Bifurcation from Discrete Rotating Waves

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September 8, 1998

Abstract

Discrete rotating waves are periodic solutions that have discrete spatiotemporal symmetries in addition to their purely spatial symmetries. We present a systematic approach to the study of local bifurcation from discrete rotating waves. The approach centers around the analysis of diffeomorphisms that are equivariant with respect to distinct group actions in the domain and the range.

Our results are valid for dynamical systems with finite symmetry group, and more generally for bifurcations from *isolated* discrete rotating waves in dynamical systems with compact symmetry group.

1 Introduction

In systems of ordinary differential equations without symmetry, there is a complete theory of the generic local bifurcations that occur as a single bifurcation parameter is varied, see for example Guckenheimer and Holmes [8, Chapter 3]. Local bifurcations are by definition the bifurcations that occur in the neighborhood of a nonhyperbolic equilibrium or a nonhyperbolic periodic solution.

Equivariant bifurcation theory [7] is concerned with the generalization of these results to the situation where the vector field is equivariant with respect to the action of a compact Lie group Γ. A systematic approach to bifurcation from equilibria is

*Supported by a Talent Stipendium of the Netherlands Organization for Scientific Research (NWO)

†Supported in part by NSF Grant DMS-9704980

laid out in [7]. In contrast, such an approach to bifurcation from periodic solutions has previously proved elusive.

Suppose that P is a periodic solution of (minimal) period T, and let $x_0 \in P$. Let $x(t)$ be the trajectory with initial condition x_0, so $P = \{x(t) : 0 \le t < T\}$. The symmetries that leave the periodic solution P invariant come in two forms. First, there is the group of *spatial symmetries*

$$\Delta = \{\gamma \in \Gamma : \gamma x_0 = x_0\}.$$

By definition, Δ is the isotropy subgroup of x_0. In fact, Δ is the isotropy subgroup of each point in P. Second, there is the group of *spatiotemporal symmetries*

$$\Sigma = \{\gamma \in \Gamma : \gamma P = P\}.$$

It is easy to see that for each $\sigma \in \Sigma$, there is a unique $T_\sigma \in [0, T)$ such that $\sigma x(t) = x(t + T_\sigma)$ for all t. Thus each spatiotemporal symmetry is the combination of a symmetry element σ composed with a time-shift by T_σ. The spatial symmetries are those spatiotemporal symmetries σ for which $T_\sigma = 0$. Moreover, Δ is a normal subgroup of Σ and either $\Sigma/\Delta \cong S^1$ or $\Sigma/\Delta \cong \mathbb{Z}_m$ for some $m \ge 1$.

When $\Sigma/\Delta \cong S^1$, the periodic solution P is called a *rotating wave*. Krupa [9] generalized the results of [7] to include local bifurcation from rotating waves (and more generally, from relative equilibria).

When $\Sigma/\Delta \cong \mathbb{Z}_m$, the periodic solution P is called a *discrete rotating wave* [5]. Special cases that have been studied include the case when $m = 1$, Chossat and Golubitsky [4], and the case when Σ is cyclic, Fiedler [5]. (As verified in [3], many of Fiedler's results apply equally well when Σ is abelian.) For general discrete rotating waves, there are partial results due to [17, 20, 24, 25]. However, these results rely explicitly on hypotheses that appear plausible but are not shown to be valid. In fact, these hypotheses can be shown to be valid generically using the theory established in the paper. Details together with comparison of our methods and those of [17, 20, 24, 25] can be found in [12].

Discrete rotating waves occur in a variety of situations, notably through Hopf bifurcation from equilibria. For example, Hopf bifurcation from an equilibrium with $\mathbf{O}(2)$ symmetry [7, Chapter XVII] leads to branches of rotating waves and standing waves (see [7, Chapter XI, Figure 1.4]). The rotating waves have no spatial symmetry, $\Delta = 1$, and rotational spatiotemporal symmetry $\Sigma = \mathbf{SO}(2)$ so that time evolution is equivalent to rigid rotation. The standing waves have spatial symmetry $\Delta = \mathbb{Z}_2$ and spatiotemporal symmetry $\Sigma = \mathbb{Z}_2 \times \mathbb{Z}_2$, corresponding to a state that is fixed at all times by a single reflection and for which evolution through half a period is the same as rotation through 180°. These rotating waves and standing waves occur, for example, in the motion of a flexible hosepipe with circular cross-section [1]. In particular, the standing wave is an example of a discrete rotating wave. Hopf bifurcation with dihedral symmetry [7, Chapter XVIII] produces a particularly rich set of examples of discrete rotating waves. An example of such a discrete rotating wave is shown schematically in Figure 2(b).

The purpose of this paper is to understand secondary bifurcation from discrete rotating waves such as those that occur in Hopf bifurcation from equilibria. Throughout, we make the simplifying assumption that $\dim \Delta = \dim \Sigma = \dim \Gamma$. (It follows that generically the discrete rotating wave is isolated in the usual sense that there is a neighborhood U of the discrete rotating wave such that no further periodic solutions are contained entirely in U.) There are technical problems that arise in bifurcation from continuous group orbits of discrete rotating waves (or more generally from relative periodic solutions); these issues are addressed in Sandstede et al [22] and Wulff [27]. In fact, the work of [22, 27] in conjunction with the results in this paper, lead to a general theory for compact (and noncompact) symmetry groups [13].

Chossat and Golubitsky [4] consider the case $m = 1$ ($\Sigma = \Delta$), where periodic solutions have spatial symmetry but no further spatiotemporal symmetry. They proceed by analyzing the Poincaré map G which in this case is a general Δ-equivariant diffeomorphism. The resulting theory is similar to the theory of [7] for bifurcation from equilibria.

When $m > 1$, the map G remains Δ-equivariant, but there are additional restrictions coming from the spatiotemporal symmetries in $\Sigma - \Delta$. In general, the restrictions on the Poincaré map are not of a type amenable to direct analysis. It is useful to consider the diffeomorphism f obtained by integrating for approximately $1/m$'th of the period and pulling back to the original cross-section by the appropriate symmetry element (see Figure 1). For example, when $\Sigma = \mathbb{Z}_2$, $\Delta = 1$, it can be shown that $G = f^2$, from which it follows that generically the linearized Poincaré map has no eigenvalues at -1 leading to the phenomenon known as 'suppression of period-doubling' [23].

Fiedler [5] considers in detail the case when Σ is cyclic (with the emphasis on global, rather than local, bifurcation theory). For Σ cyclic, f is a general Δ-equivariant diffeomorphism. It is now possible to apply the methods in [4] to f and then to reinterpret the results in terms of the full problem. In Subsection 4(c), we describe a natural generalization of a local bifurcation-theoretic result of Fiedler [5, Theorem 5.11] that includes all cases when f is Δ-equivariant.

For general discrete rotating waves, the mapping f is not Δ-equivariant in the usual sense. Rather, there is an automorphism $\phi \in \mathrm{Aut}(\Delta)$ associated with the discrete rotating wave such that

$$f(\delta x) = \phi(\delta) f(x),$$

(cf. [18, 11]). The diffeomorphism f is completely general subject to this 'twisted equivariance' condition. The structure of such diffeomorphisms (referred to as k-symmetric maps) has been studied in [14, 10]. (We note that these references consider also time-reversal symmetries, but such issues are not the subject of the present paper.)

In this paper, we systematically study bifurcation from discrete rotating waves by studying bifurcation from fixed points for a twisted equivariant diffeomorphism. In particular, we classify the local bifurcations in terms of the structure of the center subspace of the linearized twisted equivariant diffeomorphism. The nonlinear theory then follows the setup suggested in [11] (see also [20]).

In Section 2, we describe the basic setup and introduce the twisted equivariant map f that governs the local bifurcations. Our main results are presented in Sections 3, 4 and 6. There are two kinds of bifurcation, nonHopf and Hopf bifurcation, which are classified at the linear level in Section 3. Based on the linear theory, we show in Sections 4 and 6 that generically nonHopf and Hopf bifurcation reduce to bifurcations of the kind studied in [4]. Examples that illustrate our theory for nonHopf bifurcation can be found in Section 5. Example 6.5 illustrates the theory for Hopf bifurcation.

To apply the results of [4], it is necessary to compute the irreducible representations of certain cyclic extensions of the spatial symmetry group Δ. (We note that the appropriate cyclic extensions are related to the spatiotemporal symmetry group Σ but are not the group Σ itself.) In practice, it is often the case that the irreducible representations of Δ are 'known'. It is then a standard technique in representation theory, the theory of *induced representations*, to build up the irreducible representations of the cyclic extension of Δ from the irreducible representations of Δ. This is the subject of Section 7. We then reanalyze nonHopf bifurcation from this perspective in Section 8.

2 First hit maps and twisted equivariance

Let $\Gamma \subset \mathbf{O}(n)$ be a compact Lie group acting orthogonally on $\mathbb{R}^n$ and let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a Γ-equivariant vector field. Suppose that P_0 is a periodic solution with spatial symmetry group Δ and spatiotemporal symmetry group Σ. Suppose also that P_0 is an isolated discrete rotating wave so that $\Sigma/\Delta \cong \mathbb{Z}_m$, for some $m \geq 1$, and $\dim \Delta = \dim \Sigma = \dim \Gamma$.

Let T be the period of P_0 and choose $x_0 \in P_0$. Then $\psi(x_0, T/m) = \sigma x_0$ for some $\sigma \in \Sigma$ where ψ is the flow corresponding to the vector field F. It follows from the above considerations that Σ is generated by Δ and σ. In particular, $\sigma^m \in \Delta$. Note that σ is defined only up to multiplication by elements of Δ. The following result ensures that σ can be chosen to have finite order.

Lemma 2.1 *Suppose that Σ is a compact Lie group and that Δ is a closed normal subgroup such that $\Sigma/\Delta \cong \mathbb{Z}_m$. Then there is an element $\sigma \in \Sigma$ of finite order such that Σ is generated by Δ and σ.*

Proof Choose $\sigma \in \Sigma$ so that Σ is generated by Δ and σ. We show that there is an element $\delta \in \Delta$ such that $\sigma' = \delta\sigma$ has finite order.

Let C be the closed subgroup of Δ generated by σ^m and let C^0 be the connected component of the identity in C. Then C^0 is a torus and $\sigma^{jm} \in C^0$ for some $j \geq 1$. Choose $\alpha \in C^0$ such that $\alpha^{jm} = \sigma^{jm}$, and set $\delta = \alpha^{-1} \in \Delta$, $\sigma' = \delta\sigma$. We claim that α commutes with σ. It then follows that $(\sigma')^{jm} = (\alpha^{-1}\sigma)^{jm} = \alpha^{-jm}\sigma^{jm} = 1$ as required.

It remains to prove the claim. Let D be the closed subgroup of Σ generated by σ. Then D is abelian and contains σ. Further, $\alpha \in C \subset D$ and hence α commutes with σ. ∎

Let X be a Δ-invariant local section containing x_0 and define the first hit maps $g^{(j)} : X \to \sigma^j X$. Let $f = \sigma^{-1} g^{(1)}$, see Figure 1. The diffeomorphism $f : X \to X$ is equivariant with respect to distinct actions of Δ in the domain and the range. Indeed $f(\delta x) = (\sigma^{-1}\delta\sigma) f(x)$.

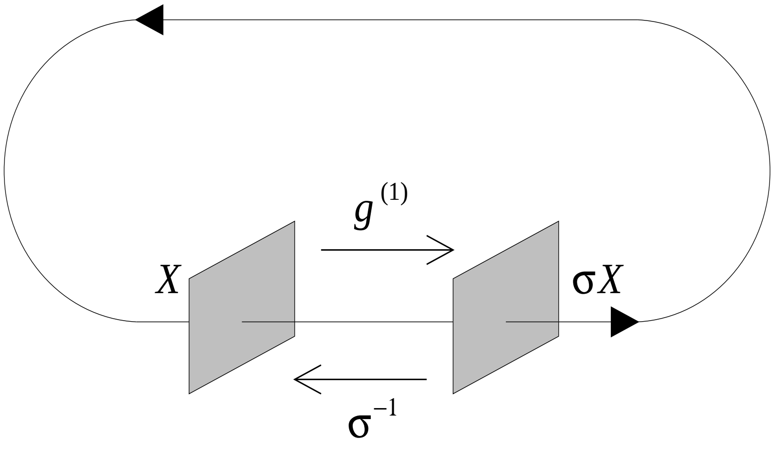


Figure 1: The twisted equivariant diffeomorphism $f = \sigma^{-1} g^{(1)} : X \to X$

Note that σ does not act on the section X, but $\sigma^{-1}\delta\sigma \in \Delta$ and hence acts on X. We introduce the automorphism $\phi \in \mathrm{Aut}(\Delta)$ defined by $\phi(\delta) = \sigma^{-1}\delta\sigma$. More generally, ϕ^s denotes the automorphism induced by σ^s. The *twisted equivariance* condition satisfied by f now takes the form

$$f(\delta x) = (\sigma^{-1}\delta\sigma) f(x) = \phi(\delta) f(x). \tag{2.1}$$

Remark 2.2 (a) The automorphism ϕ depends on the choice of σ but is defined uniquely up to inner automorphism. In other words, there is a unique outer automorphism associated with the discrete rotating wave P_0.
(b) There is an integer k which depends on the specific choice of ϕ (or σ) and which plays a significant role in this paper. We define k to be the least positive integer such that σ^k lies in the centralizer of Δ. Equivalently, the automorphism ϕ has order k. (In [14], a twisted equivariant map f is called k-symmetric, reflecting the fact that the k'th iterate of f is equivariant.)

By Lemma 2.1, we can choose ϕ (or σ) so that k is finite. When Σ is a semidirect product of Δ and $\mathbb{Z}_m$, it is natural to choose σ to be a generator of $\mathbb{Z}_m$ so that $\sigma^m = 1$ and hence k divides m. When Σ is a direct product of Δ and $\mathbb{Z}_m$, this choice of σ gives $k = 1$.

When Σ is not a semidirect product of Δ and $\mathbb{Z}_m$, there are examples where $k > m$. Let $\mathbb{Z}_2^4$ be the group of diagonal 4×4 matrices with entries ± 1 on the diagonal and set $\Sigma = \mathbb{Z}_2^4 \rtimes \mathbb{Z}_4$ where $\mathbb{Z}_4$ is generated by a 4×4 cyclic permutation matrix σ. Define $\Delta = \mathbb{Z}_2^4 \rtimes \mathbb{Z}_2$ where $\mathbb{Z}_2$ is generated by σ^2. Clearly, Σ is a cyclic extension of Δ and $m = 2$. Moreover, σ^2 does not commute with elements of $\mathbb{Z}_2^4$ and it follows that the automorphism ϕ corresponding to σ has order $k = 4$. Replacing σ by another element of the coset $\sigma\Delta$ does not reduce k.

The first hit maps are recovered from f by the relation $g^{(j)} = \sigma^j f^j$. In particular, the Poincaré map for P_0 is given by

$$G = g^{(m)} = \sigma^m f^m. \tag{2.2}$$

Note that f and G define discrete dynamical systems on X, whereas in general, iteration of the maps $g^{(j)}$ is undefined.

Since σ has finite order, certain iterates of the Poincaré map G and the derived map f coincide. It follows that the dynamics for these maps are very closely related. In the following, we use the terminology 'periodic solution' specifically to refer to a periodic solution for the flow, whereas 'periodic orbit' refers to a periodic trajectory for the discrete dynamical system defined by f or G.

Proposition 2.3 *A point $x \in X$ (close to x_0) lies on a periodic orbit for f if and only if x lies on a periodic solution P lying in a small enough neighborhood of P_0. Moreover, x is hyperbolic, a periodic sink, a periodic saddle, etc for f if and only if P has the corresponding property for the underlying flow.*

Proof Suppose that σ has order q. Then $G^q = f^{qm}$ and it follows that periodic points for f coincide with periodic points for G and hence with periodic solutions that lie in the neighborhood where the Poincaré maps are defined.

Specifically, suppose that $f^j(x) = x$. Then x is a fixed point for $f^{qmj} = G^{qj}$ and hence has the same stability type for each map. It follows that the periodic point x has the same stability type for f as for G. ∎

Let $x \in X$ be a periodic point for f and let P be the corresponding periodic solution. Let $\Delta_x \subset \Delta$ denote the isotropy subgroup of x. The next result shows how to recover the symmetry group $\Sigma_P \subset \Sigma$ that fixes P as a set.

Lemma 2.4 *Suppose that $x \in X$ is a periodic point for f with corresponding periodic solution P. Let $p \geq 1$ be least such that $f^p(x) = \delta x$ for some $\delta \in \Delta$. Then Σ_P is the group generated by Δ_x and $\sigma^p\delta$. The spatiotemporal symmetry $\sigma^p\delta$ corresponds to a time-shift by approximately the period of P_0 multiplied by p/m.*

Proof We have $g^{(p)}(x) = \sigma^p f^p(x) = \sigma^p\delta x$ and hence $\sigma^p\delta x \in P$. It follows that $\sigma^p\delta \subset \Sigma_P$. Note also that $\psi(x, T) = \sigma^p\delta x$ for some $T > 0$ least and that this T is also least such that $\psi(x, T) \in \Sigma x$. Hence the symmetry group of P is generated by Δ_x and $\sigma^p\delta$. ∎

Remark 2.5 It is natural to measure time-shifts as a fraction of the period of the periodic solution P (as opposed to the old periodic solution P_0). Note that the period of P is approximately the period of P_0 multiplied by some positive integer q. Hence, the spatiotemporal symmetry $\sigma^p\delta$ in Lemma 2.4 corresponds to a phase-shift by (exactly) the period of P multiplied by $\frac{p}{qm}$. It is convenient to speak of the spatiotemporal symmetry $(\sigma^p\delta, \frac{p}{qm})$ signifying that $g^{(p)}(x) = \sigma^p\delta x$. In particular, the original periodic solution P_0 (for which $q = 1$) has the spatiotemporal symmetry $(\sigma, \frac{1}{m})$.

In this subsection, we have effectively reduced the study of bifurcations from a discrete rotating wave to the study of bifurcation from a fixed point for the map f. Note that the Poincaré map G is Δ-equivariant in the usual sense, whereas f is equivariant only with respect to distinct actions in the domain and range. On the other hand, f is completely arbitrary up to the twisted equivariance condition (2.1), whereas the Poincaré map is constrained by the relation (2.2). It turns out to be simpler to apply genericity arguments to the twisted equivariant map f than to the constrained equivariant map G.

3 Classification of local bifurcations

Recall that $G = \sigma^m f^m$ is the Poincaré map associated with the discrete rotating wave P_0. Let $L = (df)_{x_0}$. It follows from the proof of Proposition 2.3 that the center subspaces of L and $(dG)_{x_0} = \sigma^m L^m$ coincide and we denote the common center subspace by E^c. Since E^c is the center subspace of L, it is immediate that L acts on E^c. Also, E^c is the center subspace of the Δ-equivariant linear map $(dG)_{x_0}$ from which it follows that Δ acts on E^c. We define Δ_L to be the group generated by the actions of Δ and L on E^c. The twisted equivariance condition $L\delta = \phi(\delta)L$, $\delta \in \Delta$, implies that Δ is a normal subgroup of Δ_L. In general, Δ_L need not be compact since L need not be semisimple. It is a consequence of Theorem 3.2 below that twisted equivariant linear maps are generically semisimple.

By the center manifold theorem [8], we can reduce to the center manifold associated with the periodic solution P_0. After center manifold reduction, the center subspace E^c is identified with the cross-section X. In the sequel, G denotes the restriction of G to the center manifold of x_0 and so on.

In Remark 2.2, we introduced the integer k given by the order of the automorphism ϕ in equation (2.1). The integer k and the linear map L depend on the choice of σ. Of course, $(dG)_{x_0}$ is independent of the choice of σ.

Proposition 3.1 *The linear maps $(dG)_{x_0}$ and L^k commute with the action of Δ_L on E^c.*

Proof Clearly, L^k commutes with L and $L^k\delta = \phi^k(\delta)L^k = \delta L^k$ by definition of k so that L^k is Δ-equivariant. The Poincaré map G is Δ-equivariant so that $(dG)_{x_0}$ commutes with Δ. Moreover, $L\sigma^m = \phi(\sigma^m)L = \sigma^m L$ and so $(dG)_{x_0} = \sigma^m L^m$ commutes with L. ∎

Theorem 3.2 *Generically, the action of Δ_L on E^c is irreducible. Moreover, Δ_L is compact.*

Proof Suppose that E^c is not Δ_L-irreducible, and let $V \subset E^c$ be a Δ_L-invariant subspace. If V is not Δ_L-irreducible, then we may choose an invariant subspace of lower dimension than V. Hence we may suppose without loss that V is Δ_L-irreducible. Since Δ is compact, we have a splitting $E^c = V \oplus W$ where W is a Δ-invariant subspace.

Since Δ_L cannot be assumed to be compact, it is not necessarily the case that L leaves W invariant. By construction, $L(V) \subset V$ so that $L : V \oplus W \to V \oplus W$ has the block-matrix form

$$L = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}.$$

Since Δ leaves V and W invariant, the twisted equivariance of L is given (in an obvious notation) by

$$A\delta_V = \phi(\delta)_V A, \qquad B\delta_W = \phi(\delta)_V B, \qquad D\delta_W = \phi(\delta)_W D. \tag{3.1}$$

Now define

$$L_\epsilon = \begin{pmatrix} A & B \\ 0 & (1+\epsilon)D \end{pmatrix}.$$

It is evident from the linearity of conditions (3.1) that L_ϵ is twisted equivariant for all ϵ. Moreover, the center subspace of E^c is given by the Δ_L-irreducible subspace V for all nonzero ϵ.

Next, we verify that Δ_L is compact. By Proposition 3.1, L^k commutes with the action of Δ_L. Hence, elements of Δ_L preserve the eigenspaces of L^k. It follows from irreducibility that L^k is a scalar multiple (real or complex) of the identity. Moreover, L has eigenvalues on the unit circle, so that L^k is a unit scalar multiple of the identity. In particular, the closed group $\langle L^k \rangle$ generated by L^k is compact.

Since Δ is compact, it follows immediately that $\Delta \times \langle L^k \rangle$ is compact. But Δ_L is a finite cyclic extension (by L) of $\Delta \times \langle L^k \rangle$ and hence is compact. ∎

Theorem 3.2 is the first step towards understanding generic bifurcation from discrete rotating waves. We note that this result, taken alone, is not so useful since *a priori* we do not know Δ_L. Part of the theory is to give a concrete description of Δ_L. This is done in Theorems 4.2 and 6.4 below. In addition, Theorem 3.4 below classifies the types of bifurcation that can occur in terms of the action of Δ_L.

A local bifurcation from P_0 is signaled by eigenvalues of $(dG)_{x_0}$ crossing the unit circle. There are three cases:

- **Period preserving bifurcation:** Eigenvalues at $+1$.
- **Period doubling bifurcation:** Eigenvalues at -1.
- **Hopf bifurcation:** Complex conjugate eigenvalues on the unit circle.

Remark 3.3 The terminology above is justified by the results in this paper (see for example, Proposition 4.5). It is important to note that these cases are distinguished by the eigenvalues of $(dG)_{x_0}$ and *not* by the eigenvalues of $L = (df)_{x_0}$. For many purposes, it is convenient to combine the period preserving and period doubling cases into a single bifurcation which we call *nonHopf* bifurcation.

Let V be an irreducible representation of a group Γ and let $\mathrm{Hom}_\Gamma(V)$ denote the space of linear maps $A : V \to V$ that commute with the action of Γ. Then it is well-known (see for example [7, Chapter XII,3]) that $\mathrm{Hom}_\Gamma(V)$ has the structure of a real division ring and hence is isomorphic to the reals $\mathbb{R}$, the complex numbers $\mathbb{C}$ or the quaternions $\mathbb{H}$. The representation is accordingly said to be irreducible of real, complex or quaternionic type. Irreducible representations of real type are also called absolutely irreducible and irreducible representations of complex or quaternionic type are said to be nonabsolutely irreducible.

Theorem 3.4 *Suppose that the discrete rotating wave P_0 undergoes local bifurcation. Generically, either*

(a) Δ_L acts absolutely irreducibly on E^c and there is a nonHopf bifurcation (either period-preserving or period-doubling), or

(b) The action of Δ_L on E^c is irreducible of complex type and there is a Hopf bifurcation.

Proof By Theorem 3.2, generically Δ_L acts irreducibly on E^c. A priori, the action could be irreducible of real, complex or quaternionic type. We consider the three cases in turn.

If the action of Δ_L on E^c is absolutely irreducible, then it follows from Proposition 3.1 that $(dG)_{x_0} = \pm I$ and hence that there is nonHopf bifurcation.

Next suppose that the action of Δ_L on E^c is irreducible of complex type. By Proposition 3.1, L^k is a complex scalar multiple of the identity. Write $L^k = \alpha I$ where $\alpha \in \mathbb{C}$. Since E^c is the center subspace of L, and hence is the center subspace of L^k, it follows that $|\alpha| = 1$.

We claim that α is a general complex number with absolute value one. To see this, let $\beta \in \mathbb{C}$ be a general complex number with absolute value one. Since the action of Δ_L is of complex type, βI commutes with L. It follows that $(\beta L)^k = \beta^k L^k = \beta^k \alpha I$. This proves the claim since $\beta^k\alpha$ is general.

It follows from the claim that generically the eigenvalues of L are not roots of unity. Since certain powers of L and $(dG)_{x_0} = \sigma^m L^m$ coincide, it must be the case that the eigenvalues of $(dG)_{x_0}$ are not real.

Finally, we show that the remaining possibility, namely that the action of Δ_L is irreducible of quaternionic type, is nongeneric. Suppose for contradiction that E^c is irreducible of quaternionic type. By Proposition 3.1, we can write $L^k = \alpha I$ where $\alpha \in \mathbb{H}$. Proceeding just as in the complex case, we can perturb L so that α is a general unit quaternion. In particular, by the noncommutativity of $\mathbb{H}$, there is a quaternion $\beta \in \mathbb{H}$ that does not commute with α. But βI commutes with the action of Δ_L and hence commutes with L^k. This leads to the equation $\alpha\beta I = L^k\beta I = \beta L^k = \beta\alpha I$ contradicting the fact that α and β do not commute. ∎

4 Analysis of NonHopf bifurcation

In this section, we analyze nonHopf bifurcation from discrete rotating waves. There is a systematic approach to bifurcation from periodic solutions with only spatial symmetry due to Chossat and Golubitsky [4]. We obtain a similarly systematic approach by reducing our problem to the one studied in [4].

There are three subsections. In Subsection (a), we obtain an explicit description of the representation of the group Δ_L from Theorem 3.4(a). The nonlinear analysis is given in Subsection (b) and relies on Birkhoff normal form theory for twisted equivariant maps [10]. Generalized versions of 'suppression of period-doubling' are given in Remark 4.6. In Subsection (c), we consider the simplest case ($k = 1$) where twisted equivariance reduces to equivariance in the usual sense.

(a) Structure of the group Δ_L

In the nonHopf case, Δ_L acts absolutely irreducibly on E^c. It follows from Proposition 3.1 that

$$(dG)_{x_0} = \pm I, \qquad L^k = \pm I.$$

We stress that the signs of $(dG)_{x_0}$ and L^k are independent and play different roles. The sign of $(dG)_{x_0}$ determines whether there is a period preserving or period doubling bifurcation, see Proposition 4.5. The sign of L^k determines the group Δ_L, see Theorem 4.2.

Define the abstract group $\Delta \rtimes \mathbb{Z}_k$ to be the cyclic extension of Δ by an element ρ of order k such that $\rho\delta = \phi(\delta)\rho$. We stress that this group depends only on Δ and the choice of ϕ.

A second cyclic extension $\Delta \rtimes \mathbb{Z}_{2k}$ is defined similarly, except that ρ now has order $2k$. In this case, ρ^k is a nontrivial element that commutes with the elements of Δ.

Remark 4.1 (a) When k is odd, $\Delta \rtimes \mathbb{Z}_{2k} = (\Delta \rtimes \mathbb{Z}_k) \times \mathbb{Z}_2$. No such decomposition exists when k is even, as demonstrated below in Example 5.3. There, we have $\Sigma = \mathbb{D}_4$, $\Delta = \mathbb{D}_2$. It turns out that $k = 2$ and $\Delta \rtimes \mathbb{Z}_2 \cong \mathbb{D}_4$. However $\Delta \rtimes \mathbb{Z}_4 \ncong \mathbb{D}_4 \times \mathbb{Z}_2$.
(b) In the case that Σ is a semidirect product of Δ and $\mathbb{Z}_m$, it is natural to take σ to be a generator of $\mathbb{Z}_m$. In particular, σ has order m and k divides m. Moreover, we have the group isomorphism

$$\Delta \rtimes \mathbb{Z}_k \cong \frac{\Sigma}{\mathbb{Z}_{\frac{m}{k}}},$$

where $\mathbb{Z}_{\frac{m}{k}}$ is generated by σ^k. In the extreme cases $k = 1$ and $k = m$, we have $\Delta \rtimes \mathbb{Z}_k = \Delta$ and $\Delta \rtimes \mathbb{Z}_k \cong \Sigma$ respectively.

The group isomorphism is obtained as followed. Elements of Σ can be represented uniquely as $\delta\sigma^j$ where j is computed $\bmod m$. Similarly, elements of $\Delta \rtimes \mathbb{Z}_k$ can be represented uniquely as $\delta\rho^j$ where j is computed $\bmod k$. Since k divides m, the map $\chi : \Sigma \to \Delta \rtimes \mathbb{Z}_k$ given by $\chi(\delta\sigma^j) = \delta\rho^j$ is well-defined. Since σ and ρ induce the same automorphism ϕ of Δ, the map χ is a group homomorphism. Moreover, it is clear that χ is onto and $\ker \chi = \mathbb{Z}_{\frac{m}{k}}$. Hence χ induces the required isomorphism.

Theorem 4.2 *Suppose that Δ_L acts absolutely irreducibly on E^c. Then either*

(i) $\Delta_L = \Delta \rtimes \mathbb{Z}_k$ (if $L^k = I$), or

(ii) $\Delta_L = \Delta \rtimes \mathbb{Z}_{2k}$ (if $L^k = -I$).

Moreover, the action of $\mathbb{Z}_k$ (resp. $\mathbb{Z}_{2k}$) is generated by L.

Proof By Proposition 3.1, L^k commutes with the action of Δ_L. Hence $L^k = \pm I$. Moreover $L\delta = \phi(\delta)L$. It follows from the definitions of $\Delta \rtimes \mathbb{Z}_k$ and $\Delta \rtimes \mathbb{Z}_{2k}$ that Δ_L is isomorphic to one of these groups with L playing the role of ρ. ∎

Remark 4.3 To apply Theorem 4.2, it is necessary only to consider the absolutely irreducible representations of $\Delta \rtimes \mathbb{Z}_{2k}$, since these representations incorporate the absolutely irreducible representations of $\Delta \rtimes \mathbb{Z}_k$, but with L^k acting trivially. (It should be noted that in general the order of L may be any integer that divides $2k$. For example, Δ may act trivially on the center subspace in which case the center subspace is one-dimensional and $L = \pm I$ regardless of the size of k.)

In the nonequivariant context, we have $\Delta = \Sigma = \mathbf{1}$ and $\Delta_L = \mathbf{1}$ or $\Delta_L = \mathbb{Z}_2$. Although it is usual practice to consider these cases separately, one could equally say that $\Delta_L = \Delta \rtimes \mathbb{Z}_{2k} = \mathbb{Z}_2$ acts absolutely irreducibly in each case — either by the one-dimensional trivial representation ($L = I$), or by the one-dimensional nontrivial representation ($L = -I$).

In Section 8 we work throughout with the group $\Delta \rtimes \mathbb{Z}_{2k}$. Using methods developed in Section 7, it is possible to derive the absolutely irreducible representations of $\Delta \rtimes \mathbb{Z}_{2k}$ from the irreducible representations of Δ, see Theorem 8.1.

(b) Nonlinear analysis of nonHopf bifurcation

Following Lamb [10], we write $f = Lh$. Since f and L are twisted equivariant, it follows that h is Δ-equivariant in the usual sense. Moreover, h is a general Δ-equivariant diffeomorphism satisfying $h(x_0) = x_0$ and $(dh)_{x_0} = I$.

Lemma 4.4 ([10]) *Up to arbitrarily high order, h is Δ_L-equivariant.*

Proof We have already observed that h is Δ-equivariant (to all orders). Moreover, L is semisimple (indeed $L^k = \pm I$). Hence it follows from Birkhoff normal form theory for twisted equivariant maps [10] that f can be transformed (by changes of coordinates that preserve twisted equivariance) so that f is L-equivariant up to any specified order in its Taylor expansion. In particular, $h = L^{-1}f$ is Δ_L-equivariant up to any specified order. ∎

We proceed by supposing that h is Δ_L-equivariant to *all* orders. This simplification is lifted at the end of the subsection. Thus $h : E^c \to E^c$ is a general diffeomorphism, satisfying $h(x_0) = x_0$ and $(dh)_{x_0} = I$, equivariant under an absolutely irreducible representation of the compact Lie group Δ_L. This is precisely the setting discussed in Chossat and Golubitsky [4] — bifurcation from a fixed point with eigenvalue passing through 1 for a diffeomorphism with Δ_L symmetry.

Since $f = Lh$ where h and L commute and $L^k = \pm I$, we have $h^{2k} = f^{2k}$. Therefore, periodic points for h coincide with periodic points for f and their stabilities are identical. It follows from Proposition 2.3 that locally the existence and stability of bifurcating periodic solutions for the underlying flow is reduced to the existence and stability of periodic points for the diffeomorphism h.

The methods of [4] lead to a partial determination of the local dynamics of h. In particular, bifurcation of fixed points is equivalent to bifurcation of steady-states for Δ_L-equivariant vector fields. Hence the branching and stability of fixed points for h can be read off from the corresponding results for steady-state bifurcations [7].

In the remainder of this subsection, we discuss how properties of periodic solutions for the flow can be read off from properties of the fixed points for h.

First we determine the periods of the periodic solutions corresponding to fixed points for h. In the process, we justify the terminology 'period preserving/doubling bifurcation' introduced in Section 3.

Proposition 4.5 *In the period preserving case ($(dG)_{x_0} = I$), branches of fixed points x_λ for h correspond to branches of periodic solutions P_λ for the flow with period approximately equal to the period of P_0. In the period doubling case ($(dG)_{x_0} = -I$), branches of fixed points x_λ for h correspond to branches of periodic solutions P_λ for the flow with period approximately twice the period of P_0.*

Proof By our assumption of Birkhoff normal form symmetry to all orders, h is L-equivariant. The Poincaré map satisfies

$$G = \sigma^m f^m = \sigma^m (Lh)^m = \sigma^m L^m h^m = (dG)_{x_0} h^m = \pm h^m.$$

Hence, a fixed point of h corresponds to a fixed point of G in the period preserving case and to a period two point of G in the period doubling case. ∎

Remark 4.6 It is well-known that in certain cases, the spatiotemporal symmetry of a discrete rotating wave *suppresses* the possibility of period-doubling. The simplest case is when $\Sigma = \mathbb{Z}_2$ and $\Delta = \mathbf{1}$, see [23].

We describe a generalized version of this phenomenon. Suppose that k divides m. Suppose further that m/k is even and that σ^m has odd order. Then we have suppression of period-doubling (independent of the action of Δ_L).

To see this, recall that $(dG)_{x_0} = \sigma^m L^m$. Since $L^k = \pm I$ and m/k is even, it follows that $L^m = I$. But then $\sigma^m = (dG)_{x_0} = \pm I$. Since σ^m has odd order, we must be in the case $(dG)_{x_0} = +I$.

An important special case is when Σ is a semidirect product of Δ and $\mathbb{Z}_m$ (so that $\sigma^m = 1$). Suppression of period-doubling occurs when m/k is even.

Next, we discuss the symmetries of the periodic solutions obtained in Proposition 4.5. Suppose that x is a fixed point for h with isotropy subgroup $J \subset \Delta_L$. Then the spatial symmetry group of the corresponding periodic solution P is given by $\Delta_x = J \cap \Delta$. The spatiotemporal symmetries Σ_P can be read off from J as follows [11].

Lemma 4.7 *Suppose that $x \in E^c$ is a fixed point for h with isotropy subgroup $J \subset \Delta_L$. Let P be the corresponding periodic solution. Let $p \geq 1$ be least such that $L^{-p}\delta \in J$ for some $\delta \in \Delta$. Then Σ_P is the group generated by Δ_x and $(\sigma^p\delta, \frac{p}{qm})$, where $q = 1$ in the period-preserving case and $q = 2$ in the period-doubling case.*

Proof We apply Lemma 2.4, searching for the least p such that $f^p(x) = \delta x$ where $\delta \in \Delta$. Since x is a fixed point for h, we compute that $f^p(x) = L^p h^p(x) = L^p x$. Hence $f^p(x) = \delta x$ if and only if $L^p x = \delta x$. Equivalently, $L^{-p}\delta \in J$. ∎

When computing the quantity $\frac{p}{qm}$, it should be understood that m is known from the outset and that q takes the values 1 or 2 depending on whether $(dG)_{x_0} = \sigma^m L^m = \pm I$.

We note that many fixed points for h (and hence periodic solutions for the flow) can be obtained from the equivariant branching lemma [7]. Let J be an axial isotropy subgroup of Δ_L (that is, J is an isotropy subgroup with one-dimensional fixed-point subspace). Then the equivariant branching lemma predicts the existence generically of a branch of fixed points with isotropy J for the mapping h.

As promised, we end this subsection by dropping the assumption that h is Δ_L-equivariant to all orders. A fundamental determinacy theorem of Field [6] states that branches of fixed points for Δ_L-equivariant diffeomorphisms are generically determined and hyperbolic for the truncation of h at some fixed finite order (dependent only on the particular absolutely irreducible representation of the compact Lie group

Δ_L). Moreover, Field [6] shows that there is a (higher) finite order at which perturbations that break the Δ_L-equivariance do not affect the branches of fixed points for h. Since we consider only perturbations that preserve Δ-equivariance of h, it follows easily that our statements about spatiotemporal symmetries are unchanged by these perturbations.

(c) The untwisted case ($k = 1$)

In this section, we obtain a natural generalization of the results of Fiedler [5] who studied the case when the spatiotemporal symmetry group Σ is cyclic. Indeed, we consider the case when the automorphism $\phi \in \mathrm{Aut}(\Delta)$ in equation (2.1) is the trivial automorphism. In other words, σ lies in the centralizer of Δ and $k = 1$. This includes the case when Σ is abelian [3] and the case $\Sigma \cong \Delta \times \mathbb{Z}_m$.

The condition that ϕ is the trivial automorphism implies that f is Δ-equivariant. We analyze nonHopf bifurcations, applying the results of Sections 3 and 4.

Proposition 4.8 *Let P_0 be a discrete rotating wave with spatial symmetry group Δ and spatiotemporal symmetry group Σ and suppose that $k = 1$. Suppose also that P_0 undergoes a nonHopf bifurcation. Then the center subspace is generically Δ-absolutely irreducible and either $\Delta_L = \Delta$ (if $L = I$) or $\Delta_L = \Delta \times \mathbb{Z}_2$ (if $L = -I$).*

Proof By Theorem 3.4, the center subspace of $L = (df)_{x_0}$ (equivalently of $(dG)_{x_0}$) is generically Δ_L-absolutely irreducible. It follows from Proposition 3.1 that $L = L^k = \pm I$. Since the automorphism ϕ is trivial, Theorem 4.2 implies that $\Delta_L = \Delta$ if $L = I$ and that $\Delta_L = \Delta \times \mathbb{Z}_2$ if $L = -I$. In particular, Δ_L acts absolutely irreducibly if and only if Δ acts absolutely irreducibly. ∎

It follows also from Proposition 3.1 that $(dG)_{x_0} = \pm I$. Once again, we stress that the distinction between period preserving or period doubling is unrelated to the distinction between normal form symmetry Δ or $\Delta \times \mathbb{Z}_2$.

Finally, we note that σ^m lies in the center of Δ and hence acts as $\pm I$ on E^c. In the following result, the notation $\sigma^m = \pm I$ refers to the action of σ^m on E^c.

Theorem 4.9 *Let P_0 be a discrete rotating wave with spatial symmetry group Δ and spatiotemporal symmetry group Σ where $\Sigma/\Delta \cong \mathbb{Z}_m$ and suppose that $k = 1$. Suppose also that P_0 undergoes a generic nonHopf bifurcation. Each branch of fixed points for h corresponds to a branch of periodic solutions for the flow, and the periods of the periodic solutions are determined as follows:*

$\sigma^m = I$, m **even.** *Period preserving.*

$\sigma^m = I$, m **odd.** *Period preserving if $L = I$, period doubling if $L = -I$.*

$\sigma^m = -I$**,** m **even.** *Period doubling.*

$\sigma^m = -I$**,** m **odd.** *Period doubling if* $L = I$*, period preserving if* $L = -I$*.*

Proof By Proposition 4.5, it is sufficient to determine whether $(dG)_{x_0} = I$ or $(dG)_{x_0} = -I$. But $(dG)_{x_0} = \sigma^m L^m = \sigma^m(\pm I)^m$ from which the result is immediate. ∎

We note that the sign of σ^m depends on the absolutely irreducible representation of Δ, whereas the parity of m is purely group theoretic.

Remark 4.10 (a) Fiedler [5, Theorem 5.11] obtained the results described in Theorem 4.9 in the case when Σ is cyclic. For example, Fiedler's 'flip-flop doubling' corresponds to the case when $\sigma^m = -I$ and m is even. Buono [3] extended Fiedler's results to the case when Σ is abelian.
(b) Suppose that x is a fixed point for h with isotropy subgroup $J \subset \Delta_L$. Then the corresponding periodic solutions consist of points with isotropy $\Delta_x = J \cap \Delta$. The spatiotemporal group Σ_P can be determined using Lemma 4.7. Note that $L^{-1} = L$ in all cases. When $L = I$, trivially $L \in J$ so that $\Sigma_P = \langle \Delta_x, \sigma \rangle$. When $L = -I$, there are two subcases. If there exists an element $\tau \in \Delta$ such that $L\tau \in J$, then $\Sigma_P = \langle \Delta_x, \tau\sigma \rangle$. Otherwise, $\Sigma_P = \langle \Delta_x, \sigma^2 \rangle$.

Next we make explicit the interpretation of Σ_P as spatiotemporal symmetries. First suppose that we are in the period-preserving case ($(dG)_{x_0} = \sigma^m L^m = I$). Then we have $\Sigma_P = \langle \Delta_x, (\sigma, \frac{1}{m}) \rangle$ if $L = I$; $\Sigma_P = \langle \Delta_x, (\tau\sigma, \frac{1}{m}) \rangle$ or $\Sigma_P = \langle \Delta_x, (\sigma^2, \frac{2}{m}) \rangle$ if $L = -I$.

Similarly, in the period-doubling case ($(dG)_{x_0} = \sigma^m L^m = -I$) we have $\Sigma_P = \langle \Delta_x, (\sigma, \frac{1}{2m}) \rangle$ if $L = I$; $\Sigma_P = \langle \Delta_x, (\tau\sigma, \frac{1}{2m}) \rangle$ or $\Sigma_P = \langle \Delta_x, (\sigma^2, \frac{1}{m}) \rangle$ if $L = -I$.

5 Examples of nonHopf bifurcation

In this section, we illustrate the theory obtained in this paper with examples of nonHopf bifurcation from a discrete rotating wave. In Subsection (a), we give a complete analysis of examples in the untwisted case ($k = 1$). Examples to illustrate the genuinely twisted equivariant case ($k \geq 2$) are given in Subsection (b).

A complete treatment of the examples for $k \geq 2$ relies on the computation of the absolutely irreducible representations of Δ_L. The required representation theory is given in Section 7 and a complete analysis of the examples for $k \geq 2$ is deferred to Section 8.

(a) Examples in the untwisted case ($k = 1$)

Example 5.1 We consider nonHopf bifurcation from a discrete rotating wave with spatial symmetry $\Delta = \mathbb{D}_n$ and spatiotemporal symmetry $\Sigma = \mathbb{D}_{2n}$ where n is odd.

Let R_θ denote counterclockwise rotation through angle θ and let κ denote a specific reflection in $\mathbb{D}_n$ (in the x-axis say). Then $\mathbb{D}_n$ is generated by $R_{2\pi/n}$ and κ. Moreover $\mathbb{D}_{2n} = \mathbb{D}_n \times \mathbb{Z}_2$ where $\mathbb{Z}_2$ is generated by R_π. Evolution on the discrete rotating wave through half a period is thus the same as rotation through π (equivalently, by any other element of $\mathbb{D}_{2n}$ that does not lie in $\mathbb{D}_n$).

Choosing $\sigma = R_\pi$, we have $k = 1$ and it follows from Proposition 4.8 that $\Delta_L = \mathbb{D}_n$ (if $L = I$) or $\Delta_L = \mathbb{D}_{2n} = \mathbb{D}_n \times \mathbb{Z}_2$ (if $L = -I$). In the second case, we caution that it makes no sense to identify Δ_L with Σ. (As a further cautionary note, we observe that there exist 'bad' choices of σ for which $k \geq 2$.)

Since $k = 1$, we can apply the methods of Section 4(c). We note that the decomposition $\mathbb{D}_{2n} = \mathbb{D}_n \times \mathbb{Z}_2$ is no longer valid when n is even. In that case, we have $k = 2$ and require the full strength of the theory presented in this paper. The case n even is analyzed partially in Example 5.4 and the complete analysis is given in Example 8.3.

Note that Σ is a direct product of Δ and $\mathbb{Z}_m$ with $m = 2$ and hence $m/k = 2$ is even. It follows from Remark 4.6 that we have suppression of period doubling.

The first step is to enumerate the absolutely irreducible representations of Δ_L. By Proposition 4.8, it suffices to enumerate the absolutely irreducible representations of $\Delta = \mathbb{D}_n$. The irreducible representations of the group $\mathbb{D}_n$, n odd, are all absolutely irreducible and can be enumerated as follows. There are 2 one-dimensional representations $V_{0,\pm}$ and $(n-1)/2$ two-dimensional representations V_j, where the action of the generators $R_{2\pi/n}$ and κ is given by

	$V_{0,+}$	$V_{0,-}$	$V_j,\ 1 \leq j < n/2$
$R_{2\pi/n}$	1	1	$\begin{pmatrix} \cos 2\pi j/n & -\sin 2\pi j/n \\ \sin 2\pi j/n & \cos 2\pi j/n \end{pmatrix}$
κ	1	-1	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

(The integer subscript j indicates the action of $R_{2\pi/n}$. There is a unique irreducible representation of $\mathbb{D}_n$ for each $j \geq 1$ and two irreducible representations when $j = 0$ which are distinguished by the action of κ, hence the subscript $\pm$.)

The branches of periodic solutions (and their symmetries) corresponding to the absolutely irreducible representations of $\mathbb{D}_n$ are summarized in Table 1. We now describe the details behind these results for the representations V_j, $1 \leq j < n/2$. (The cases $V_{0,\pm}$ are easier and we omit the details.)

First suppose that $L = I$. Then $\Delta_L = \mathbb{D}_n$. In particular, when $j = 1$ we have the standard action of $\mathbb{D}_n$ on the plane. It follows from [7, Chapter XIII,5] that

Space	L	Δ_x	Σ_P	Remarks
$V_{0,+}$	1	$\mathbb{D}_n$	$\langle\kappa,(R_{\pi/n},1/2)\rangle \cong \mathbb{D}_{2n}$	saddle-node
$V_{0,+}$	-1	$\mathbb{D}_n$	$\mathbb{D}_n$	
$V_{0,-}$	1	$\mathbb{Z}_n(R_{2\pi/n})$	$\langle(R_{\pi/n},1/2)\rangle \cong \mathbb{Z}_{2n}$	
$V_{0,-}$	-1	$\mathbb{Z}_n(R_{2\pi/n})$	$\langle R_{2\pi/n},(R_\pi\kappa,1/2)\rangle \cong \mathbb{D}_n$	
V_j $1 \le j < n/2$	I	$\mathbb{D}_{j'}(R_{2\pi/j'},\kappa)$	$\langle\kappa,(R_{\pi/j'},1/2)\rangle \cong \mathbb{D}_{2j'}$	transcritical if $n/j' = 3$
V_j $1 \le j < n/2$	$-I$	$\mathbb{D}_{j'}(R_{2\pi/j'},\kappa)$ $\mathbb{Z}_{j'}(R_{2\pi/j'})$	$\mathbb{D}_{j'}$ $\langle R_{2\pi/j'},(R_{\pi/j'}\kappa,1/2)\rangle \cong \mathbb{D}_{j'}$	

Table 1: Spatiotemporal symmetry of bifurcating solutions in nonHopf bifurcation from a discrete rotating wave with $(\mathbb{D}_{2n},\mathbb{D}_n)$ symmetry, n odd. All bifurcations are period-preserving. All bifurcations are pitchforks unless stated otherwise. Notation: $j' = \gcd(j,n)$.

(up to conjugacy) there is a unique axial isotropy subgroup $J = \mathbb{D}_1(\kappa) \subset \Delta_L$. By Remark 4.10(b), the corresponding periodic solutions have spatial symmetry $\Delta_x = J \cap \Delta = \mathbb{D}_1$ and spatiotemporal symmetry $\Sigma_P = \langle\Delta_x,(\sigma,1/2)\rangle = \langle\kappa,(R_\pi,1/2)\rangle \cong \mathbb{D}_2$. In general, for $j \ge 1$ the action has a kernel $\mathbb{Z}_{j'}$ where $j' = \gcd(j,n)$ so that $J = \mathbb{D}_{j'}(R_{2\pi/j'},\kappa)$. By Remark 4.10(b), the corresponding periodic solutions have spatial symmetry $\Delta_x = J \cap \Delta = \mathbb{D}_{j'}$ and spatiotemporal symmetry $\Sigma_P = \langle\Delta_x,(\sigma,1/2)\rangle = \langle R_{2\pi/j'}\kappa,(R_\pi,1/2)\rangle = \langle\kappa,(R_{\pi/j'},1/2) \cong \mathbb{D}_{2j'}$.

Next, suppose that $L = -I$. Then $\Delta_L = \mathbb{D}_{2n}$. Suppose for simplicity that $j = 1$ (this restriction is lifted just as in the case $L = I$ by incorporating the kernel $\mathbb{Z}_{j'}$). By [7], there are now two conjugacy classes of axial isotropy subgroups $J = \mathbb{D}_1(\kappa)$ and $J = \mathbb{D}_1(L\kappa)$. The respective spatial symmetry groups are given by $\Delta_x = \mathbb{D}_1(\kappa)$ and $\Delta_x = \mathbf{1}$. The respective spatiotemporal symmetry groups are $\Sigma_P = \langle\kappa\rangle \cong \mathbb{D}_1$ and $\Sigma_P = \langle(R_\pi\kappa,1/2) \cong \mathbb{D}_1$.

The stability of the bifurcating periodic solutions, and the form of the branches, can be read off from [7, Chapter XIII,5]. To reduce the number of possibilities, we assume that the underlying discrete rotating wave is asymptotically stable subcritically and unstable supercritically. When $L = I$ and $n/j' = 3$, the branches are transcritical and unstable. When $L = I$ and $n/j' \ge 5$, the branches are asymmetric pitchforks (asymmetric in the sense that the two states on the pitchfork are unrelated by symmetry). If the pitchfork bifurcation is subcritical, then the periodic solutions are unstable. If the pitchfork bifurcation is supercritical, then each pitchfork consists of a periodic sink and a periodic saddle.

When $L = -I$, both branches of periodic solutions are symmetric pitchforks. Provided both solutions bifurcate supercritically, precisely one of the solutions is stable. Otherwise, both solutions are unstable.

Finally, in all cases in Table 1, the entire local dynamics consists of the enumerated periodic solutions together with their stabilities.

Example 5.2 We consider nonHopf bifurcation from a discrete rotating wave with spatial symmetry $\Delta = \mathbf{O}(2)$ and spatiotemporal symmetry $\Sigma = \mathbf{O}(2) \times \mathbb{Z}_2$. Evolution on the discrete rotating wave through half a period is the same as applying the nontrivial element of $\mathbb{Z}_2$.

Denote the elements of $\mathbf{O}(2)$ by R_θ and κ and let τ be the generator of $\mathbb{Z}_2$. Choosing $\sigma = \tau$, we ensure that $k = 1$. (Observe that in this example there exist 'bad' choices of σ for which k is arbitrarily large or for which $k = \infty$.) As in Example 5.1, we have suppression of period doubling.

The irreducible representations of $\mathbf{O}(2)$ (all of which are absolutely irreducible) are as follows:

	$V_{0,+}$	$V_{0,-}$	$V_j, j \geq 1$
R_θ	1	1	$\begin{pmatrix} \cos j\theta & -\sin j\theta \\ \sin j\theta & \cos j\theta \end{pmatrix}$
κ	1	-1	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

The corresponding branches of periodic solutions are summarized in Table 2. The details are similar to, but easier than, the arguments in Example 5.1. The main point is to recall from [7] that in the standard irreducible representation V_1 of $\mathbf{O}(2)$, there is up to conjugacy a unique axial isotropy subgroup $J = \mathbb{D}_1$.

This particular nonHopf bifurcation has been observed in experiments, numerical simulations and numerical linear stability calculations associated with three-dimensional wake flows behind cylinders. See [2, 26] and references therein. In particular, von Kármán vortices (a periodic two-dimensional state with spatiotemporal symmetry $(\mathbf{O}(2) \times \mathbb{Z}_2)$) undergo period-preserving bifurcations to fully three dimensional states where the continuous $\mathbf{SO}(2)$ symmetry is broken. (Such bifurcations correspond to the representations V_j, $j \geq 1$ in Table 2.) The bifurcating states have two different forms, mode A and mode B [26]. As shown in [2], mode A has the spatiotemporal symmetry $\langle \mathbb{D}_j, (\tau, 1/2)$ associated with $L = I$ whereas mode B has the spatiotemporal symmetry $\langle \mathbb{D}_j, (\tau R_{\pi/j}, 1/2) \rangle$ associated with $L = -I$. For mode A, [2, Figure 10 and equation (3.3)] shows that evolution through half a period is the

Space	L	Δ_x	Σ_P	Remarks
$V_{0,+}$	1	$\mathbf{O}(2)$	$\langle \mathbf{O}(2), (\tau, 1/2)\rangle \cong \mathbf{O}(2) \times \mathbb{Z}_2$	saddle-node
$V_{0,+}$	-1	$\mathbf{O}(2)$	$\mathbf{O}(2)$	
$V_{0,-}$	1	$\mathbf{SO}(2)$	$\langle \mathbf{SO}(2), (\tau, 1/2)\rangle \cong \mathbf{SO}(2) \times \mathbb{Z}_2$	
$V_{0,-}$	-1	$\mathbf{SO}(2)$	$\langle \mathbf{SO}(2), (\tau\kappa, 1/2)\rangle \cong \mathbf{O}(2)$	
$V_j,\ j \geq 1$	I	$\mathbb{D}_j$	$\langle \mathbb{D}_j, (\tau, 1/2)\rangle \cong \mathbb{D}_j \times \mathbb{Z}_2$	
$V_j,\ j \geq 1$	$-I$	$\mathbb{D}_j$	$\langle \mathbb{D}_j, (\tau R_{\pi/j}, 1/2)\rangle \cong \mathbb{D}_{2j}$	

Table 2: Spatiotemporal symmetry of bifurcating solutions in nonHopf bifurcation from a discrete rotating wave with $(\mathbf{O}(2) \times \mathbb{Z}_2, \mathbf{O}(2))$ symmetry. All bifurcations are period-preserving. All bifurcations are pitchforks unless stated otherwise.

same as applying the 'obvious' up-down symmetry τ. For mode B, [2, equation (3.4)] shows that evolution through half a period is the same as applying a glide-reflection consisting of τ composed with the translation $R_{\pi/j}$ in $\mathbf{SO}(2)$.

(b) Examples in the twisted equivariant setting ($k \geq 2$)

We now illustrate the general nonHopf theory with a few examples. A complete treatment of these examples (using induced representation theory, Section 7) is deferred to Section 8. For ease of exposition, we suppose throughout that the underlying discrete rotating wave is asymptotically stable subcritically and unstable supercritically.

Example 5.3 We consider local bifurcations from a discrete rotating wave with spatial symmetry $\Delta = \mathbb{D}_2$ and spatiotemporal symmetry $\Sigma = \mathbb{D}_4$. The group $\mathbb{D}_4$ is the symmetry group of the square and is generated by the four-fold counterclockwise rotation $R_{\pi/2}$ together with a reflection κ. The four reflections in $\mathbb{D}_4$ are shown in Figure 2(a). We take $\mathbb{D}_2$ to be the subgroup of $\mathbb{D}_4$ generated by R_π and κ (alternatively by the diagonal reflections κ and $R_\pi\kappa$ shown in Figure 2(a)). The discrete rotating wave is shown schematically in Figure 2(b) and has spatial symmetry $\mathbb{D}_2$ together with the half-period spatiotemporal symmetry $(R_{\pi/2}\kappa, 1/2)$. It is natural to choose $\sigma = R_{\pi/2}\kappa$ so that $k = m = 2$.

Observe that $(dG)_{x_0} = \sigma^2 L^2 = L^2$. Hence, we have period preserving bifurcations when $L^2 = I$ and period doubling bifurcations when $L^2 = -I$.

First, we consider the case $L^2 = I$. By Remark 4.1(b), $\Delta_L = \Delta \rtimes \mathbb{Z}_2$ is isomorphic to $\mathbb{D}_4$. The irreducible representations of $\mathbb{D}_4$ are of dimension one and two. We concentrate attention on the standard two-dimensional faithful representation of $\Delta_L \cong \mathbb{D}_4$. Generically, steady-state bifurcation with $\mathbb{D}_4$ symmetry [7, Chapter XIII,5] is determined locally by the steady-states with axial isotropy subgroup. It follows that, generically, bifurcation for the $\mathbb{D}_4$-equivariant diffeomorphism h is determined locally by the fixed points with axial isotropy. Up to conjugacy, we have the axial isotropy subgroups $J = \mathbb{D}_1(\kappa)$ and $J = \mathbb{D}_1(L)$. If these axial fixed points both bifurcate supercritically, then precisely one of these states is asymptotically stable (depending on the sign of a cubic coefficient in h). If one or both of the states bifurcates subcritically, then there is no asymptotically stable dynamics.

The corresponding periodic solutions for the flow are of approximately the same period as the underlying discrete rotating wave and inherit the stability of the fixed points for h. Their spatial symmetry $\Delta_x = J \cap \Delta$ is given by $\mathbb{D}_1(\kappa)$ and $\mathbf{1}$ respectively. Applying Lemma 4.7, we have that Σ_P is generated by κ and by $(R_{\pi/2}\kappa, 1/2)$ respectively. Thus, each of the bifurcating periodic solutions has a single reflection symmetry, but for one solution this is a spatial symmetry and for the other solution this is a spatiotemporal symmetry.

In alternative notation, we can write $(\Sigma_P, \Delta_x) = (\mathbb{D}_1, \mathbb{D}_1)$ and $(\Sigma_P, \Delta_x) = (\mathbb{D}_1, \mathbf{1})$ for the two periodic solutions. These solutions are shown in Figure 3.

It should be noted that the bifurcation diagrams of these bifurcations (including stability properties) can be read off directly from the results on $\mathbb{D}_4$-equivariant steady-state bifurcations, see [7].

In the period doubling case (where $\sigma^2 L^2 = L^2 = -I$), the group $\Delta_L = \Delta \rtimes \mathbb{Z}_4$ is less familiar. We consider the two-dimensional absolutely irreducible representation given by

$$\kappa = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad R_\pi = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad L = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

This action is isomorphic to the standard two-dimensional absolutely irreducible representation of $\mathbb{D}_4$ but with kernel generated by the order two element $L^2 R_\pi$.

The bifurcating fixed points for h have axial isotropy subgroups $J = \mathbb{D}_2(\kappa, L^2 R_\pi)$ and $J = \mathbb{D}_2(L\kappa, L^2 R_\pi)$ (up to conjugacy). The corresponding period-doubled periodic solutions have spatiotemporal symmetry $\Sigma_P = \langle \kappa, (R_\pi, \frac{1}{2}) \rangle$ and $\Sigma_P = \langle (R_{3\pi/2}, \frac{1}{4}) \rangle$, so that respectively $(\Sigma_P, \Delta_x) = (\mathbb{D}_2, \mathbb{D}_1)$ or $(\Sigma_P, \Delta_x) = (\mathbb{Z}_4, \mathbf{1})$. These solutions are shown in Figure 4. [1]

[1] In [11], the $L^2 = +I$ case was discussed in some detail. However, in the footnote on p. 602 it was erroneously claimed that the $+I$ and $-I$ cases lead to the same result. The computations for the case $L^2 = -I$ are carried out correctly in [17].

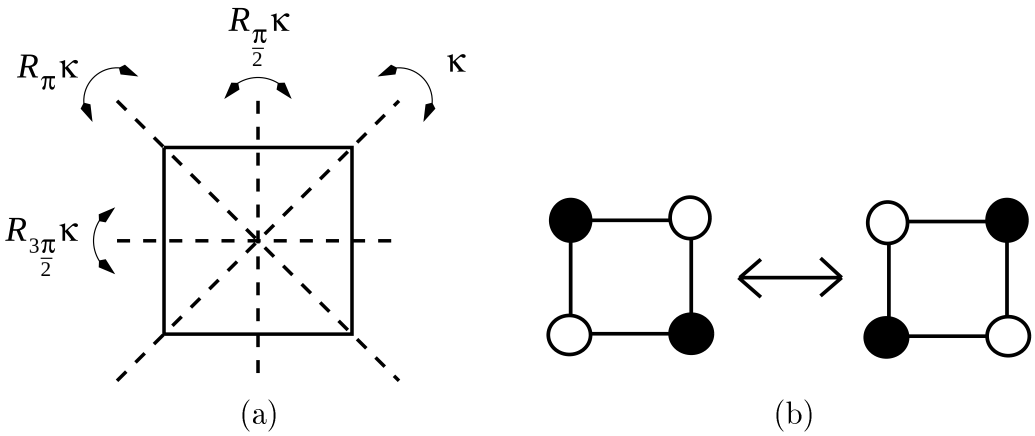
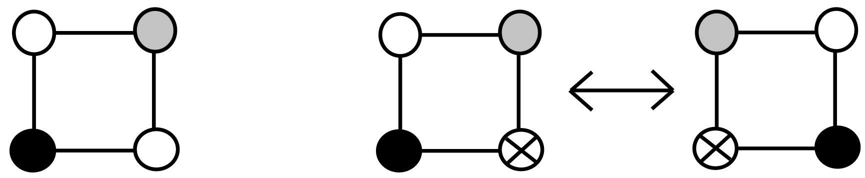


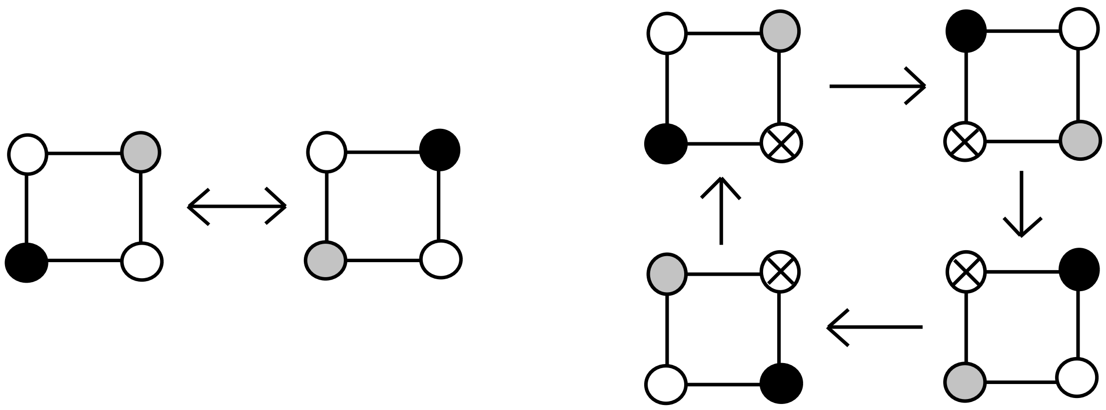
Figure 2: (a) The symmetry group $\mathbb{D}_4$. (b) Schematic stroboscopic picture of a $(\Sigma, \Delta) = (\mathbb{D}_4, \mathbb{D}_2)$ discrete rotating wave on a system of four coupled cells



$\Sigma_P = \langle \kappa \rangle$ $\qquad$ $\Sigma_P = \langle (R_{\pi/2}\kappa, \frac{1}{2}) \rangle$

$(\Sigma_P, \Delta_x) = (\mathbb{D}_1, \mathbb{D}_1)$ $\qquad$ $(\Sigma_P, \Delta_x) = (\mathbb{D}_1, 1)$

Figure 3: Schematic stroboscopic picture of solutions in a period-preserving bifurcation from a $(\Sigma, \Delta) = (\mathbb{D}_4, \mathbb{D}_2)$ discrete rotating wave on a system of four coupled cells.



$\Sigma_P = \langle \kappa, (R_\pi, \frac{1}{2}) \rangle$

$(\Sigma_P, \Delta_x) = (\mathbb{D}_2, \mathbb{Z}_2)$

$\Sigma_P = \langle (R_{3\pi/2}, \frac{1}{4}) \rangle$

$(\Sigma_P, \Delta_x) = (\mathbb{Z}_4, 1)$

Figure 4: Schematic stroboscopic picture of solutions in a period-doubling bifurcation from a $(\Sigma, \Delta) = (\mathbb{D}_4, \mathbb{D}_2)$ discrete rotating wave on a system of four coupled cells.

Example 5.4 We consider nonHopf bifurcations of discrete rotating waves with symmetry $(\Sigma, \Delta) = (\mathbb{D}_{2n}, \mathbb{D}_n)$, $n \geq 4$ even. (See Example 5.1 for the case n odd.) Write $\Sigma = \langle R_{\pi/n}, \kappa \rangle$, $\Delta = \langle R_{2\pi/n}, \kappa \rangle$, and choose $\sigma = R_{\pi/n}\kappa$. Again, $k = m = 2$.

It turns out (see Example 8.3) that there is a unique absolutely irreducible representation of Δ_L for which the subgroup $\Delta = \mathbb{D}_n$ acts on the plane by its standard faithful action. Moreover, in this representation, $L^2 = I$ so that we are in the period-preserving case. By Remark 4.1(b), $\Delta_L = \Delta \rtimes \mathbb{Z}_2 \cong \Sigma$ and we have the faithful two-dimensional absolutely irreducible representation of $\mathbb{D}_{2n}$. Applying the results of [4, 7], we obtain pitchfork branches of of fixed points with axial isotropy subgroup (up to conjugacy) $J = \mathbb{D}_1(\kappa)$ and $J = \mathbb{D}_1(L)$. Hence, the spatiotemporal symmetry group Σ_P of the corresponding periodic solutions is generated by κ and by $(R_{\pi/n}\kappa, 1/2)$ respectively. That is, $(\Sigma_P, \Delta_x) = (\mathbb{D}_1, \mathbb{D}_1)$ or $(\Sigma_P, \Delta_x) = (\mathbb{D}_1, \mathbf{1})$.

Example 5.5 We consider discrete rotating waves with symmetry $(\Sigma, \Delta) = (\mathbb{D}_n, \mathbb{Z}_n)$, $n \geq 3$. Here, Δ is generated by $R_{2\pi/n}$ and Σ is generated by $R_{2\pi/n}$ together with κ. Choosing $\sigma = \kappa$, we have $k = m = 2$.

Again, it turns out (see Example 8.5) that there is a unique absolutely irreducible representation of Δ_L in which $\Delta = \mathbb{Z}_n$ acts by the standard faithful action. Moreover, in this case $L^2 = I$ so that we have a period-preserving bifurcation. By Remark 4.1(b), $\Delta_L \cong \Sigma$ and we have the standard two-dimensional absolutely irreducible representation of $\mathbb{D}_n$.

When n is even, there are precisely two axial isotropy subgroups up to conjugacy, namely $J = \mathbb{D}_1(L)$ and $J = \mathbb{D}_1(R_{2\pi/n}L)$, see [7]. The corresponding periodic solutions have spatiotemporal symmetry $(\kappa, 1/2)$ and $(R_{2\pi/n}\kappa, 1/2)$ respectively, so that in either case $(\Sigma_P, \Delta_x) = (\mathbb{D}_1, \mathbf{1})$. Generically, both solutions occur as pitchfork bifurcations, and one of these solutions is asymptotically stable if and only if both branches are supercritical.

When n is odd, there is a unique axial isotropy subgroup $J = \mathbb{D}_1(L)$ up to conjugacy so the corresponding periodic solution has spatiotemporal symmetry $(\kappa, 1/2)$ and $(\Sigma_P, \Delta_x) = (\mathbb{D}_1, \mathbf{1})$. The bifurcation is transcritical when $n = 3$ yielding unstable solutions. When $n \geq 5$, n odd, there are asymmetric pitchforks to pairs of solutions. One out of each pair of solutions is asymptotically stable if the branches are supercritical.

6 Analysis of Hopf bifurcation

In this section, we analyze Hopf bifurcation from discrete rotating waves. Again, we show how to reduce problems with spatiotemporal symmetry to previously studied problems with only spatial symmetry [4].

There are two subsections. In Subsection (a), we consider explicitly the representation of the group Δ_L from Theorem 3.4(b). Birkhoff normal form theory (with the usual distinction between weak and strong resonances) is presented in Subsection (b) leading to a systematic nonlinear analysis.

(a) Structure of the group Δ_L

In the Hopf case, the action of Δ_L on E^c is generically irreducible of complex type. It follows from Proposition 3.1 that $(dG)_{x_0}$ and L^k are complex scalar multiples of the identity. At bifurcation, these complex numbers have absolute value 1. Generically, these scalars are nonreal. In particular, generically $(dG)_{x_0}$ has complex eigenvalues at bifurcation guaranteeing Hopf bifurcation.

Write $L^k = \alpha I$ where $\alpha \in \mathbb{C}$, $|\alpha| = 1$. Define $L_0 = \alpha^{-1/k} L$. Observe that L_0 has order k and is twisted equivariant. It follows that the group Δ_{L_0} generated by Δ and L_0 is given by $\Delta_{L_0} = \Delta \rtimes \mathbb{Z}_k$ (this time with L_0 playing the role of the generator of $\mathbb{Z}_k$).

Proposition 6.1 *Suppose that $L^k = \alpha I$ where α has infinite order. Then*

$$\Delta_L = \Delta_{L_0} \times S^1 = (\Delta \rtimes \mathbb{Z}_k) \times S^1.$$

Proof Let T denote the closed group generated by L. We claim that $T = \mathbb{Z}_k \times S^1$ where $\mathbb{Z}_k$ is generated by L_0 and S^1 is generated by L^k. Since L^k commutes with the action of Δ, it then follows that $\Delta_L = \Delta \rtimes T = \Delta \rtimes (\mathbb{Z}_k \times S^1) = (\Delta \rtimes \mathbb{Z}_k) \times S^1$ as required.

It remains to verify the claim. Since α has infinite order, the closed group generated by L^k is isomorphic to the circle group S^1. In particular $\alpha^{-1/k} I \in T$, and hence $L_0 = \alpha^{-1/k} L \in T$. It follows that T is generated by the elements L_0 and L^k. These elements in turn generate the subgroups $\mathbb{Z}_k$ and S^1 as required. ∎

Proposition 6.2 *Suppose that the action of Δ_L on E^c is irreducible of complex type. Then the center subspace E^c is Δ_{L_0}-simple (either the direct sum of two absolutely irreducible subspaces or a nonabsolutely irreducible subspace).*

Proof Observe that L_0 is independent of L, to the extent that multiplying L by a complex scalar leaves L_0 unchanged. Hence we may suppose without loss that the assumptions of Proposition 6.1 are satisfied, so that $\Delta_L = \Delta_{L_0} \times S^1$. The rest is an elementary argument in representation theory, and we refer to [7] for any basic definitions and concepts that are not defined in the following proof.

Write $E^c = W_1 \oplus \cdots \oplus W_r$ where $W_1, \ldots, W_r$ are the isotypic components for the action of Δ_{L_0}. Since elements of S^1 commute with this action, it follows that W_j is S^1-invariant for each j. But then W_j is Δ_L-invariant for each j. Since E^c is Δ_L-irreducible,it follows that E^c consists of a single isotypic component ($r = 1$). This isotypic component consists of s copies of an irreducible representation of type $\mathcal{D}$, where $\mathcal{D} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. In particular, $\mathrm{Hom}_{\Delta_{L_0}}(E^c) = \mathrm{M}_s(\mathcal{D})$ (the space of $s \times s$ matrices with entries in $\mathcal{D}$).

Now S^1 consists of matrices of the form αI where $\alpha \in \mathbb{C}$ and $|\alpha| = 1$ and the actions of Δ_{L_0} and S^1 commute. In particular, we can identify S^1 with a subset of $\mathrm{M}_s(\mathcal{D})$. It follows immediately that s is even if $\mathcal{D} = \mathbb{R}$.

The space $\mathrm{Hom}_{\Delta_L}(E^c)$ of linear maps that commute with Δ_L consists of those elements of $\mathrm{M}_s(\mathcal{D})$ that commute with S^1. At the same time, Δ_L acts irreducibly, so that $\mathrm{Hom}_{\Delta_L}(E^c) \cong \mathbb{C}$.

When $\mathcal{D} = \mathbb{R}$, we compute that $\mathrm{Hom}_{\Delta_L}(E^c) = \mathrm{M}_{s/2}(\mathbb{C})$ so that $s = 2$. When $\mathcal{D} = \mathbb{C}$ or $\mathbb{H}$, we compute that $\mathrm{Hom}_{\Delta_L}(E^c) = \mathrm{M}_s(\mathbb{C})$ so that $s = 1$. ∎

Remark 6.3 The results that we have described are valid 'generically' and we wish to clarify our use of this terminology. Genericity, as used in Theorems 3.2 and 3.4 is used in the strongest possible sense. For example, in Theorem 3.2 there is an open and dense set of nonsingular twisted equivariant linear maps L satisfying $E^c \neq \{0\}$ for which the center subspace E^c is Δ_L-irreducible. Moreover, this open and dense set is semialgebraic (defined by polynomial equalities and inequalities).

Recall that in the Hopf case, $L^k = \alpha I$, where $\alpha \in \mathbb{C}$, $|\alpha| = 1$. We note that in Theorem 3.4(b) it is assumed that α is not a root of unity (and hence has infinite order). In fact we require only that α has sufficiently high order so that the eigenvalues of $(dG)_{x_0} = \sigma^m L^m$ are not real. This again is genericity in the strong sense described above.

Genericity in Proposition 6.1 is used somewhat differently. There, it is required that α has infinite order. This is a dense condition, but not an open condition. In this case, genericity means that there is a 'residual' set of twisted equivariant linear maps for which the conclusions of Proposition 6.1 are valid. When α has finite order, the Hopf bifurcation is said to satisfy a resonance condition and Proposition 6.1 fails (even though Theorem 3.4(b) and Proposition 6.2 hold).

In Theorem 6.4, we show that the full strength of Proposition 6.1 is not required for the nonlinear analysis to go through, so our final results are valid generically in the strong meaning of the the word. There are a finite number of 'strong resonances' (where α has particularly low order) which we exclude in our theory. The remaining 'weak resonances' do not affect our results, cf. [8].

(b) Nonlinear analysis of Hopf bifurcation

Modifying slightly the approach of Lamb [10], we write $f = L_0 h$ (not $f = Lh$). Since f and L_0 are twisted equivariant, it follows as in the nonHopf case that h is Δ-equivariant. Moreover, as in Lemma 4.4, we may suppose (after changes of coordinates that preserve twisted equivariance of f) that f commutes with the semisimple matrix $L = (df)_{x_0}$ to arbitrarily high order. Since L_0 commutes with L, h is Δ_L-equivariant to arbitrarily high order.

Under the assumptions of Proposition 6.1 (α of infinite order), we have $\Delta_L = \Delta_{L_0} \times S^1$. However, if α has finite order, then Δ_L is strictly contained in $\Delta_{L_0} \times S^1$. As indicated in Remark 6.3, the condition that α has infinite order is too strong an assumption and can be relaxed as follows.

Theorem 6.4 *Suppose that the action of Δ_L on E^c is irreducible of complex type. Then generically, the diffeomorphism $h = L_0^{-1} f$ is $\Delta_{L_0} \times S^1$-equivariant through any desired finite order.*

More precisely, for each integer $\ell \geq 2$ there is a finite subset $S_\ell \subset \{\beta \in \mathbb{C} : |\beta| = 1\}$ with the following property: Write $L^k = \alpha I$. Provided $\alpha \notin S_\ell$, the diffeomorphism $h = L^{-1} f$ is $\Delta_{L_0} \times S^1$-equivariant through order ℓ.

Proof By the above discussion, we may assume that h is Δ_L-equivariant to any finite order. In particular, h commutes with $L^k = \alpha I$ to any finite order. It then follows from standard theory that there is a finite set S_ℓ such that h commutes with S^1 through order ℓ provided $\alpha \notin S_\ell$. Hence, through order ℓ, the diffeomorphism h commutes with the group generated by Δ_L and S^1. As in the proof of Proposition 6.1, this is the group $\Delta_{L_0} \times S^1$. ∎

This result shows that there is the standard distinction between weak and strong resonances. As usual, most rational angles are weak resonances and lead to the same results as in the irrational case.

We proceed, as in the nonHopf case, by assuming that h is $\Delta_{L_0} \times S^1$-equivariant to all orders. (Again, this assumption is relaxed at the end of the subsection.) Thus, h is a general diffeomorphism satisfying $h(x_0) = x_0$, $(dh)_{x_0} = \alpha^{1/k} I$, $\alpha \in \mathbb{C}$ and $\Delta_{L_0} \times S^1$-equivariant with respect to a Δ_{L_0}-simple action of the compact Lie group Δ_{L_0}. Hence, we are in the situation of generic Hopf bifurcation for diffeomorphisms with Δ_{L_0} symmetry [4, 21]. In particular, the equivariant branching lemma guarantees branches of invariant circles (invariant tori for the flow) corresponding to each $\mathbb{C}$-axial isotropy subgroup $J \subset \Delta_{L_0} \times S^1$. (Recall that a $\mathbb{C}$-axial isotropy subgroup is an isotropy subgroup with two-dimensional fixed-point subspace.) Note that J is the symmetry group that fixes the invariant circles pointwise. We introduce also the subgroup $J_C \subset \Delta_{L_0} \times S^1$ that fixes the invariant circles setwise. Observe that J is a normal subgroup of J_C.

As in the nonHopf case, we can read off from the isotropy subgroup J the symmetries of the bifurcating flow-invariant tori. Suppose that C is an invariant circle for h consisting of points with isotropy $J \subset \Delta_{L_0} \times S^1$ (not necessarily $\mathbb{C}$-axial). Choose $x \in C$. Then C and the corresponding flow-invariant 2-torus T consist of points with symmetry $\Delta_x = J \cap \Delta$. Next, we compute the symmetry group Σ_T that fixes the torus T setwise. Define $\Delta_C = J_C \cap \Delta$. Also, let $p \geq 1$ be least such that $L_0^{-p}\delta \subset J_C$ for some $\delta \in \Delta$. Then $\Sigma_T = \langle \Delta_C, \sigma^p\delta \rangle$.

The expected dynamics on the torus T takes the form of either a quasiperiodic irrational flow or phase-locked periodic solutions. In the case of irrational flow, the torus is a single ω-limit set. The group Σ_T is the largest subgroup of Σ that fixes this ω-limit set as a set and is called the 'symmetry on average' of the ω-limit set [16]. Also in the terminology of [16], Δ_x is the 'instantaneous symmetry' of the ω-limit set.

When there is phase-locking to a periodic solution $P \subset T$, the instantaneous/spatial symmetry is still given by Δ_x but the symmetry on average/spatiotemporal symmetry Σ_P is some group satisfying $\Delta_x \subset \Sigma_P \subset \Sigma_T$.

Finally, we note that under the flow, the circle $C \subset T$ is transformed into new circles on T. Occasionally, the new circle is a symmetric image of the circle C. Hence, viewing the flow on the torus as a periodic cycling of invariant circles, there is a spatiotemporal symmetry structure on T analogous to the one in the nonHopf case (regardless of whether the flow on T is quasiperiodic or phase-locked). Equivalently, we have the spatiotemporal symmetry of the periodic solution obtained by collapsing the invariant circle to a point. With this interpretation, the spatial symmetry and spatiotemporal symmetry is given by Δ_C and Σ_T.

It remains to drop the assumption that h is $\Delta_{L_0} \times S^1$-equivariant to all orders. By the results of Field [6], which build upon work of Ruelle [21], there is an integer ℓ such that the branches of invariant circles for $\Delta_{L_0} \times S^1$-equivariant diffeomorphisms h are generically robust to symmetry-breaking perturbations of order exceeding ℓ. By excluding strong resonances as in Theorem 6.4, h is a general $\Delta_{L_0} \times S^1$-equivariant diffeomorphism through order ℓ ensuring that the above analysis goes through.

Example 6.5 We consider the discrete rotating wave shown in Figure 2(b). This solution has spatial symmetry $\Delta = \mathbb{D}_2(R_\pi, \kappa)$ and spatiotemporal symmetry $\Sigma = \mathbb{D}_4(R_{\pi/2}, \kappa)$. Again, we choose $\sigma = R_{\pi/2}\kappa$ so that $k = m = 2$.

NonHopf bifurcation was considered in Example 5.3. We now consider Hopf bifurcation. By Remark 4.1(b), $\Delta_{L_0} = \Delta \rtimes \mathbb{Z}_2 \cong \mathbb{D}_4$ generated by the 'reflections' κ and L_0.

In Birkhoff normal form, the diffeomorphism h is $\Delta_{L_0} \times S^1$-equivariant and we are in the situation of $\mathbb{D}_4$-equivariant Hopf bifurcation which is studied at some length in [7, Chapter XVIII]. The Δ_{L_0}-simple representations are the direct sum of two isomorphic absolutely irreducible representations and hence are of dimension two or four. Again, we concentrate on the standard action of $\mathbb{D}_4$ so that we have a four-dimensional representation of Δ_{L_0}.

Hopf bifurcation for the diffeomorphism h leads to invariant circles corresponding to three $\mathbb{C}$-axial isotropy subgroups $J \subset \Delta_L = \mathbb{D}_4 \times S^1$ up to conjugacy, namely $J = \mathbb{Z}_4(R_{\pi/2}, -\pi/2)$, $J = \mathbb{D}_1(\kappa, 0) \times \mathbb{Z}_2(R_\pi, \pi)$ and $J = \mathbb{D}_1(L_0, 0) \times \mathbb{Z}_2(R_\pi, \pi)$. We consider these three cases in turn.

(i) Let C be the invariant circle consisting of points with $\mathbb{C}$-axial isotropy $J = \mathbb{Z}_4(R_{\pi/2}, -\pi/2)$. The invariant circle is preserved setwise by $J_C = \mathbb{Z}_4(R_{\pi/2}) \times S^1$. Note that C corresponds to a flow-invariant 2-torus T for the flow. Taking intersections with Δ yields the groups $\Delta_x = \mathbf{1}$ and $\Delta_C = \mathbb{Z}_4$. We also have $\Sigma_T = \langle \Delta_C, R_{\pi/2}\kappa \rangle \cong \mathbb{D}_4$ (since $L_0^{-p}\delta \in J_C$ with $p = 1$, $\delta = 1$).

The subgroups

$$\Delta_x = \mathbf{1}, \qquad \Delta_C = \mathbb{Z}_4, \qquad \Sigma_T = \langle \mathbb{Z}_4, (R_{\pi/2}\kappa, 1/2) \rangle \cong \mathbb{D}_4,$$

have the following interpretation. The flow-invariant torus T consists of points with no spatial symmetry ($\Delta_x = \mathbf{1}$) and $\Sigma_T = \mathbb{D}_4$ preserves the torus as a set. Hence, the dynamics on the torus has trivial instantaneous symmetry and, in the case of quasiperiodic flow, $\mathbb{D}_4$ symmetry on average. In addition, we can regard the flow on the torus as being a periodic evolution of the invariant circle C leading to spatial symmetry $\Delta_C = \mathbb{Z}_4$ and spatiotemporal symmetry $\Sigma_T = \langle \mathbb{Z}_4, (R_{\pi/2}\kappa, 1/2) \rangle$.

(ii) In the case of the $\mathbb{C}$-axial subgroup $J = \mathbb{D}_1(\kappa, 0) \times \mathbb{Z}_2(R_\pi, \pi)$, we have $J_C = \mathbb{D}_2 \times S^1$ and

$$\Delta_x = \mathbb{D}_1(\kappa), \qquad \Delta_C = \mathbb{D}_2, \qquad \Sigma_T = \mathbb{D}_2.$$

(iii) In the case of the $\mathbb{C}$-axial subgroup $J = \mathbb{D}_1(L_0, 0) \times \mathbb{Z}_2(R_\pi, \pi)$, we have $J_C = \mathbb{D}_2(R_\pi, L_0) \times S^1$ and

$$\Delta_x = \mathbf{1}, \qquad \Delta_C = \mathbb{Z}_2(R_\pi), \qquad \Sigma_T = \langle R_\pi, (R_{\pi/2}\kappa, 1/2) \rangle \cong \mathbb{D}_2.$$

7 Induced representations for $G = H \rtimes \mathbb{Z}_\ell$

The theory presented in Sections 4 and 6 for bifurcation from a discrete rotating wave relies on knowledge of the irreducible representations of certain finite cycle extensions

(Δ_L and Δ_{L_0} respectively) of a compact Lie group Δ. In this section, we show how to compute the irreducible representations of such a finite cyclic extension, given that the irreducible representations of Δ are known.

Let G be a compact Lie group. We suppose that G is a semidirect product $G = H \rtimes \mathbb{Z}_\ell$ where H is a closed normal subgroup of G and the cyclic group $\mathbb{Z}_\ell$ has generator ρ. Write $\rho h \rho^{-1} = \phi(h)$ where $\phi \in \mathrm{Aut}(H)$. Define k to be the order of ϕ (so $k \geq 1$ is least such that ρ^k lies in the centralizer of H). Note that k divides ℓ.

Using the methods of induced representations (over $\mathbb{R}$), we compute the irreducible representations of G in terms of the irreducible representations of H. Presumably, the methods of this section are known, but most presentations of the theory of induced representations are given for complex representations.

Suppose that G acts linearly on a real vector space X and define

$$T^j(X) = \{R : X \to X \text{ linear} : Rh = \phi^j(h)R \text{ for all } h \in H\}.$$

(In this definition, h is identified with the linear map on X given by the representation of H.) Let $T^j(X)^*$ consist of the nonsingular elements of $T^j(X)$. Note that $T^k(X) = \mathrm{Hom}_H(X)$ so in particular, $T^k(X)^* \neq \emptyset$. In general, $T^j(X)^*$ may be empty for certain values of j. We define the *order* s of X to be the least positive integer such that $T^s(X)^* \neq \emptyset$.

Observe that $R_1 \in T^{j_1}(X)$, $R_2 \in T^{j_2}(X)$ implies that $R_1R_2 \in T^{j_1+j_2}(X)$. It follows that the set of integers j for which $T^j(X)^* \neq \emptyset$ is a subgroup of $\mathbb{Z}$ generated by s. In particular, the order s divides k (and hence s divides ℓ).

In the case that $T^j(X)^* \neq \emptyset$, let $R_0 \in T^j(X)^*$. Then $R \in T^j(X)$ if and only if $R = AR_0$ where $A \in \mathrm{Hom}_H(X)$. In other words, $T^j(X)$ is a one-dimensional module over $\mathrm{Hom}_H(X)$.

Elementary irreducible representations of G

In this subsection, we show how to build up a simple class of irreducible representations of G, starting from an irreducible representation of H.

Suppose that V is an irreducible representation of H. Specifically, suppose that the irreducible representation is defined by the group homomorphism $\chi^{(1)} : H \to \mathbf{GL}(V)$. We have the derived representations $\chi^{(j)} : H \to \mathbf{GL}(V)$ defined by $\chi^{(j)}_h = \chi^{(1)}_{\phi^{j-1}(h)}$, $j \in \mathbb{Z}$. It is easily verified that each of these representations is irreducible. Moreover, $\chi^{(j_1)}$ is isomorphic to $\chi^{(j_2)}$ if and only if $j_1 \equiv j_2 \bmod s$ where s is the order of V. Thus we have the distinct irreducible representations $\chi^{(j)}$, $j = 1, \ldots, s$.

Now form the space

$$\Omega(V) = \bigoplus_{j=1}^{s} V = V \oplus V \oplus \cdots \oplus V,$$

and define an action $\chi : H \to \mathbf{GL}(\Omega(V))$ by

$$\begin{aligned} \chi_h(v_1, v_2, \ldots, v_s) &= (\chi_h^{(1)} v_1, \chi_h^{(2)} v_2, \ldots, \chi_h^{(s)} v_s) \\ &= (\chi_h^{(1)} v_1, \chi_{\phi(h)}^{(1)} v_2, \ldots, \chi_{\phi^{s-1}(h)}^{(1)} v_s). \end{aligned} \tag{7.1}$$

Chose $r \in T^s(V)^*$ and define

$$\chi_\rho(v_1, v_2, \ldots, v_s) = (v_2, v_3, \ldots, v_s, r v_1). \tag{7.2}$$

We compute that $\chi_r \chi_h = \chi_{\phi(h)} \chi_r$. In particular, (7.1,7.2) defines an action $\chi : G \to \mathbf{GL}(\Omega(V))$ if and only if $(\chi_\rho)^\ell = I_{\Omega(V)}$, equivalently, $r^{\ell/s} = I_V$.

Proposition 7.1 *Suppose that V is H-irreducible of order s, Suppose further that $r \in T^s(V)^*$ can be chosen so that $r^{\ell/s} = I_V$. Then $\Omega(V) = \oplus_{j=1}^s V$ is G-irreducible under the action (7.1,7.2).*

Proof Suppose that $Z \subset \Omega(V)$ is a nonempty G-invariant subspace. Then Z is H-invariant and we can write $Z = U_1 \oplus \cdots \oplus U_n$ where $U_1, \ldots, U_n$ are H-irreducible subspaces of Z. Since $\Omega(V)$ is a sum of distinct H-irreducible representations, it follows that each U_j transforms as $\chi^{(j_i)}$ under H for some j_i. Since ρ permutes the representations $\chi^{(j)}$ cyclically, invariance of Z under ρ implies that each of these representations is present so that $Z = \Omega(V)$. Hence $\Omega(V)$ is G-irreducible. ∎

When the hypotheses of Proposition 7.1 are satisfied, we call $\Omega(V)$ an *elementary irreducible representation* of G. The remaining G-irreducible representations that contain the H-irreducible representation V are called *nonelementary irreducible representations* of G.

Example 7.2 Consider the example $G = \mathbb{Z}_3$, $H = \mathbf{1}$. There is a single H-irreducible representation V, namely the trivial one-dimensional representation. Moreover, there is a unique elementary irreducible representation of G, again the trivial representation. In addition, there is a nonelementary irreducible representation given by the standard action of $\mathbb{Z}_3$ on $\mathbb{R}^2$.

Next, consider the example $G = \mathbb{Z}_4$, $H = \mathbb{Z}_2$. If we start with the trivial representation of H, then there are now two elementary irreducible representations of G: the trivial representation, and the representation where the generator of G acts as -1. However, there are no nonelementary irreducible representations of G. Conversely, if we start with the nontrivial irreducible representation of H, there are no elementary irreducible representations, though there are is a nonelementary irreducible representation given by the standard action of $\mathbb{Z}_4$ on $\mathbb{R}^2$.

Although Example 7.2 is clearly too simple to be representative of the general theory, it turns out that certain aspects are completely general. In particular, it turns out that if we start with an absolutely irreducible representation of H, then the elementary irreducible representations of G are always of real type (Corollary 7.6) and the nonelementary irreducible representations of G are always of complex type (Proposition 7.12 and Remark 7.14). Moreover, when ℓ/s is odd (as in the first example) then there is a unique elementary irreducible representation, whereas when ℓ/s is even (as in the second example) then there is either zero or two elementary irreducible representations (Corollary 7.9).

More generally, three issues arise:

(i) Determine the type (real, complex or quaternionic) of each elementary irreducible representation $\Omega(V)$.

(ii) Enumerate the elementary irreducible representations $\Omega(V)$ of G (if any) given an irreducible representation V of H.

(iii) Enumerate the nonelementary irreducible representations of G.

We begin with issue (i). Naturally, the solution depends partly on the type of the H-irreducible representation V. It turns out that we require a finer classification of H-irreducible representations that takes into account the automorphism $\phi \in \operatorname{Aut} H$. The ideas described below were developed in the context of linear reversible equivariant vector fields [15].

Let V be H-irreducible of order s and type $\mathcal{D} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Observe that $r \in T^s(V)$ induces an automorphism $\hat{r} : \operatorname{Hom}_H(V) \to \operatorname{Hom}_H(V)$ given by $\hat{r}a = rar^{-1}$. This means that $\hat{r} \in \operatorname{Aut}(\mathcal{D})$. As shown in Porteous [19], either $\hat{r}$ is an inner automorphism or $\hat{r}$ is complex conjugation. In more detail, $\hat{r}$ is the identity when $\mathcal{D} = \mathbb{R}$, the identity or complex conjugation when $\mathcal{D} = \mathbb{C}$, and conjugation by a quaternion when $\mathcal{D} = \mathbb{H}$. When $\mathcal{D} = \mathbb{C}$, we distinguish the possibilities that $\hat{r}$ is the identity or complex conjugation:

Definition 7.3 An irreducible representation of complex type V is of *inner complex type* if $\hat{r}$ is the identity and is of *outer complex type* if $\hat{r}$ is complex conjugation.

By the structure of $T^s(V)^*$ (as a module over $\operatorname{Hom}_H(V)$) it follows that this definition is independent of the choice of r.

Example 7.4 Let $n \geq 3$ and consider the groups $G_1 = \mathbb{Z}_n \times \mathbb{Z}_2$ and $G_2 = \mathbb{D}_n = \mathbb{Z}_n \rtimes \mathbb{Z}_2$, both of which are index two extensions of the normal subgroup $H = \mathbb{Z}_n$. The irreducible representations for the groups G_1 and G_2 are quite different since G_1 has two-dimensional irreducible representations of complex type whereas the irreducible representations of G_2 are all of real type. By Corollary 7.6 below, this distinction between the irreducible representations of G_1 and G_2 is accounted for by the distinction between irreducible representations of H of inner and outer complex type.

Indeed, the two-dimensional irreducible representations of H are of inner complex type in the case of the group G_1 and are of outer complex type in the case of the group G_2.

We verify the above statement in the case of the standard action on $\mathbb{R}^2 \cong \mathbb{C}$ given by $z \mapsto e^{2\pi i/n}z$. Let ρ denote an element of order two that generates the extension G_1, G_2 of H. Since G_1 is abelian, the automorphism $\phi \in \mathrm{Aut}(H)$ induced by ρ is the trivial automorphism so that $s = k = 1$. Moreover, $T^1(\mathbb{C})$ consists of the linear maps $R : \mathbb{C} \to \mathbb{C}$ that commute with the action of $\mathbb{Z}_n$. In particular, we can choose $r \in T^1(\mathbb{C})^*$ to be the identity. Clearly, the automorphism $\hat{r} \in \mathrm{Aut}(\mathbb{C})$ induced by r is the identity automorphism. Hence the irreducible representation is of inner complex type.

Next we turn to the group G_2. Since G_2 is not abelian, it follows that $k = 2$. However, we still have $s = 1$. To see this, recall the relation $\phi(h) = \rho h \rho^{-1} = h^{-1}$ for $h \in \mathbb{Z}_n$. It follows that $T^1(\mathbb{C})$ consists of linear maps $R : \mathbb{C} \to \mathbb{C}$ satisfying $Re^{2\pi i/n} = e^{-2\pi i/n}R$. In particular, we have that $r \in T^1(\mathbb{C})^*$, where $rz = \bar{z}$. Moreover, it is immediate that the automorphism $\hat{r} \in \mathrm{Aut}(\mathbb{C})$ is complex conjugation. Hence the irreducible representation is of outer complex type.

Proposition 7.5 *Suppose that V is irreducible of order s.*

(a) If V is of real, inner complex or quaternionic type, then there exists an element $r_0 \in T^s(V)^*$ such that $r_0 a = a r_0$ for all $a \in \mathrm{Hom}_H(V)$.

(b) If V is of outer complex type, then there exists an element $r_0 \in T^s(V)^*$ such that $r_0 a = \overline{a} r_0$ for all $a \in \mathrm{Hom}_H(V)$.

Proof The commutativity properties of r_0 follow immediately from the classification of $\mathrm{Aut}(H)$ described above, except in the case when V is of quaternionic type. But then $\hat{r}_0$ is conjugation by a quaternion. The equivariant maps are quaternionic multiples of the identity, so by composing r_0 with the appropriate element of $\mathrm{Hom}_H(V) \cong \mathbb{H}$, we can arrange that $\hat{r}_0$ is the identity automorphism as required. ∎

Corollary 7.6 *Assume the set up of Proposition 7.1. Then $\Omega(V)$ is G-absolutely irreducible if and only if V is H-irreducible of real or outer complex type.*

If V is of inner complex type, then $\Omega(V)$ is of complex type. If V is of quaternionic type, then $\Omega(V)$ is either of quaternionic type (if $\hat{r} = I_H$) or of complex type (if $\hat{r}$ is not the identity automorphism).

Proof Write $V \cong \mathcal{D}^d$ where $\mathcal{D} = \mathrm{Hom}_H(V)$ is the type of the H-irreducible representation V.

Suppose that $B \in \mathrm{Hom}_G(\Omega(V))$. Since $\Omega(V) = \oplus_1^\ell V$ is a direct sum of nonisomorphic H-irreducible subspaces, we have $B = (\beta_1 I_d) \oplus \cdots \oplus (\beta_\ell I_d)$ where $\beta_1, \ldots, \beta_\ell \in \mathcal{D}$. Such a linear map B lies in $\mathrm{Hom}_G(\Omega(V))$ if and only if $B\rho = \rho B$. In other words, $\beta_1 = \cdots = \beta_\ell$ and $\beta_1 I_d$ commutes with r.

When V is absolutely irreducible, we have $\beta_1 \in \mathbb{R}$ so that $\mathrm{Hom}_G(\Omega(V)) \cong \mathbb{R}$ and $\Omega(V)$ is absolutely irreducible.

It remains to consider the cases when V is nonabsolutely irreducible. Since $T^s(V)^*$ is a module over $\mathrm{Hom}_H(V)$, it follows that $r = (\alpha I_d) r_0$ where r_0 is as constructed in Proposition 7.5 and $\alpha \in \mathcal{D}$. From now on, we shall adopt the abuse of notation $r = \alpha r_0$. We have the condition

$$(\alpha r_0)\beta_1 = \beta_1(\alpha r_0). \tag{7.3}$$

In the inner complex case, $\alpha I_d, \beta_1 I_d$ and r_0 all commute so that condition (7.3) is automatically satisfied. Hence $\beta_1 \in \mathbb{C}$ is arbitrary and so $\Omega(V)$ is of complex type.

In the outer complex case, α and β_1 commute so that condition (7.3) reduces to $r_0\beta_1 = \beta_1 r_0 = r_0\overline{\beta_1}$. In other words, $\beta_1 \in \mathbb{R}$ and $\Omega(V)$ is absolutely irreducible.

Finally, in the quaternionic case, r_0 commutes with $\beta_1 I_d$ so that condition (7.3) reduces to $\alpha\beta_1 = \beta_1\alpha$. When $\alpha \in \mathbb{R}$ (equivalently the automorphism $\hat{r} = \widehat{\alpha r_0} : \mathbb{H} \to \mathbb{H}$ is trivial), $\beta_1 \in \mathbb{H}$ is arbitrary and $\Omega(V)$ is of quaternionic type. Otherwise, β_1 lies in a copy of $\mathbb{C} \subset \mathbb{H}$ and $\Omega(V)$ is of complex type. ∎

Issue (ii) centers around the order of elements $r \in T^s(V)^*$. This is addressed in the next result.

Lemma 7.7 *Suppose that V is irreducible of order s. Then the element $r_0 \in T^s(V)^*$ constructed in Proposition 7.5 can be chosen so that, in addition, $r_0^{k/s} = \pm I_V$.*

Furthermore, r_0 can be chosen so that $r_0^{k/s} = I_V$ provided V is of inner complex type, or if k/s is odd. (However, note that for V of outer complex type, k/s is automatically even.)

Proof Write $V \cong \mathcal{D}^d$ where $\mathrm{Hom}_H(V) \cong \mathcal{D}$. Observe that $r_0^{k/s} \in \mathrm{Hom}_H(V)$ and so $r_0^{k/s} = \alpha I_d$ for some nonzero $\alpha \in \mathcal{D}$.

In the real case, we scale r_0 by $1/|\alpha|^{s/k}$ so that $r_0^{k/s} = \pm I_V$.

In the inner complex case, we scale r_0 by a k/s'th root of unity so that $r_0^{k/s} = I_V$.

In the quaternionic case, $r_0^{k/s} = \alpha I_d$ commutes with elements of $\mathrm{Hom}_H(V) \cong \mathbb{H}$ and it follows that $\alpha \in \mathbb{R}$. Hence we may scale as in the real case so that $r_0^{k/s} = \pm I_V$ (while preserving commutativity of r_0).

In the outer complex case, observe that $r_0^{k/s} = \alpha I_d$ commutes with all elements of $\mathrm{Hom}_H(V)$. Since odd powers of r_0 do not have this property, it follows that k/s is even. In addition, $r_0^{k/s}$ commutes with r_0. Since complex scalars do not commute with r_0 we have that $\alpha \in \mathbb{R}$. Hence we can scale as in the real case so that $r_0^{k/s} = \pm I_V$.

In all cases, if k/s is odd, we can replace r_0 by $-r_0$ if necessary so that $r_0^{k/s} = I_V$. ∎

Remark 7.8 If V is a faithful representation of H, then the results in Lemma 7.7 are optimal. Otherwise, we may replace k by k_V where k_V is the order of the automorphism ϕ modulo the kernel of the representation of H on V.

Corollary 7.9 *Suppose that V is H-irreducible of order s. The number of nonisomorphic elementary irreducible representations $\Omega(V)$ for G is given as follows.*

Type of V		Nonisomorphic $\Omega(V)$
Real	ℓ/s odd	1
	ℓ/s even	0 or 2
Outer complex		0 or 1
Inner complex		ℓ/s
Quaternionic		$[\ell/2s]$

Proof In the real case, we have $r = \alpha r_0$ where $\alpha \in \mathbb{R}$ and r_0 is as in Proposition 7.5 and Lemma 7.7. Hence $r^{\ell/s} = \pm\alpha^{\ell/s} I_V$ with finite order if and only if $\alpha = \pm 1$. If ℓ/s is odd, precisely one of the choices $\alpha = \pm 1$ yields an element r such that $r^{\ell/s} = I_V$. If ℓ/s is even, either both choices or neither choice works.

To complete the analysis of the real case, suppose that ℓ/s is even and $r_0^{\ell/s} = I_V$. We have two elementary irreducible representations $\Omega(V)$ depending on whether we choose $r = r_0$ or $r = -r_0$ in defining the action of ρ in Proposition 7.1. Label the choices of ρ by ρ_+ and ρ_-. The resulting representations of G are isomorphic if there is a nonsingular linear map $A : \Omega(V) \to \Omega(V)$ such that A commutes with the action of H and $A\rho_+ = \rho_- A$. Commutativity with H implies that $A = a_1 I_V \oplus \cdots \oplus a_s I_V$ where $a_j \in \mathbb{R}$. The condition $A\rho_+ = \rho_- A$ guarantees that $a_1 = \cdots = a_s$ and that $a_1 r_0 = -r_0 a_1$ which is impossible. Hence the two representations are nonisomorphic.

In the inner complex case, we have $r = \alpha r_0$ where $\alpha \in \mathbb{C}$ and $r^{\ell/s} = \alpha^{\ell/s} I_V$. Hence, we can choose α to be any ℓ/s'th root of unity. The same argument as in the real case shows that distinct choices of α define nonisomorphic elementary irreducible representations.

In the outer complex case, we can write $r = \alpha r_0$ where $\alpha \in \mathbb{C}$ so that $r^{\ell/s} = |\alpha|^{\ell/s} r_0^{\ell/s}$. In particular, if $r_0^{\ell/s} = -I_V$, then there are no elementary irreducible representations. If $r_0^{\ell/s} = I_V$, then we obtain an elementary irreducible representation for each choice of $\alpha \in \mathbb{C}$ with $|\alpha| = 1$. However, these are isomorphic representations. For example, the map $Av = (\beta v_0, \ldots, \beta v_{s-1})$ where $\beta^2 = \alpha_0$ gives an isomorphism between the cases $\alpha = 1$ and $\alpha = \alpha_0$.

In the quaternionic case, we can write $r = \alpha r_0$ where $\alpha \in \mathbb{H}$ so that $r^{\ell/s} = \pm\alpha^{\ell/s} I_d$. Choosing α so that $\pm\alpha^{\ell/s} = 1$ yields an elementary irreducible representation $\Omega(V)$. The representations are isomorphic if and only if the choices of α are conjugate in $\mathbb{H}$. Hence we may suppose that α lies in a fixed copy of $\mathbb{C} \subset \mathbb{H}$ (spanned by 1 and i, say). Moreover, conjugating by j corresponds to complex conjugation of α, so that α can be assumed to have the form $e^{i\theta}$, $0 \le \theta \le \pi$. ∎

Remark 7.10 In the real case with ℓ/s even, and in the outer complex case, a sufficient condition for the existence of elementary irreducible representations is that ℓ/k is even. By Lemma 7.7, $r_0^{k/s} = \pm I_V$ so that $r_0^{\ell/s} = (\pm 1)^{\ell/k} I_V$.

Classification of irreducible representations of G

In this subsection, we complete the enumeration of the irreducible representations of G in terms of the irreducible representations of H, thereby resolving issue (iii) above.

Theorem 7.11 *Suppose that G acts irreducibly on a vector space X. Then either*

(a) $X = \Omega(V)$ is an elementary irreducible representation corresponding to an H-irreducible representation V, or

(b) $X = \Omega(V) \oplus \Omega(V)$ where V is an H-irreducible representation of real or outer complex type.

Proof Consider the isotypic decomposition $X = \oplus_\alpha W_\alpha$ for the action of H on X. It is easy to verify that ρ permutes the H-isotopic components W_α. Hence, by irreducibility of the action of G, there is an H-isotypic component $W \subset X$ of order s such that $X = W \oplus \rho W \oplus \cdots \oplus \rho^{s-1} W$.

Now, define $G' = H \rtimes \mathbb{Z}_{\ell/s}$ to be the subgroup of G generated by H and ρ^s. Then G' acts on W. Moreover, if $U \subset V$ is a G'-invariant subspace, then $U \oplus \rho U \oplus \cdots \oplus \rho^{s-1} U$ is G-invariant. It follows that G' acts irreducibly on W.

Write $W = \oplus_1^n V$ where V is H-irreducible and define $R_0 = \oplus_1^n r_0$ where $r_0 \in T^s(V)^*$ is as in Proposition 7.5. Then $R_0 \in T^s(W)^*$ and has the same commutativity properties as does r_0. Moreover, $\rho^s = A R_0$ where $A \in \mathrm{Hom}_H(W) \cong \mathrm{M}_n(\mathcal{D})$. Indeed, $A = \{a_{ij} I_d\}$ where $a = \{a_{ij}\} \in \mathrm{M}_n(\mathcal{D})$. Writing $V \cong \mathcal{D}^d$, we have $\rho^s = \{a_{ij} r_0\}$.

Suppose that $S \subset \mathcal{D}^n$ is an invariant subspace for a, $S \ne \{0\}$. Define the H-invariant subspace $S \otimes V \subset W$ spanned by $(\beta_1 v, \ldots, \beta_n v)$ for $\beta = (\beta_1, \ldots, \beta_n) \in S$ and $v \in V$.

In the real, inner complex and quaternionic cases, it is easy to see that $S \otimes V$ is an invariant subspace for ρ^s. Indeed, in tensor product notation, we have $\rho^s = a \otimes r_0$ and we compute that $\rho^s(\beta \otimes v) = (a \otimes r_0)(\beta \otimes v) = a\beta \otimes r_0 v \subset S \otimes V$ whenever $\beta \in S$ and $v \in V$.

By irreducibility of the G' action, we have that $S \otimes V = W$ and hence that $S = \mathcal{D}^n$. In other words, a has no proper invariant subspaces in $\mathcal{D}^n$.

By algebraic completeness of $\mathbb{C}$, in the inner complex case, $a \in \mathrm{M}_n(\mathbb{C})$ has an eigenvector in $\mathbb{C}^n$ and hence there is a one-dimensional invariant subspace $S \subset \mathbb{C}^n$. But $S = \mathbb{C}^n$ and so $n = 1$. Hence, $W = V$ and $X = \Omega(V)$ is elementary. Similarly, in the quaternionic case, it is known that $a \in \mathrm{M}_n(\mathbb{H})$ has an eigenvector in $\mathbb{H}^n$ leading to the same conclusion as in the inner complex case. In the real case, $a \in \mathrm{M}_n(\mathbb{R})$ need not have an eigenvector in $\mathbb{R}^n$, but the complexification of a has an eigenvector in $\mathbb{C}^n$ yielding a one-dimensional or two-dimensional invariant subspace $S \subset \mathbb{R}^n$. We conclude that $n = 1$ or $n = 2$ so that X is either elementary or has the form $X = \Omega(V) \oplus \Omega(V)$.

Finally, in the outer complex case, we can write $(\rho^s)^2 = (a\bar{a}) \otimes r_0^2$. Arguing as in the inner complex case, we obtain a one-dimensional invariant subspace $S \subset \mathbb{C}^n$ for $a\bar{a}$ leading to a H-invariant, $(\rho^s)^2$-invariant subspace $S \otimes V \subset W$. The subspace $S \otimes V + \rho^s(S \otimes V)$ is G'-invariant of complex dimension at most $2 \dim V$. By irreducibility, this subspace coincides with W which has dimension $n \dim V$. Hence, $n = 1$ or $n = 2$. ∎

Proposition 7.12 *Suppose that X is a nonelementary G-irreducible representation. Then X is nonabsolutely irreducible.*

Proof Write $X = \Omega(V) \oplus \Omega(V) = \oplus_{j=1}^{s}(V \oplus V)$ where V is H-irreducible of real or outer complex type. The action of G on X can be written as

$$h(u_1, v_1, u_2, v_2, \dots, u_s, v_s) = (hu_1, hv_1, \phi(h)u_2, \phi(h)v_2, \dots, \phi^{s-1}(h)u_s, \phi^{s-1}(h)v_s)$$
$$\rho(u_1, v_1, u_2, v_2, \dots, u_s, v_s) = (u_2, v_2, u_3, v_3, \dots, u_s, v_s, A(u_1, v_1))$$

where $A \in T^s(V \oplus V)^*$ satisfies $A^{\ell/s} = I_{V \oplus V}$.

Now suppose that $B \in \mathrm{Hom}_G(X)$. Since the s copies of $V \oplus V$ carry distinct representations of H, we have that $B = B_1 \oplus \cdots \oplus B_s$ where $B_j \in \mathrm{Hom}_H(V \oplus V)$. The two copies of V in $V \oplus V$ are isomorphic irreducible representations of H, and so $B_j = b_j \otimes I_V$ where $b_j \in \mathrm{M}_2(\mathcal{D})$ (where $\mathcal{D} = \mathbb{R}$ or $\mathbb{C}$). In addition, we have that B commutes with ρ. It follows that $B_1 = \cdots = B_s$ and $B_1 A = A B_1$.

It remains to compute the structure of the centralizer of the matrix A. By irreducibility, we have that the centralizer is one of the division rings $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, so it is sufficient to show that the centralizer has real dimension at least two.

First suppose that V is H-absolutely irreducible. Then $A = a \otimes r_0$ where $a \in \mathrm{M}_2(\mathbb{R})$ and $r_0 \in T^s(V)^*$ is as in Proposition 7.5 and Lemma 7.7. Write $B_1 = b_1 \otimes I_V$ where $b_1 \in \mathrm{M}_2(\mathbb{R})$. Then B_1 commutes with A if and only if b_1 commutes with a and we have reduced the problem to computing the centralizer of the 2×2 real matrix a. Moreover, since A and r_0 have finite order, it follows that a has finite order and hence is semisimple. Transforming a into Jordan normal form, it is easy to see that the centralizer of a is isomorphic either to $\mathbb{R}^2$ (if a has real eigenvalues) or to $\mathbb{C}$ (if a has complex conjugate eigenvalues). In particular, the centralizer is not isomorphic to $\mathbb{R}$ or $\mathbb{H}$ and so X is G-irreducible of complex type. (If the centralizer of a is isomorphic to $\mathbb{R}^2$, then X splits into two elementary irreducible representations, cf. the proof of Theorem 7.11.)

Next suppose that V is H-irreducible of outer complex type. We continue to write $A = a \otimes r_0$, $B_1 = b_1 \otimes I_V$, where $a, b_1 \in \mathrm{M}_2(\mathbb{C})$, with the cautionary note that products of these tensor products must be handled with care since r_0 does not commute with complex scalars. (More precisely, these are tensor products of semilinear maps, see [15].) Commutativity of A and B_1 leads to the equation $b_1 a = a\overline{b_1}$. Again, it is sufficient to show that the space of matrices b_1 that satisfy this condition is at least two-dimensional (real). The required (somewhat tedious) calculation is given in Lemma 7.13 below. ∎

Lemma 7.13 *Let $A \in M^2(\mathbb{C})$ be nonsingular. Define $Y(A) = \{B \in M^2(\mathbb{C}) : BA = A\overline{B}\}$. Then $\dim_{\mathbb{R}} Y(A) \geq 2$.*

Proof Let $P \in M^2(\mathbb{C})$ be nonsingular, and set $A' = P^{-1}A\overline{P}$, $B' = P^{-1}BP$. If $BA = A\overline{B}$, then we compute that

$$B'A' = P^{-1}BA\overline{P} = P^{-1}A\overline{BP} = A'\overline{B'}.$$

It follows that $Y(P^{-1}A\overline{P}) \cong Y(A)$ for all P.

We show how to choose P so that $P^{-1}A\overline{P}$ has a form convenient for calculations. Indeed, we claim that A can be transformed so that $a_{11} = 0$, $a_{12} = 1$, $a_{22} \in \mathbb{R}$.

First, we transform so that $a_{11} = 0$. Suppose that $a_{11} \neq 0$. If $a_{21} \neq 0$, take $P = \begin{pmatrix} 1 & 1 \\ 0 & a_{21}/a_{11} \end{pmatrix}$. A computation shows that $P^{-1}A\overline{P}$ has the required form. The case $a_{12} \neq 0$ is similar. This leaves the case $a_{11}, a_{22} \neq 0$, $a_{12} = a_{21} = 0$. If $a_{11} \neq a_{22}$, we can choose $P = \begin{pmatrix} 1 & 1 \\ 1 & a_{22}/a_{11} \end{pmatrix}$. Finally, if $a_{11} = a_{22}$, then take $P = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$.

After this transformation, the new matrix A has $a_{11} = 0$ and $a_{12} \neq 0$. Let $P = \begin{pmatrix} a_{12} & 0 \\ 0 & 1 \end{pmatrix}$. We now have $a_{11} = 0$ and $a_{12} = 1$.

If $a_{22} = 0$ then we are finished. Otherwise, choose s so that $s^2 = a_{22}$ and take $P = \begin{pmatrix} \overline{s} & 0 \\ 0 & s \end{pmatrix}$. Then $P^{-1}A\overline{P} = \begin{pmatrix} 0 & 1 \\ a_{21} & a_{22}\overline{s}/s \end{pmatrix}$. Compute that $a_{22}\overline{s}/s = a_{22}\overline{s}s/s^2 = \overline{s}s \in \mathbb{R}$ as required.

This calculation verifies that A can be transformed so as to satisfy $a_{11} = 0$, $a_{12} = 1$ and $a_{22} \in \mathbb{R}$ (and $a_{21} \neq 0$ since A is invertible). It remains to compute the dimension of $Y(A)$ for such an A.

First, we suppose that $a_{21} \notin \mathbb{R}$. Set $c = -a_{22}/\operatorname{Im} a_{21}$. A calculation shows that $B \in Y(A)$ has the form

$$B = \begin{pmatrix} \alpha & c \operatorname{Im} \alpha \\ c\overline{a}_{21} \operatorname{Im} \alpha & \overline{\alpha} + ca_{22} \operatorname{Im} \alpha \end{pmatrix},$$

where $\alpha \in \mathbb{C}$ is arbitrary and hence $\dim_{\mathbb{R}} Y(A) = 2$.

Next, we suppose that $a_{21} \in \mathbb{R}$. If $a_{22} = 0$, then

$$B = \begin{pmatrix} \alpha & \beta \\ a_{21}\overline{\beta} & \overline{\alpha} \end{pmatrix},$$

where α, $\beta \in \mathbb{C}$ are arbitrary so that $\dim_{\mathbb{R}} Y(A) = 4$. If $a_{22} \neq 0$, then

$$B = \begin{pmatrix} \alpha & \beta \\ \beta a_{21} & \alpha + \beta a_{22} \end{pmatrix},$$

where $\alpha, \beta \in \mathbb{R}$ are arbitrary so that $\dim_{\mathbb{R}} Y(A) = 2$.

In all cases, we have shown that the space of commuting linear maps has real dimension two or four as required. ∎

Remark 7.14 Suppose that $X = \Omega(V) \oplus \Omega(V)$, where $\Omega(V)$ is an elementary irreducible representation of G. When V is H-irreducible of real type, it follows from the proof of Proposition 7.12 that X is either reducible or irreducible of complex type. Moreover irreducibility occurs precisely when the eigenvalues of the matrix a in the proof of Proposition 7.12 are complex.

The analogous results when V is H-irreducible of outer complex type are more complicated. However, observe that X is G-irreducible if and only if $Y(A)$ is a division ring, that is, if every nonzero element of $Y(A)$ has an inverse. We deduce from the proof of Lemma 7.13 that

(i) If $a_{22}^4 - 4(\operatorname{Im} a_{21})^2 + 4a_{22}^2 \operatorname{Re} a_{21} < 0$, then X is irreducible of complex type.

(ii) If $a_{22} = 0$ and $a_{21} < 0$, then X is irreducible of quaternionic type.

(iii) Otherwise, X is reducible.

The conclusions of Corollary 7.6, Theorem 7.11, Proposition 7.12 and Remark 7.14 are summarized in Table 3 below.

V	$\Omega(V)$	$\Omega(V) \oplus \Omega(V)$
real	$\mathbb{R}$	$\mathbb{C}$
outer complex	$\mathbb{R}$	$\mathbb{C}$, $\mathbb{H}$
inner complex	$\mathbb{C}$	
quaternionic	$\mathbb{C}$, $\mathbb{H}$	

Table 3: The types of elementary and nonelementary irreducible representations corresponding to an H-irreducible representation V.

8 Examples of nonHopf bifurcation — revisited

In this section, we show how to obtain complete results in the examples of nonHopf bifurcation considered in Subsection 5(b). The main stumbling block was the computation of the absolutely irreducible representations of Δ_L. This computation can be carried out using the results of Section 7.

Suppose that we have a discrete rotating wave with spatial symmetry Δ and automorphism $\phi \in \mathrm{Aut}(\Delta)$ of order k. We show how to compute the absolutely irreducible representations of the semidirect product Δ_L. As pointed out in Remark 4.3, we may suppose without loss that $\Delta_L = \Delta \rtimes \mathbb{Z}_{2k}$. We now apply the methods of Section 7 with $H = \Delta$ and $\ell = 2k$.

Theorem 8.1 *The absolutely irreducible representations of $\Delta \rtimes \mathbb{Z}_{2k}$ are given by the elementary irreducible representations $\Omega(V)$ corresponding to Δ-irreducible representations V of real or outer complex type. Moreover, (up to isomorphism) there are precisely two absolutely irreducible representations $\Omega(V)$ for each V of real type, and precisely one absolutely irreducible representation $\Omega(V)$ for each V of outer complex type.*

Proof We apply the results of Section 7 with $H = \Delta$ and $\ell = 2k$. By Theorem 7.11 and Proposition 7.12, it suffices to compute absolutely irreducible elementary representations. These arise, by Corollary 7.6, from absolutely irreducible and outer complex representations of Δ. (The proof so far can be read off from Table 3.) By Remark 7.10 there exist absolutely irreducible actions of $\Delta \rtimes \mathbb{Z}_{2k}$ on $\Omega(V)$. By Corollary 7.9 there are precisely two such actions up to isomorphism when V is absolutely irreducible and a unique action when V is outer complex. ∎

Remark 8.2 The analogous result for Hopf bifurcation can be read off from Table 3. By the results in Section 6, we are interested in 'simple' representations of $\Delta_{L_0} = \Delta \rtimes$

$\mathbb{Z}_k$: those representations that either are the direct sum of two isomorphic irreducible representations or are nonabsolutely irreducible. By Table 3, such representations are given by $\Omega(V) \oplus \Omega(V)$ where V is Δ-irreducible of real or outer complex type and by $\Omega(V)$ where V is Δ-irreducible of inner complex or quaternionic type.

Example 8.3 (Example 5.4 revisited.) We consider a discrete rotating wave with spatial symmetry $\Delta = \mathbb{D}_n(R_{2\pi/n}, \kappa)$ and spatiotemporal symmetry $\Sigma = \mathbb{D}_{2n}(R_{\pi/n}, \kappa)$.

The case n odd was studied in its entirety in Example 5.1. When n is even, the best we can do is $k = 2$ taking $\sigma = R_{\pi/n}\kappa$. This situation was studied partially in Example 5.4. We now complete the analysis of the case n even.

We note that Remark 4.6 does not apply and period-doubling does occur, though to a somewhat limited extent.

By Remark 4.3, nonHopf bifurcation is governed by the absolutely irreducible representations of the group $\Delta_L = \mathbb{D}_n \rtimes \mathbb{Z}_4$. We emphasize that L (acting on the representation space) is not required to have minimal order 4. Indeed, $L^2 = \pm I$ and it may be the case that $L = I$. Using the methods of Section 7 we obtain the following result. (The rationale behind the notation V_* is identical to that employed in Example 5.1.)

Proposition 8.4 *The group $\Delta_L = \mathbb{D}_n \rtimes \mathbb{Z}_4$ has $n+4$ absolutely irreducible representations:*

(a) 4 one-dimensional absolutely irreducible representations given by

$$\begin{array}{llll} V_{0,+}: & R_{2\pi/n} = 1, & \kappa = 1, & L = \pm 1. \\ V_{0,-}: & R_{2\pi/n} = 1, & \kappa = -1, & L = \pm 1. \end{array}$$

(b) $n-2$ two-dimensional absolutely irreducible representations given by

$$V_j: \quad R_{2\pi/n} = \begin{pmatrix} \cos 2\pi j/n & -\sin 2\pi j/n \\ \sin 2\pi j/n & \cos 2\pi j/n \end{pmatrix}, \quad \kappa = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$L = \pm \begin{pmatrix} \cos \pi/n & \sin \pi/n \\ \sin \pi/n & -\cos \pi/n \end{pmatrix}, \quad 1 \le j < n/2.$$

(c) 2 further two-dimensional absolutely irreducible representations given by

$$V_{n/2,+} \oplus V_{n/2,-}: \quad R_{2\pi/n} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \kappa = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 1 \\ \pm 1 & 0 \end{pmatrix}.$$

Proof We use the methods of Section 7 to compute the absolutely irreducible representations of the abstract group $\mathbb{D}_n \rtimes \mathbb{Z}_4$.

The group $\mathbb{D}_n$ has $n/2+3$ irreducible representations, each of which is absolutely irreducible. Four of the representations are one-dimensional and the remaining representations are two-dimensional. The generators $R_{2\pi/n}$, κ act in these representations as follows:

	$V_{0,+}$	$V_{0,-}$	$V_j,\ 1\le j<n/2$	$V_{n/2,+}$	$V_{n/2,-}$
$R_{2\pi/n}$	1	1	$\begin{pmatrix}\cos 2\pi j/n & -\sin 2\pi j/n\\ \sin 2\pi j/n & \cos 2\pi j/n\end{pmatrix}$	-1	-1
κ	1	-1	$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$	1	-1

The automorphism $\phi\in\mathrm{Aut}(\Delta)$ defined by conjugation by σ interchanges R_θ and $R_{-\theta}$. It follows that ϕ preserves the representations V_j, $1\le j<n/2$. Clearly, ϕ preserves the trivial representation $V_{0,+}$ and therefore preserves also the representation $V_{0,-}$. There remains the representations $V_{n/2,+}$ and $V_{n/2,-}$. Observe that κ acts trivially in $V_{n/2,+}$ whereas $\phi(\kappa)=R_{2\pi/n}\kappa$ acts nontrivially. It follows that ϕ interchanges the representations $V_{n/2,+}$ and $V_{n/2,-}$.

In particular, $V_{n/2,+}$ and $V_{n/2,-}$ have order $s=2$, whereas the remaining representations have order $s=1$. (In the notation of Section 7, we have the irreducible representations $\chi_*:\mathbb{D}_n\to V_*$, and these satisfy $\chi^{(1)}_{0,\pm}=\chi_{0,\pm}$, $\chi^{(1)}_j=\chi_j$, $\chi^{(1)}_{n/2,\pm}=\chi_{n/2,\mp}$.)

It follows from Theorem 8.1 that the absolutely irreducible representations of $\mathbb{D}_n\rtimes\mathbb{Z}_4$, are the elementary irreducible representations $\Omega(V_*)$ corresponding to each irreducible representation V_*. Moreover, by Theorem 8.1, there are precisely two nonisomorphic absolutely irreducible representations of $\mathbb{D}_n\rtimes\mathbb{Z}_4$ for each of the spaces $\Omega(V_*)$.

First, we consider the one-dimensional $\mathbb{D}_n$-irreducible representations of order $s=1$, namely $V_{0,\pm}$. It is immediate that $L=\pm1$ and hence we obtain the representations listed in (a).

Next, we consider the two-dimensional $\mathbb{D}_n$-irreducible representations, namely V_j. These have order $s=1$ and the action of $\mathbb{D}_n$ is given. Moreover, the action of $\mathbb{D}_n$ extends to an action of $\mathbb{D}_{2n}$ where $\sigma=R_{\pi/n}\kappa$ acts as $\begin{pmatrix}\cos\pi/n & \sin\pi/n\\ \sin\pi/n & -\cos\pi/n\end{pmatrix}$. In particular, this choice of linear map defines the correct automorphism and so L (which is defined uniquely up to $\pm I$) is as given.

Finally, we consider the one-dimensional $\mathbb{D}_n$-irreducible representations of order $s=2$, namely $V_{n/2,\pm}$. The corresponding elementary representations are defined on $V_{n/2,+}\oplus V_{n/2,-}$. Moreover, the action of $\mathbb{D}_n$ follows from the construction, so it remains to deduce the action of L. By Proposition 7.1, coordinates can be chosen so that $L = \begin{pmatrix} 0 & 1 \\ r & 0 \end{pmatrix}$ where $r \in T^2(V_2)^*$. Since L has eigenvalues on the unit circle, we have the choices $r = \pm 1$. ∎

The branches of periodic solutions corresponding to the absolutely irreducible representations of Δ_L are summarized in Table 4. The verification of these results proceeds in the same way as the arguments in Example 5.4. Note that Example 5.4 corresponds to the representation V_1.

Space	L	Δ_x	Σ_P	Remarks
$V_{0,+}$	1	$\mathbb{D}_n$	$\langle \kappa, (R_{\pi/n}, 1/2)\rangle \cong \mathbb{D}_{2n}$	saddle-node
$V_{0,+}$	-1	$\mathbb{D}_n$	$\mathbb{D}_n$	
$V_{0,-}$	1	$\mathbb{Z}_n(R_{2\pi/n})$	$\langle R_{2\pi/n}, (R_{\pi/n}\kappa, 1/2)\rangle \cong \mathbb{D}_n$	
$V_{0,-}$	-1	$\mathbb{Z}_n(R_{2\pi/n})$	$\langle (R_{\pi/n}, 1/2)\rangle \cong \mathbb{Z}_{2n}$	
V_j		$\mathbb{D}_{j'}(R_{2\pi/j'}, \kappa)$	$\mathbb{D}_{j'}$	
$1 \le j < n/2$		$\mathbb{Z}_{j'}(R_{2\pi/j'})$	$\langle R_{2\pi/j'}, (R_{\pi/n}\kappa, 1/2)\rangle \cong \mathbb{D}_{j'}$	
$V_{n/2+} \oplus V_{n/2,-}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\mathbb{D}_{n/2}(R_{4\pi/n}, \kappa)$	$\mathbb{D}_{n/2}$	
		$\mathbb{Z}_{n/2}(R_{4\pi/n})$	$\langle R_{4\pi/n}, (R_{\pi/n}\kappa, 1/2)\rangle \cong \mathbb{D}_{n/2}$	
$V_{n/2+} \oplus V_{n/2,-}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\mathbb{D}_{n/2}(R_{4\pi/n}, \kappa)$	$\langle \kappa, (R_{2\pi/n}, 1/2)\rangle \cong \mathbb{D}_n$	doubling
		$\mathbb{Z}_{n/2}(R_{4\pi/n})$	$\langle (R_{-\pi/n}, 1/4)\rangle \cong \mathbb{Z}_{2n}$	doubling

Table 4: Spatiotemporal symmetry of bifurcating solutions in nonHopf bifurcation from a discrete rotating wave with $(\mathbb{D}_{2n}, \mathbb{D}_n)$ symmetry, n even. All bifurcations are pitchforks and period-preserving unless stated otherwise. Notation: $j' = \gcd(j, n)$.

Example 8.5 (Example 5.5 revisited.) We consider a discrete rotating wave with spatial symmetry $\Delta = \mathbb{Z}_n(R_{2\pi/n})$ and spatiotemporal symmetry $\Sigma = \mathbb{D}_n(R_{2\pi/n}, \kappa)$.

When n is even, the group $\mathbb{Z}_n$ has $n/2 + 1$ irreducible representations, two of which are one-dimensional of real type, while the remaining $n/2 - 1$ representations are two-dimensional of complex type. When n is odd, there are $(n + 1)/2$ irreducible representations, and all of the nontrivial representations are two-dimensional of complex type. To summarize we have

	V_0	$V_j,\ 1 \leq j < n/2$	$V_{n/2}$ (n even)
$R_{2\pi/n}$	1	$\begin{pmatrix} \cos 2\pi j/n & -\sin 2\pi j/n \\ \sin 2\pi j/n & \cos 2\pi j/n \end{pmatrix}$	-1

As shown in Example 7.4, the two-dimensional irreducible representations V_j are of outer complex type. Since $s = 1$, we have $L = r_0$ and L can be taken to be the reflection $L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

It follows from Theorem 8.1 that $\Delta_L = \mathbb{Z}_n \rtimes \mathbb{Z}_4$ has 2 one-dimensional absolutely irreducible representations corresponding to each of V_0 and $V_{n/2}$ (taking $L = \pm 1$) and 1 two-dimensional absolutely irreducible representation corresponding to each V_j, $1 \leq j < n/2$ (choosing L to be a reflection as above).

In particular, the possibility that L has order 4 is never realized. Thus we have suppression of period-doubling even though Remark 4.6 does not apply. We note that [17] shows that period doubling is suppressed for this example, but under a simplifying hypothesis on the action of a certain group.

The branches of periodic solutions are summarized in Table 5. The verification of these results proceeds in the same way as the arguments in Example 5.5. Note that Example 5.5 corresponds to the representation V_1.

Space	L	Δ_x	Σ_P	Remarks
V_0	1	$\mathbb{Z}_n$	$\langle R_{2\pi/n}, (\kappa, 1/2)\rangle \cong \mathbb{D}_n$	saddle-node
V_0	-1	$\mathbb{Z}_n$	$\mathbb{Z}_n$	
V_j $1 \leq j < n/2$		$\mathbb{Z}_{j'}(R_{2\pi/j'})$	$\langle R_{2\pi/j'}, (\kappa, 1/2)\rangle \cong \mathbb{D}_{j'}$	n/j' odd transcritical if $n/j' = 3$
V_j $1 \leq j < n/2$		$\mathbb{Z}_{j'}(R_{2\pi/j'})$ $\mathbb{Z}_{j'}(R_{2\pi/j'})$	$\langle R_{2\pi/j'}, (\kappa, 1/2)\rangle \cong \mathbb{D}_{j'}$ $\langle R_{2\pi/j'}, (\kappa R_{2\pi/n}, 1/2)\rangle \cong \mathbb{D}_{j'}$	n/j' even
$V_{n/2}$	1	$\mathbb{Z}_{n/2}(R_{4\pi/n})$	$\langle R_{4\pi/n}, (\kappa, 1/2)\rangle \cong \mathbb{D}_{n/2}$	n even
$V_{n/2}$	-1	$\mathbb{Z}_{n/2}(R_{4\pi/n})$	$\langle R_{4\pi/n}, (\kappa R_{2\pi/n}, 1/2)\rangle \cong \mathbb{D}_{n/2}$	n even

Table 5: Spatiotemporal symmetry of bifurcating solutions in nonHopf bifurcation from a discrete rotating wave with $(\mathbb{D}_n, \mathbb{Z}_n)$ symmetry. All bifurcations are period-preserving. All bifurcations are pitchforks unless stated otherwise. Notation: $j' = \gcd(j, n)$.

Acknowledgments It is a great pleasure to thank Marty Golubitsky and Mark Roberts for valuable discussions.

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