)-heteroclinic orbits does not depend on the dimension of the phase space. In $\mathbb{R}^3$ however there are no such non-symmetric heteroclinic orbits for simple geometrical reason: In this case namely the unstable manifold of p_2 is one-dimensional – and non-symmetric orbits arise in pairs.

Remark 4.13. In order to prove the first statement of Lemma 1.3 we consider the "general version" of Equation (4.4):

$$\begin{aligned}
\Xi_1(\boldsymbol{\omega}, \mu) &= \left\{ \begin{array}{ll} \mu_1 + L_{11}(\omega_{1,2}, \mu) + r_{1,1}(\boldsymbol{\omega}, \mu) & = 0 \\ \mu_2 + L_{12}(\omega_{1,2}, \mu) + r_{2,1}(\boldsymbol{\omega}, \mu) & = 0 \end{array} \right. \\
\Xi_i(\boldsymbol{\omega}, \mu) &= \left\{ \begin{array}{ll} \mu_1 + L_{11}(\omega_{1,i+1}, \mu) - L_{21}(\omega_{2,i}, \mu) + r_{1,i}(\boldsymbol{\omega}, \mu) & = 0 \\ \mu_2 + L_{12}(\omega_{1,i+1}, \mu) - L_{22}(\omega_{2,i}, \mu) + r_{2,i}(\boldsymbol{\omega}, \mu) & = 0 \end{array} \right. \\
& \qquad i = 2, \ldots, k-1 \\
\Xi_k(\boldsymbol{\omega}, \mu) &= \left\{ \begin{array}{ll} \mu_1 - L_{21}(\omega_{2,k}, \mu) + r_{1,k}(\boldsymbol{\omega}, \mu) & = 0 \\ \mu_2 - L_{22}(\omega_{2,k}, \mu) + r_{2,k}(\boldsymbol{\omega}, \mu) & = 0. \end{array} \right.
\end{aligned} \tag{4.10}$$

Further, instead of (4.8) we consider a system consisting of the second equation of $\Xi_1 = 0$, the equations $\Xi_i = 0$, $i = 2, \ldots, k-1$, and the first equation of $\Xi_k = 0$. Correspondingly we choose $(\omega_{1,i+1}, \omega_{2,i}) \in \Omega_{n_0}$ which we defined in Section 3.2.2, $i = 2, \ldots, k-1$. Finally the equivalent of (4.9) is a fixed point equation for (μ_1, μ_2). □

4.3.4 k-(1,2)-heteroclinic orbits

Next we search for k-(1,2)-heteroclinic orbits for $k \geq 2$. A k-(1,2)-heteroclinic orbit is associated to a k-periodic Lin orbit of a k-periodic sequence $\boldsymbol{\omega}$ with $\omega_{1,1} = \omega_{2,k} = \infty$. So the jump Ξ_k belonging to such a k-periodic Lin orbit describes the splitting of the unstable manifold of p_2 and the stable manifold of p_1. In order to create a k-(1,2)-heteroclinic orbit out of such a Lin orbit only the jumps Ξ_i, $i = 1, \dots, k-1$ have to be equal to zero. Hence the bifurcation equation for these orbits reads, see also Section 3.1.

$$\Xi_i(\boldsymbol{\omega}, \mu) = \left\{ \begin{array}{rl} L_{11}(\omega_{1,i+1}, \mu) - L_{11}(\omega_{2,i}, \mu) + r_{1,i}(\boldsymbol{\omega}, \mu) & = 0 \\ \mu + L_{12}(\omega_{1,i+1}, \mu) + L_{12}(\omega_{2,i}, \mu) + r_{2,i}(\boldsymbol{\omega}, \mu) & = 0 \end{array} \right. \qquad i = 1, \dots, k-1. \tag{4.11}$$

In order to solve these equations we can proceed similar as in the previous section. Again we introduce a parameter ϵ:

$$\hat{\Xi}_i(\boldsymbol{\omega}, \mu, \epsilon) = \left\{ \begin{array}{rl} L_{11}(\omega_{1,i+1}, \mu) - L_{11}(\omega_{2,i}, \mu) + \epsilon r_{1,i}(\boldsymbol{\omega}, \mu) & = 0 \\ \mu + L_{12}(\omega_{1,i+1}, \mu) + L_{12}(\omega_{2,i}, \mu) + \epsilon r_{2,i}(\boldsymbol{\omega}, \mu) & = 0 \end{array} \right. \qquad i = 1, \dots, k-1.$$

Consider first $\hat{\Xi}_i(\boldsymbol{\omega}, 0, 0) = 0$. For $\hat{\boldsymbol{\omega}}$ with $(\omega_{1,i+1}, \omega_{2,i}) \in \Omega_{n_0}$ we can solve, as in Section 4.3.3, $\hat{\Xi}_i(\boldsymbol{\omega}, \mu, \epsilon) = 0$, $i = 1, \dots, k-1$, for $\boldsymbol{\omega} = \boldsymbol{\omega}_{\hat{\boldsymbol{\omega}}}(\mu, \epsilon)$ near $(\boldsymbol{\omega}, \mu, \epsilon) = (\hat{\boldsymbol{\omega}}, 0, 0)$. Again, there is an n_0 such that for all $n \geq n_0$ this solution is defined for $\epsilon = 1$.

In this way we find countably many solutions of $\Xi_i(\boldsymbol{\omega}, 0) = 0$, $i = 1, \dots, k-1$ because there are countably many different $\boldsymbol{\omega}$ at which $\hat{\Xi}_i = 0$ can be solved. Therefore there are countably many k-(1,2)-heteroclinic orbits (for each k) at $\mu = 0$.

By construction $q_1(0)$ is an accumulation point of the set of intersections of these orbits with Σ_1 because $\{(n\pi - \varphi_2)/\phi(0), n \geq n_0\}$ is unbounded.

Let, for fixed $\hat{\boldsymbol{\omega}}$, $\mathcal{O}_k$ be the k-(1,2)-heteroclinic orbit which corresponds to the solution $(\boldsymbol{\omega}_{\hat{\boldsymbol{\omega}}}(0, 1), 0)$ of (4.11). Since $\boldsymbol{\omega}_{\hat{\boldsymbol{\omega}}}(\cdot, 1)$ is also defined for μ close to 0, the orbit $\mathcal{O}_k$ can be continued for those μ.

For fixed $\mu \neq 0$ equation $\Xi_i(\boldsymbol{\omega}, \mu) = 0$ has only finitely many solutions, see also Section 4.3.2 and Figure 4. So there are only finitely many k-(1,2)-heteroclinic orbits for those μ.

Remark 4.14. For the second statement of Lemma 1.3 we consider

$$\Xi_i(\boldsymbol{\omega}, \mu) = \left\{ \begin{array}{rl} \mu_1 + L_{11}(\omega_{1,i+1}, \mu) - L_{21}(\omega_{2,i}, \mu) + r_{1,i}(\boldsymbol{\omega}, \mu) & = 0 \\ \mu_2 + L_{12}(\omega_{1,i+1}, \mu) - L_{22}(\omega_{2,i}, \mu) + r_{2,i}(\boldsymbol{\omega}, \mu) & = 0 \end{array} \right. \qquad i = 1, \dots, k-1. \tag{4.12}$$

These equations can be treated in the same way as described above. □

4.3.5 Homoclinic Orbits

It is sufficient to consider k-homoclinic orbits to p_1 because their R-images are just the k-homoclinic orbits to p_2.

As we already mentioned in Section 3.1 a k-homoclinic orbit to p_1 corresponds to a k-periodic Lin orbit with k-periodic sequence $\boldsymbol{\omega}$ with $\omega_{1,1} = \infty$. So the bifurcation equation for such orbits reads:

$$\begin{aligned} \Xi_i(\boldsymbol{\omega}, \mu) &= \begin{cases} L_{11}(\omega_{1,i+1}, \mu) - L_{11}(\omega_{2,i}, \mu) + r_{1,i}(\boldsymbol{\omega}, \mu) &= 0 \\ \mu + L_{12}(\omega_{1,i+1}, \mu) + L_{12}(\omega_{2,i}, \mu) + r_{2,i}(\boldsymbol{\omega}, \mu) &= 0 \end{cases} \\ & \qquad\qquad i = 1, \dots, k-1 \\ \Xi_k(\boldsymbol{\omega}, \mu) &= \begin{cases} -L_{11}(\omega_{2,k}, \mu) + r_{1,k}(\boldsymbol{\omega}, \mu) &= 0 \\ \mu + L_{12}(\omega_{2,k}, \mu) + r_{2,k}(\boldsymbol{\omega}, \mu) &= 0. \end{cases} \end{aligned} \tag{4.13}$$

These equations have the same structure as (4.7); they differ only in the number of equations and variables. So the discussion of (4.13) runs along the same lines as the discussion in Section 4.3.3.

Remark 4.15. In order to prove the corresponding statement for general vector fields we perform the same modifications as described in Remark 4.13 and Remark 4.14. ☐

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