Local bifurcations in k-symmetric dynamical systems

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Abstract

A map $U : \mathbb{R}^d \mapsto \mathbb{R}^d$ is called a (reversing) k-symmetry of the dynamical system represented by the map $L : \mathbb{R}^d \mapsto \mathbb{R}^d$ if k is the smallest positive integer for which U is a (reversing) symmetry of the kth iterate of L. We study generic local bifurcations of fixed points that are invariant under the action of a compact Lie group Σ that is a reversing k-symmetry group of the map L, on the basis of a normal form approach.

We derive normal forms relating the local bifurcations of k-symmetric maps to local steady-state bifurcations of symmetric flows of vector fields. Alternatively, we also discuss the derivation of normal forms entirely within the framework of Taylor expansions of maps.

We illustrate our results with some examples.

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Contents

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1 Introduction

In dynamical systems, symmetry properties have received much attention in recent years as they occur in many applications and may cause anomalous generic dynamical behaviour. In [27] the conventional notions of symmetries and reversing symmetries (i.e. symmetries involving a time reversal) have been extended to k-symmetries and reversing k-symmetries. This generalization takes into account the fact that a map L may possess fewer symmetries and reversing symmetries than its kth iterate L^k.

k-Symmetry has been observed to occur in various dynamical systems of physical interest such as web maps modelling the dynamics of a kicked charged particle in a magnetic field [17, 22, 42], a classical and quantum kicked top [16], and trace maps [33, 41]. In particular, general mechanisms have pointed out the occurrence and relevance of k-symmetry in resonantly driven dynamical systems [24, 25].

Because of the extensive work in recent years on symmetric dynamical systems, cf. e.g. [7, 13, 14, 35, 39], the introduction of the concept of k-symmetry has given rise to the question what the differences and similarities are between phenomena that occur in symmetric and k-symmetric dynamical systems. In this paper this question will be addressed in the context of local bifurcations of fixed points that are invariant under the action of a reversing k-symmetry group that is a compact Lie group. Before formulating our objectives and results more precisely, we will first give a short introduction into the concept of k-symmetry.

1.1 k-Symmetry in dynamical systems

We consider invertible dynamical systems with discrete time of which the evolution in time is given by a smooth map L from a manifold Ω to itself, $L : \Omega \mapsto \Omega$. An invertible map $M : \Omega \mapsto \Omega$ that commutes with L, [1]

$$M \circ L = L \circ M, \tag{1.1}$$

is called a *symmetry* of L. This property is also known as *equivariance*. The set $\mathcal{G}$ containing all symmetries of a dynamical system forms a group under composition: the *symmetry group* of the dynamical system.

Apart from the notion of symmetries (or equivariance) there exists a separate notion of reversibility. An invertible map $S : \Omega \mapsto \Omega$ is called a *reversing symmetry* of L if[2]

$$S \circ L^{-1} = L \circ S. \tag{1.2}$$

Unlike the set of symmetries $\mathcal{G}$, the set of reversing symmetries $\mathcal{R}$ does not form a group under composition. However, the composition of two reversing symmetries yields a symmetry and the composition of a symmetry and a reversing symmetry yields a reversing symmetry. This is contained in the notion of the *reversing symmetry group* [21], that we will denote $\mathcal{E}$ and consists of the union of the symmetry group $\mathcal{G}$ and the set of reversing symmetries $\mathcal{R}$,

$$\mathcal{E} := \mathcal{G} \cup \mathcal{R}. \tag{1.3}$$

Let $\mathcal{G}_k$ denote the symmetry group of the kth iterate of L, i.e. L^k. Then $\mathcal{G} = \mathcal{G}_1$ is a subgroup of $\mathcal{G}_k$ for all $k \geq 1$. In case $\mathcal{G}_k \neq \mathcal{G}_1$ there exist symmetries of L^k that are not

[1] $\circ$ denotes composition.

[2] It should be noted that we do not require any of the reversing symmetries S to be involutions, i.e. to satisfy $S^2 = \mathrm{Id}$.

symmetries of L. In order to describe this, in [27] the notion of k-symmetry was introduced: a map $M \in \mathcal{G}_k$ is called a *k-symmetry* of L if k is the smallest positive integer for which

$$M \circ L^k = L^k \circ M. \tag{1.4}$$

In analogy to the above notation, let $\mathcal{E}_k$ denote the reversing symmetry group of L^k consisting of $\mathcal{G}_k$ and a reversing set $\mathcal{R}_k$. A map $S \in \mathcal{R}_k$ is called a *reversing k-symmetry* of L if k is the smallest positive integer such that

$$S \circ L^k = L^{-k} \circ S. \tag{1.5}$$

If a map L possesses (reversing) k-symmetries, the dynamics induces a nontrivial permutation of the elements of $\mathcal{E}_k$. This action is represented by an operator ϕ_L which is defined as follows: if $M \in \mathcal{G}_k$, then

$$\phi_L(M) := L \circ M \circ L^{-1}, \tag{1.6}$$

and if $S \in \mathcal{R}_k$ then

$$\phi_L(S) := L \circ S \circ L. \tag{1.7}$$

With this definition of ϕ_L on $\mathcal{E}_k$ we have,

$$\phi_L(U) \circ L^{\varepsilon_U} = L \circ U, \tag{1.8}$$

where $\varepsilon_U = +1$ if $U \in \mathcal{G}_k$ and $\varepsilon_U = -1$ if $U \in \mathcal{R}_k$.

The operator ϕ_L generates orbits in $\mathcal{E}_k$. These ϕ_L-orbits contain at most k elements and provide an elegant approach towards k-symmetry: k-symmetries and reversing k-symmetries correspond to elements of ϕ_L-orbits of length k.

From the above setting, we will now give a more precise definition of a *reversing k-symmetry group*.

Definition 1.1 *A group Σ is called a (reversing) k-symmetry group of a map L if and only if k is the smallest positive integer such that $\phi_L^k(U) = U$ for all $U \in \Sigma$, <u>and</u> $\phi_L(\Sigma) = \Sigma$.*

Importantly, we require reversing k-symmetry groups to be invariant under ϕ_L. This ensures that ϕ_L-orbits of elements of Σ remain within Σ. In this paper we will study local bifurcations of smooth dynamical systems posessing a reversing k-symmetry group Σ that is a compact Lie group.

Remark 1.1 The requirement that reversing k-symmetry groups Σ must be invariant under ϕ_L is motivated by the fact that one seeks a setting in which all (unavoidable) nontrivial symmetry properties of L are taken into account. Namely, if Σ is not invariant under ϕ_L, then one can find nontrivial (reversing) symmetries of L that are not contained in Σ by applying ϕ_L to elements of Σ. In this way one can always extend Σ to a larger group Σ' that is invariant under ϕ_L (in [27] the resulting group Σ' was called the ϕ_L-orbit group of Σ). The extension one obtains in this way is nontrivial, in the sense that the additional symmetries one finds are not just powers (or roots) of the map L.

In the known mechanisms that explain the occurrence of k-symmetry in driven systems [24, 25], the ϕ_L-invariance of reversing k-symmetry groups arises naturally.

1.2 Objectives and outline of the result

The objective of this paper is to provide a framework for studying local bifurcations occurring in the context of k-symmetric dynamical systems. In the presence of a certain structure (such as symmetry or volume preservation), generic bifurcations are often different from the generic bifurcations occurring in the absense of additional structures.

Around an equilibrium point of a (sufficiently smooth) map the local dynamics may be studied on the basis of the Taylor expansion of the map at that point. The object of this approach is to simplify the Taylor expansion by successive coordinate transformations, to truncate the resulting polynomial expression at a certain order, and to obtain in this way a simple and 'representative' description of the local dynamics (or at least of that aspect of the local dynamics in which one is interested). This approach has been followed by Poincaré, Birkhoff and many others, see e.g. [1, 8, 10, 15, 37].

We will focus on equilibrium points that are fixed under the action of a compact Lie group Σ that is a (reversing) k-symmetry group of the dynamical system, i.e. Σ is an *isotropy subgroup* of an equilibrium point. We focus on isotropy subgroups Σ that are compact Lie groups and hence we may assume without loss of generality that the local $\mathbb{R}^d$ coordinates on the manifold Ω in which the Taylor expansion of the map is made are such that the action of the isotropy subgroup Σ is a subgroup of $O(d)$ (the orthogonal group in d dimensions) acting on $\mathbb{R}^d$ as a linear (orthogonal) representation [3].

In order to analyse local bifurcations in this context, our main tool will be the derivation of k-symmetric normal forms, and their symmetry properties (that mainly characterize them).

The main result of this paper is summarized as follows. Consider a map $L : \mathbb{R}^d \mapsto \mathbb{R}^d$ with a reversing k-symmetry group Σ that is an isotropy subgroup of a fixed point $\mathbf{0}$. Let A_s be the semisimple part of the linear part of L,[3] and let Γ denote the group generated by A_s and Σ. Then L can be written in *normal form* upto arbitrarily high order as $L = A_s \circ \mathcal{X}_\tau$, where $\mathcal{X}_\tau$ is the time-one map of the flow of a vector field with reversing symmetry group Γ (in which A_s acts as a symmetry). From this it follows that generic local bifurcations of k-symmetric maps are closely related to generic local bifurcations of symmetric vector fields, the dynamics of the map L being approximated by alternatingly applying A_s and the time-one map $\mathcal{X}_\tau$. A more technical and precise formulation of these results are presented in Theorem 1.1 and section 2 below.

One of the simplest nontrivial examples in which this result applies arises when $\Sigma = \mathbf{D}_2$ (acting on $\mathbb{R}^d$) is a 2-symmetry group of L. In this case $\Gamma = \mathbf{D}_4$, and $\mathcal{X}_\tau$ is $\mathbf{D}_4$-equivariant, although L is only $\mathbb{Z}_2$-equivariant and L^2 only $\mathbf{D}_2$-equivariant. Generic local bifurcations of L are now driven by local *steady-state* bifurcations of $\mathbf{D}_4$-equivariant flows of vector fields. Local bifurcation theory for equivariant vector fields can be found in [14], from which many aspects of the local bifurcations of L can be deduced almost immediately. This example will be discussed in more detail in section 3.1.

In case Σ contains not only k-symmetries but also reversing k-symmetries, the procedure is entirely analogous, although for local bifurcations of vector fields with reversing symmetry groups a general theory comparable to the equivariant one presented in [14] is not yet available . Recently, however, local bifurcations of vector fields with reversing symmetry groups were studied in [19, 26] and a more general treatment along the lines of [14] is in

[3] A linear operator is semi-simple whenever it is $\mathbb{C}$-diagonalizable. A linear operator N is nilpotent if there exists an integer n such that N^n is zero.

preparation [20].

In the derivation of k-symmetric normal forms, we follow ideas of Broer [4, 5], and use successive Σ-equivariant coordinate transformations which preserve the local symmetry properties consisting of the isotropy subgroup Σ and its ϕ_L-orbit structure.

The main result of this paper is the following k-symmetric normal form theorem. Its derivation follows an approach originally due to Takens [38] and will be discussed in section 2.1.

Theorem 1.1 *Consider a map $L : I\!R^d \mapsto I\!R^d$, with fixed point $\mathbf{0}$, possessing a reversing k-symmetry group $\Sigma \leq O(d)$ that is an isotropy subgroup of $\mathbf{0}$. Let $DL|_{\mathbf{0}}$ denote the (Frechet) derivative of L at $\mathbf{0}$ and $DL|_{\mathbf{0}} = A_s + A_n$ be the canonical splitting in its semi-simple and nilpotent part.[3] Then, in case L is at least C^m, there exists a Σ-equivariant polynomial coordinate transformation T of degree m such that*

$$T^{-1} \circ L \circ T \quad \text{and} \quad A_s \circ \mathcal{X}_\tau$$

have the same Taylor expansion up to order m at $\mathbf{0}$, where $\mathcal{X}_\tau$ is the time-one map of the flow of a vector field $\mathcal{X}$ that commutes with A_s and has Σ as a reversing symmetry group.

In section 2.1, it will furthermore be shown that the vector field $\mathcal{X}$ in theorem 1.1 can be considered to be in normal form at the same time. Alternatively, in section 2.2 we will discuss the derivation of k-symmetric normal forms by a direct approach that does not use time-one maps of vector fields but is entirely formulated within the framework of Taylor-expansions of maps.

In relation to our objective to use normal forms to study local bifurcations, it is important to note that theorem 1.1 has a natural extension to the inclusion of unfoldings, in the sense that unfoldings may be considered on the level of the vector field $\mathcal{X}$ in such a way that its symmetry properties are preserved, cf. also [6, 38]. This implies that local bifurcations of k-symmetric maps may in fact be studied via corresponding local bifurcations of symmetric vector fields. Note in this respect that in the above theorem the reversing k-symmetry group Σ of the map L has become a reversing symmetry group of the corresponding vector field $\mathcal{X}$, which moreover possesses an additional *formal* symmetry group

$$\mathcal{S} := \overline{\{(A_s)^n \mid n \in \mathbb{Z}\}}. \tag{1.9}$$

Theorem 1.1 illustrates the transparent way in which the symmetric theory generalizes to the k-symmetric one. Differences between the symmetric and k-symmetric case become clear when one takes a closer look at the group structure of the group Γ that is generated by the isotropy subgroup Σ and the formal symmetry group $\mathcal{S}$. The structure of this group depends on the fact whether $k = 1$ or $k \geq 2$ and on whether Σ contains reversing elements or not: $\mathcal{S}$ is a normal subgroup of Γ if and only if the isotropy subgroup Σ does not contain (reversing) k-symmetries (with $k > 1$), and Σ is a normal subgroup of Γ if and only if Σ is a symmetry group. Interestingly, hence, in the case that Σ contains reversing k-symmetries (with $k > 1$), neither $\tilde{\mathcal{S}}$ nor Σ is a normal subgroup of Γ. The related phenomenon of the occurrence of not simultaneously linearizable reversing k-symmetry groups will be discussed in section 4.

The rest of this paper is outlined as follows. In section 2 we will discuss the structure preserving normal form procedure, yielding theorem 1.1 and other k-symmetric normal form results. It will turn out that the mathematical framework for k-symmetry, as introduced in

[27], permits a transparent generalization of the normal form results that apply in the case of symmetric systems (i.e. in the case $k=1$). In section 3, we will illustrate the application of our results to the derivation of k-symmetric normal forms and the study of local bifurcations in the case of maps with 2-symmetry group $\mathbb{Z}_2 \times \mathbb{Z}_2$ ($\mathbf{D}_2$) and reversing 2-symmetry group $\mathbb{Z}_2 \times \mathbb{Z}_2'$ ($\mathbf{D}_2$ with one reversing generator). The paper is concluded with a discussion.

2 k-Symmetric normal forms at equilibrium points

We consider an equilibrium point $\mathbf{0}$ of a map L with an isotropy subgroup Σ that is a (reversing) k-symmetry group of L, with the operator ϕ_L permuting the elements of Σ. The aim is to derive normal forms while preserving the local symmetry properties, consisting here of the isotropy subgroup Σ and its ϕ_L-orbit structure.

We will confine the treatment to the case that the isotropy subgroup Σ is a compact Lie group and in this context we will assume without loss of generality that in local $\mathbb{R}^d$ coordinates the action of Σ is linear (and orthogonal) [3]. We furthermore define $\mathcal{H}_n$ to be the vector space of homogeneous (d-dimensional) vector polynomials over $\mathbb{R}^d$ of degree n, i.e. $P_n \in \mathcal{H}_n$ is a polynomial map $\mathbb{R}^d \mapsto \mathbb{R}^d$ with $P_n(a\mathbf{x}) = a^n P_n(\mathbf{x})$ for all $a \in \mathbb{R}$. Then, the nth order Taylor expansion of L, that is denoted $L^{(n)} := \sum_{i=1}^n L_i$ with $L_i \in \mathcal{H}_i$, satisfies

$$\phi_L(U) \circ [L^{(n)}]^{\varepsilon_U} \overset{(n)}{=} L^{(n)} \circ U, \quad \forall\, U \in \Sigma, \tag{2.1}$$

where $\varepsilon_U = +1$ if U is a k-symmetry, and $\varepsilon_U = -1$ if U is a reversing k-symmetry (for some k) and $\overset{(n)}{=}$ denotes equality up to order n.

Let us now consider a coordinate transformation $T_n = \mathrm{Id} + P_n$, with $P_n \in \mathcal{H}_n$. With such a transformation we may transform L into $\tilde{L} := T_n^{-1} \circ L \circ T_n$, such that

$$\tilde{L}^{(n-1)} = L^{(n-1)}, \tag{2.2}$$

$$\tilde{L}_n = L_n + [L_1, P_n]_n, \tag{2.3}$$

where $[L_1, \cdot]_n : \mathcal{H}_n \mapsto \mathcal{H}_n$ denotes the commutator

$$[L_1, P_n]_n := L_1 \circ P_n - P_n \circ L_1. \tag{2.4}$$

Now, we may think of $\mathcal{H}_n$ consisting of the image of $[L_1, \cdot]_n$ and some complementary space $\mathcal{V}_n$,

$$\mathcal{H}_n = \mathrm{im}([L_1, \cdot]_n) \oplus \mathcal{V}_n, \tag{2.5}$$

such that, by applying successive T_n transformations one may end up with *normal forms* that have at each order n all terms in the complementary space $\mathcal{V}_n$.

In order to preserve the local symmetry properties in the derivation of the normal form, we will require that our normal form transformations are such that they preserve the local symmetry properties as described in (2.1). It turns out that to do so it suffices to consider only Σ-equivariant transformations. This is the content of proposition 2.1.

Proposition 2.1 *Let $L^{(n)}$ satisfy* (2.1) *and let $T_n = \mathrm{Id} + P_n$ transform $L^{(n)}$ to $\tilde{L}^{(n)}$ according to* (2.2) *and* (2.3). *If $\tilde{L}^{(n)}$ also satisfies* (2.1), *then*

$$(L_n - \tilde{L}_n) \in \mathrm{im}([L_1, \cdot]_n|_{\mathcal{H}_n^{\Sigma\ \mathrm{pres}}}), \tag{2.6}$$

where

$$\mathcal{H}_n^{\Sigma\ \mathrm{pres}} := \{P_n \in \mathcal{H}_n \mid P_n \circ U = U \circ P_n, \ \forall U \in \Sigma\}. \tag{2.7}$$

Proof. Let us write $\Sigma = \Sigma_{nr} \cup \Sigma_r$, with all $S \in \Sigma_r$ being reversing, and all $M \in \Sigma_{nr}$ being non-reversing.

If we want to preserve Σ and its ϕ_L-orbit structure after applying the transformation $T_n = \mathrm{Id} + P_n$, we find at nth order that

$$\phi_{L_1}(M) \circ [L_1, P_n]_n = [L_1, P_n]_n \circ M, \quad \forall\, M \in \Sigma_{nr}, \tag{2.8}$$

and

$$[L_1, P_n]_n \circ S \circ L_1 = -L_1 \circ S \circ [L_1, P_n]_n, \quad \forall\, S \in \Sigma_r, \tag{2.9}$$

The relation (2.8) can be written as

$$[M, L_1 \circ P_n \circ L_1^{-1}]_n = [M, P_n]_n, \quad \forall\, M \in \Sigma_{nr}, \tag{2.10}$$

and (2.9) leads to

$$S^{-1} \circ P_n \circ S - L_1 \circ \phi_{L_1}(S)^{-1} \circ P_n \circ \phi_{L_1}(S) \circ L_1^{-1} = P_n - L_1 \circ P_n \circ L_1^{-1}, \quad \forall\, S \in \Sigma_r. \tag{2.11}$$

and after averaging over ϕ_L-orbits

$$\frac{1}{k}\sum_{m=1}^{k} \left(\phi_{L_1}^m(S)^{-1} \circ P_n \circ \phi_{L_1}^m(S) - L_1 \circ \phi_{L_1}^m(S)^{-1} \circ P_n \circ \phi_{L_1}^m(S) \circ L_1^{-1}\right) = P_n - L_1 \circ P_n \circ L_1^{-1}, \quad \forall\, S \in \Sigma_r, \tag{2.12}$$

where k is the length of the ϕ_L-orbit of S.

Further averaging over Σ_{nr} resp. Σ_r, and noting the fact that both (2.10) and (2.12) are linear in P_n leads to the conclusion that in both cases all solutions are of the form $P_n = A_n + B_n$ with $A_n \in \ker([L_1, \cdot]_n)$ and $B_n \in \mathcal{H}_n^{\Sigma\ \mathrm{pres}}$. Since the terms $A_n \in \ker([L_1, \cdot]_n)$ give no contribution at order n, it hence turns out that all symmetry preserving contributions at nth order may originate from Σ-equivariant T_n transformations. □

Since Σ-equivariant T_n transformations not only preserve the local symmetry properties up to order n, but up to any order, Σ-equivariant coordinate transformations are optimal for a symmetry preserving normal form procedure. This k-symmetric normal form procedure is an example of a structure preserving normal form strategy along the lines of the ideas set out in [4, 5], and is a direct extension of the preserving schemes for the reversible and ordinary symmetric cases.

Thus, starting from a Taylor expansion $L^{(n)}$ satisfying (2.1) we may use successive Σ-equivariant T_n transformations to obtain a k-symmetric normal form. We will discuss two approaches towards normal forms of maps: one approach that uses time-one maps of vector fields (following Takens [38]), and another direct approach that works completely within the framework of Taylor expansions of maps. Each one of these approaches has their own advantages: as the Takens approach yields a transparent result and insight in the relation between symmetric and k-symmetric local bifurcations, the direct approach is convenient in case one desires explicit expressions for normal forms of maps.

2.1 The Takens approach

We will first follow Takens [38], deriving normal forms of k-symmetric maps with the use of time-one maps of vector fields, yielding in the end theorem 1.1. Our proof of this theorem uses the following result:

Lemma 2.2 *Let Σ have linear action and be a reversing k-symmetry group of a map L, with equilibrium point $\mathbf{0}$, i.e. $L(\mathbf{0}) = \mathbf{0}$. Moreover, let A_s denote the semi-simple part of $DL|_{\mathbf{0}}$, then the ϕ_L-orbit structure of Σ is equal to the ϕ_{A_s}-orbit structure of Σ, i.e.*

$$\phi_L(U) = \phi_{A_s}(U), \quad \forall\, U \in \Sigma. \tag{2.13}$$

Proof. In case the action of Σ is linear, it follows that $\phi_L(U) = \phi_{L_1}(U)$ for all $U \in \Sigma$, with $L_1 := DL|_{\mathbf{0}}$. We use the splitting of the linear part L_1 into its semi-simple and nilpotent part,

$$L_1 = A_s + A_n. \tag{2.14}$$

Let M and S be linear and invertible, then

$$M \circ L_1 \circ M^{-1} = L_1 \quad \Rightarrow \quad M \circ A_s \circ M^{-1} = A_s. \tag{2.15}$$

$$S \circ L_1 \circ S^{-1} = (L_1)^{-1} \quad \Rightarrow \quad S \circ A_s \circ S^{-1} = (A_s)^{-1}. \tag{2.16}$$

The latter implication uses the fact that the semi-simple part of the inverse of L_1 is the inverse of the semi-simple part of L_1.

We write $L_1 = A_s \circ (\mathrm{Id} + (A_s)^{-1} \circ A_n)$. Noting the fact that $(A_s)^{-1} \circ A_n$ is also nilpotent we find that $(\mathrm{Id} + (A_s)^{-1} \circ A_n) = \exp(G)$ for some nilpotent G that commutes with A_s, cf. also [38].

Because the semi-simple part of $(L_1)^k$ is precisely $(A_s)^k$, we obtain

$$M \circ (L_1)^k \circ M^{-1} = (L_1)^k \;\Rightarrow\; M \circ (A_s)^k \circ M^{-1} = (A_s)^k. \tag{2.17}$$

Writing $L_1 = A_s \exp(G)$, we then find that

$$M \circ (L_1)^k \circ M^{-1} = (L_1)^k \quad \Rightarrow M \circ G \circ M^{-1} = G, \tag{2.18}$$

from which it follows that

$$L_1 \circ M \circ (L_1)^{-1} = A_s \circ M \circ (A_s)^{-1}. \tag{2.19}$$

Analogously one easily checks that

$$S \circ (L_1)^k \circ S^{-1} = (L_1)^{-k} \quad \Rightarrow \quad S \circ (A_s)^k \circ S^{-1} = (A_s)^{-k}, \tag{2.20}$$

$$\Rightarrow \quad S \circ G \circ S^{-1} = -G. \tag{2.21}$$

Hence we also have

$$L_1 \circ S \circ L_1 = A_s \circ S \circ A_s. \tag{2.22}$$

The equalities (2.19) and (2.22) are precisely (2.13). □

Proof of theorem 1.1: We use lemma 2.2 in combination with the fact that Σ-equivariant transformations form an optimal set of symmetry preserving transformations (proposition 2.1). The proof is obtained in two steps. First, following [38], one writes the expansion of the map L as an expansion of $A_s \circ \mathcal{X}_\tau$, where $\mathcal{X}_\tau$ is the time-one map of a vector field $\mathcal{X}$. By applying only Σ-equivariant coordinate transformations $\mathcal{X}_\tau$ may be modified until it commutes with A_s. Then, we find that Σ is a reversing symmetry group of the vector field $\mathcal{X}$. Namely, with $A_s \circ \mathcal{X}_\tau = \mathcal{X}_\tau \circ A_s$ it direcly follows from lemma 2.2 that

$$\phi_L^{\varepsilon_U}(U) \circ (A_s \circ \mathcal{X}_\tau) \circ U^{-1} = (A_s \circ \mathcal{X}_\tau)^{\varepsilon_U} \quad \Leftrightarrow \quad U \circ \mathcal{X}_\tau \circ U^{-1} = \mathcal{X}_\tau^{\varepsilon}, \tag{2.23}$$

where $\varepsilon_U = +1$ if U is non-reversing and $\varepsilon_U = -1$ if U is reversing. $\square$

Using only coordinate transformations that are equivariant with respect to Σ and A_s, within the context of theorem 1.1 the symmetric vector field $\mathcal{X}$ may be brought into normal form too. This can be done again in a symmetry preserving setting. In the remainder of this section we will discuss this scheme in more detail, yielding a symmetric normal form theorem which is outside the context of theorem 1.1 also of independent interest.

In the case of a dynamical system

$$\frac{d}{dt}\mathbf{x} = F(\mathbf{x}), \tag{2.24}$$

with $F : \mathbb{R}^d \mapsto \mathbb{R}^d$ a vector field with equilibrium point $\mathbf{0}$, i.e. $F(\mathbf{0}) = \mathbf{0}$, we may use coordinate transformations $T_n = \mathrm{Id} + P_n$ to simplify the Taylor expansion of this vector field $F^{(n)} = \sum_{i=1}^{n} F_i$ with $F_i \in \mathcal{H}_i$. In analogy with (2.2)-(2.3) one finds that

$$\tilde{F}^{(n-1)} = F^{(n-1)}, \tag{2.25}$$
$$\tilde{F}_n = F_n + \mathrm{ad}_n F_1(P_n), \tag{2.26}$$

where $\mathrm{ad}_n F_1 : \mathcal{H}_n \mapsto \mathcal{H}_n$ denotes a Lie derivative with respect to F_1, i.e.

$$\mathrm{ad}_n F_1(P_n)(\mathbf{x}) := F_1 \circ P_n(\mathbf{x}) - DP_n|_{\mathbf{x}} \circ F_1(\mathbf{x}). \tag{2.27}$$

A dynamical system of the form (2.24), possesses a linear symmetry M if

$$M \circ F = F \circ M, \tag{2.28}$$

and a linear reversing symmetry S if

$$S \circ F = -F \circ S. \tag{2.29}$$

In the context of vector fields possessing a reversing symmetry group Σ, again a symmetry preserving normal form procedure applies: a symmetry admitting subspace $\mathcal{H}_n^{\Sigma}$ is composed of the image of $\mathrm{ad}_n F_1$ restriced to $\mathcal{H}_n^{\Sigma\ \mathrm{pres}}$ (i.e. Σ-equivariant transformations) and a complementary subspace $\mathcal{V}_n^{\Sigma}$.

In fact, the symmetric normal form scheme follows by projection from the general one. This projection involves a weighted average over the (compact Lie) group Σ using an invariant normalized Haar measure:

$$\rho_{\Sigma}(P_n) = \int_{\Sigma} \varepsilon_U \, (U \circ P_n \circ U^{-1}) \, dU, \tag{2.30}$$

with weights $\varepsilon_U = +1$ in case U is a symmetry and $\varepsilon_U = -1$ in case U is a reversing symmetry. Clearly, ρ_{Σ} is a projection, mapping $\mathcal{H}_n \mapsto \mathcal{H}_n^{\Sigma}$ and $\mathrm{im}(\mathrm{ad}_n F_1) \mapsto \mathrm{im}(\mathrm{ad}_n F_1|_{\mathcal{H}_n^{\Sigma\ \mathrm{pres}}})$ such that indeed $\mathcal{V}_n^{\Sigma} := \rho_{\Sigma}(\mathcal{V}_n)$ is a subspace of $\mathcal{H}_n^{\Sigma}$ complementary to $\mathrm{im}(\mathrm{ad}_n F_1|_{\mathcal{H}_n^{\Sigma\ \mathrm{pres}}})$. This is the content of theorem 2.3.

Theorem 2.3 *Consider a dynamical system* (2.24) *possessing a reversing symmetry group* $\Sigma \leq O(d)$ *with linear action that is an isotropy subgroup of the equilibrium point* $\mathbf{0}$. *Then,* *in case the vector field is at least C^m, there exists a Σ-equivariant polynomial change of coordinates of degree m such that the vector field F is transformed into*

$$\sum_{n=1}^{m} \tilde{F}_n + R_m, \tag{2.31}$$

with $\tilde{F}_1 = F_1 := DF|_{\mathbf{0}}$, and for all $n = 2, \ldots, m$

$$\tilde{F}_n \in \mathcal{V}_n^{\Sigma} := \rho_{\Sigma}(\mathcal{V}_n) \tag{2.32}$$

where $\mathcal{V}_n$ is a complementary space of $\mathrm{im}(\mathrm{ad}_n F_1)$ *in $\mathcal{H}_n$, ρ_Σ is defined by* (2.30)*, and R_m indicates terms of degree at least $m+1$.*

Theorem 2.3 extends and comprises previous results on normal forms of equivariant [14, Chapter XVI §5] and reversible [18, 26, 29, 31, 32, 36] vector fields.

Remark 2.1 With respect to theorem 2.3 there are some convenient choices of the complementary space $\mathcal{V}_n$ which are well suited to obtain explicit expressions for normal forms of symmetric vector fields. Namely, in case $\mathcal{V}_n$ is chosen such that

$$U\mathcal{V}_n = \mathcal{V}_n U, \quad \forall\, U \in \Sigma, \tag{2.33}$$

then the projection ρ_Σ just yields

$$\mathcal{V}_n^{\Sigma} = \mathcal{V}_n \cap \mathcal{H}_n^{\Sigma}, \tag{2.34}$$

such that the symmetric normal form can be obtained from just *imposing* the reversing symmetry group on the general (nonsymmetric) one.

Explicit expressions may be obtained by choosing an inner product on $\mathcal{H}_n$ and requiring $\mathcal{V}_n$ to be the orthogonal complement of $\mathrm{im}(\mathrm{ad}_n F_1)$ with respect to this inner product. By a special choice of the inner product on $\mathcal{H}_n$, Elphick *et al.* [11] obtained

$$\mathcal{V}_n = \ker(\mathrm{ad}_n F_1^T), \tag{2.35}$$

where F_1^T denotes the transposed of F_1, i.e. in matrix notation $[F_1^T]_{ij} := [F_1]_{ji}$. When the action of Σ is chosen to be orthogonal (such that $\forall U \in \Sigma$, $U^T = U^{-1}$), it is easily checked that (2.33) is satisfied, leading to

$$\mathcal{V}_n^{\Sigma} = \mathcal{H}_n^{\Sigma} \cap \ker(\mathrm{ad}_n F_1^T). \tag{2.36}$$

2.2 The direct approach

Alternatively to the Takens approach, normal forms for maps may be derived without the use of time-one maps of vector fields, as Taylor expansions of maps with terms at nth order in $\mathcal{V}_n$, being a space complementary to the image of $[L_1, \cdot]_n$ in $\mathcal{H}_n$, cf. (2.5). The k-symmetry preserving strategy may be used to derive k-symmetric normal forms.

Up to here, the procedure is analogous to the vector field case, as discussed in the previous section. However, a straightforward generalization of theorem 2.3 to the case of k-symmetric maps ($k \geq 1$) is only possible in case the isotropy subgroup Σ contains no reversing elements. We shall discuss this case first. After that, a separate discussion will be given to cover the case of isotropy subgroups containing also reversing k-symmetries ($k \geq 1$).

In analogy to the vector field case, in the case of maps with a k-symmetry group Σ, at each order n one may project $\mathcal{H}_n$ onto a k-symmetric subspace $\mathcal{H}_n^\Sigma$ defined as

$$\mathcal{H}_n^\Sigma := \{F_n \in \mathcal{H}_n \mid \phi_L(U) \circ F_n = F_n \circ U, \; \forall\, U \in \Sigma\}. \tag{2.37}$$

The projection ρ_Σ should in this case be taken as

$$\rho_\Sigma(P_n) = \int_\Sigma \phi_L(U) \circ P_n \circ U^{-1}\, dU. \tag{2.38}$$

This projection transparently filters out the k-symmetric normal form procedure from the general one, leading to the following result:

Theorem 2.4 *Consider a map $L : I\!R^d \mapsto I\!R^d$, with fixed point $\mathbf{0}$, possessing a reversing k-symmetry group $\Sigma \leq O(d)$ that is an isotropy subgroup of $\mathbf{0}$. Then, if L is at least C^m, there exists a Σ-equivariant polynomial change of coordinates of degree m such that L transforms to*

$$\sum_{n=1}^{m} \tilde{L}_n + R_m, \tag{2.39}$$

with $\tilde{L}_1 = L_1 := DL|_{\mathbf{0}}$, and for all $n = 2, \ldots, m$

$$\tilde{L}_n \in \mathcal{V}_n^\Sigma := \rho_\Sigma(\mathcal{V}_n), \tag{2.40}$$

where $\mathcal{V}_n$ is a complementary space of $\mathrm{im}([L_1, \cdot]_n)$ *in $\mathcal{H}_n$, ρ_Σ is defined by* (2.38)*, and R_m indicates terms of degree at least $m+1$.*

Remark 2.2 Along the lines of the remark 2.1 we may use suitable choices of $\mathcal{V}_n$ to obtain explicit expressions for k-symmetric normal forms, in particular when $\mathcal{V}_n$ satisfies (2.33), k-symmetric normal forms can be found by imposing the k-symmetry onto the general normal form. For example, in [8] $\mathcal{V}_n = \ker([L_1^T, \cdot]_n)$ was obtained as a complementary space of $\mathrm{im}([L_1, \cdot]_n)$ in $\mathcal{H}_n$. As this complementary space satisfies (2.33), with an orthogonal action of Σ this leads to a normal form with

$$\mathcal{V}_n^\Sigma = \mathcal{H}_n^\Sigma \cap \ker([L_1^T, \cdot]_n). \tag{2.41}$$

It should be kept in mind that there are of course also other possible choices of normal form. It may depend on the application which type of normal form is most convenient.

In case the isotropy subgroup Σ contains a reversing symmetry, the situation becomes more complicated. Namely, whereas in the case of a k-symmetry group with linear action at each order n the space $\mathcal{H}_n$ can be decomposed in a k-symmetric and non-k-symmetric part, in case the isotropy subgroup contains a reversing element it may depend on the lower order terms $< n$ which terms are respecting the symmetry properties in the Taylor expansion at order n and which are not. This is due to the fact that the condition for a map to possess a reversing k-symmetry involves the inverse of the map, which causes relations between Taylor coefficients from different orders.[4] This 'mixing of orders' does not obstruct the

[4] Similar problems arise in the context of normal forms of symplectic and volume preserving maps.

symmetry preserving normal form procedure. However, it makes a direct projection from general (nonsymmetric) normal forms to k-symmetric normal forms less obvious.

If the linear part of the map is semi-simple, however, the existence of a direct projection method from a convenient general normal form is assured:

Theorem 2.5 *Consider a map $L : \mathbb{R}^d \mapsto \mathbb{R}^d$, with fixed point $\mathbf{0}$ and $L_1 := DL|_{\mathbf{0}}$ is semi-simple, possessing a reversing k-symmetry group $\Sigma \leq O(d)$ that is an isotropy subgroup of $\mathbf{0}$. Then, if L is at least C^m, there exists a Σ-equivariant polynomial change of coordinates of degree m such that L is transformed to*

$$\sum_{n=1}^{m} \tilde{L}_n + R_m, \tag{2.42}$$

with $\tilde{L}_1 = L_1 := DL|_{\mathbf{0}}$, and for all $n = 2, \ldots, m$

$$\tilde{L}_n \in \ker([L_1, \cdot]_n), \tag{2.43}$$

such that for all $U \in \Sigma$

$$\phi_L(U) \circ [\sum_{n=1}^{m} \tilde{L}_n]^{\varepsilon_U} \overset{(m)}{=} [\sum_{n=1}^{m} \tilde{L}_n] \circ U, \tag{2.44}$$

where $\overset{(m)}{=}$ denotes equality up to order m, and $\varepsilon_U = -1$ in case U is reversing and $\varepsilon_U = +1$ in case U is non-reversing. R_m indicates terms of degree at least $m+1$.

The proof of this theorem rests on theorem 1.1: in case L_1 is semi-simple one may obtain a normal form that is entirely characterized by its (formal) symmetry properties. The normal form obtained in theorem 2.5 is just a time-one map of this vector field normal form which is again fully characterized by its (formal) symmetry properties.

In case Σ contains reversing k-symmetries and L_1 is not semi-simple, then the above two theorems do not apply. In that case, unfortunately we have no transparent result in the direct approach. In [29] and [23, chapter 2] some of the problems related to this were discussed in the context of reversible maps. Concludingly, in the non semi-simple case one may either try to work out conditions for which some direct projection method is valid, extending the procedure described in [29, proposition 6], or use the Takens approach instead.

3 Local bifurcations with 2-symmetry group $\mathbb{Z}_2 \times \mathbb{Z}_2$ and reversing 2-symmetry group $\mathbb{Z}_2 \times \mathbb{Z}_2'$

In this section, to illustrate the results obtained in the previous sections, we will study generic codimension-one bifurcations of fixed points of maps with 2-symmetry group $\mathbb{Z}_2 \times \mathbb{Z}_2$ (i.e. $\mathbf{D}_2$) and reversing 2-symmetry group $\mathbb{Z}_2 \times \mathbb{Z}_2'$ ($\mathbb{Z}_2'$ denoting the subgroup generated by a reversing symmetry). These groups are among the simplest point groups that can be reversing k-symmetry groups with $k > 1$ (both must have $k = 2$).

We focus on local bifurcations occurring when an eigenvalue of the linear part of L^k at a fixed point equals ± 1.

We will now argue that in order to study these bifurcations, the bifurcation problem in $\mathbb{R}^d$ generically reduces (at worst) to a bifurcation problem in $\mathbb{R}^2$. Namely, local bifurcations that occur when an eigenvalue of the linear part of L^k at a fixed point goes through ± 1 have their counterpart in steady state bifurcations of symmetric vector fields. In the equivariant case it is well known that in such bifurcation problems the symmetry group generically acts as an irreducible representation (irrep) [14]. In the case of 2-symmetry group $\mathbb{Z}_2 \times \mathbb{Z}_2$ the bifurcation problem is related to that of a $\mathbf{D}_4$-equivariant vector field. As the irreducible representations of $\mathbf{D}_4$ are one- and two-dimensional, the bifurcation problem will generically reduce to a bifurcation problem in $\mathbb{R}^2$ (at worst). A similar argument applies in the case of reversing 2-symmetry group $\mathbb{Z}_2 \times \mathbb{Z}_2'$.

The bifurcation problems can reduce to $\mathbb{R}$ only if the action of ϕ_L in the irrep is trivial, in which case generic bifurcations arise as if the (reversing) k-symmetry group were a (reversing) symmetry group (with $k = 1$). Here we are mainly interested in bifurcation problems in which ϕ_L acts nontrivially. In the case of the groups $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2'$ this implies that we may focus our attention on the (faithful) two-dimensional representations $2m$ and $2m'$.[5] In these representations the groups are generated by a (reversing) mirror in the x-axis $M_x : (x, y) \mapsto (x, -y)$ and a twofold rotation $R_\pi : (x, y) \mapsto (-x, -y)$. In both cases there is only one possible ϕ_L-orbit structure with $k \neq 1$:

$$\phi_L^2(M_x) = \phi_L(M_y) = M_x \;\text{ and }\; \phi_L(R_\pi) = R_\pi, \tag{3.1}$$

where $M_y : (x, y) \mapsto (-x, y)$ denotes the mirror in the y-axis. The corresponding representations of the reversing symmetry groups in the related steady-state bifurcations of vector fields are $4m$ and $4'm$.

Many aspects of local bifurcations in k-symmetric dynamical systems can be found directly from applying results on generic bifurcations of vector fields with appropriately symmetry properties. In this way one can avoid explicit normal form calculations. However, explicit normal forms in terms of Taylor expansions of maps are of independent interest.

In our treatment below, we will first derive a 2-symmetric normal forms along the lines of theorem 2.4 in order to illustrate the derivation of normal forms using the direct approach. We then study the bifurcations that occur in unfoldings of these normal forms. In the spirit of theorem 1.1, we will demonstrate how these results can also be derived from local steady-state bifurcations of symmetric vector fields.

3.1 2-Symmetry group $2m$

We consider a map $L : \mathbb{R}^2 \mapsto \mathbb{R}^2$ with 2-symmetry group $2m$, generated by M_x and R_π, and ϕ_L-orbit structure (3.1).

Since R_π is a symmetry, $\mathbf{0}$ is automatically a fixed point, and $\langle M_x, R_\pi \rangle$ is an isotropy subgroup of this point. To study the local bifurcations of such a fixed point, we shall derive a k-symmetric normal form for this case along the lines of theorem 2.4.

The ϕ_L-orbit structure of the 2-symmetry group puts restrictions onto the linear part of the map. The relation

$$L_1 \circ M_x \circ L_1^{-1} = M_y, \tag{3.2}$$

[5]We borrow the Shubnikov-Belov notation from the context of (crystallographic) magnetic point groups or dichromatic color groups, cf. [30]. In this notation the generators of the point group are indicated: a '2' denotes a twofold rotation, and m denotes a mirror. If a generator is reversing it is labelled with an additional prime. Thus, $2m$ and $2m'$ are identical, apart from the fact that the mirrors of $2m'$ are reversing. The faithful representation $2'm$ can only occur as a reversing symmetry group (with $k = 1$), and is therefore not considered here.

causes L_1 to be of the form

$$L_1 = \begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix}, \tag{3.3}$$

where we have used the scaling freedom in x and y to set the upper right entry in L_1 equal to 1. Examining the complement of $\mathrm{Im}([L_1, \cdot]_n)$ in $\mathcal{H}_n$ for $n \geq 2$, we may construct a general normal form: [6]

$$\tilde{L}_c : \begin{cases} x' &= y, \\ y' &= cx + \mathrm{g}(x,y), \end{cases} \tag{3.4}$$

where $\mathrm{g}(0,0) = \partial_x \mathrm{g}(0,0) = \partial_y \mathrm{g}(0,0) = 0$.

Since in the context of (3.4) $U\mathcal{V}_n = \mathcal{V}_n U$ for all $U \in \langle M_x, R_\pi \rangle$, along the lines of remark 2.2 we may obtain a $2m$ 2-symmetric normal form directly by imposing the local symmetry properties. In (3.4), this leads to the following conditions on the function g:

$$\mathrm{g}(-x,y) = \mathrm{g}(-x,-y) = -\mathrm{g}(x,y). \tag{3.5}$$

Up to third order we thus find a normal form

$$\tilde{L}_c^{(3)} : \begin{cases} x' &= y, \\ y' &= cx + \mathrm{g}_{30}x^3 + \mathrm{g}_{12}xy^2. \end{cases} \tag{3.6}$$

If $|c| \neq 1$, the fixed point $\mathbf{0}$ is hyperbolic and structurally stable. If $|c| = 1$, the eigenvalues of the linear part are on the unit circle and local bifurcations may occur. We have to distinguish between the cases $c > 0$ and $c < 0$ separately (here, we will not discuss the case $c = 0$).

Let us first consider the orientation reversing case, i.e. $c > 0$. We parametrize $c = 1 + \varepsilon$ such that local bifurcations at $\mathbf{0}$ may occur when ε passes through zero, yielding the (codimension-one) unfolding.

In table 1 we list the periodic orbits that are involved in the local bifurcation occurring in (3.6) at $c = 1$. We find fixed points and period-2 orbits branching of the origin when it changes stability (from attractor to repeller or vice-versa). The isotropy subgroups of the orbits and the individual points on the orbits are also given.[7]

In parameter space g_{12}-g_{30} we have six regimes classifying the different codimension-one bifurcations. In Fig. 1 these regimes are indicated and in Fig. 2 the corresponding bifurcation diagrams are depicted.

In the orientation preserving case $c < 0$, we study the local bifurcations parametrizing $c = -(1 + \varepsilon)$. In table 2 the periodic orbits involved in the local bifurcation are presented. We now find period-4 orbits branching off the origin when it changes stability.

In parameter space g_{12}-g_{30} we again find the six regimes of Fig. 1 classifying the different codimension-one bifurcations. In Fig. 3 the corresponding bifurcation diagrams are depicted.

Following the approach of theorem 1.1, one can also study the above described bifurcations by relating them to local steady-state bifurcations of appropriately symmetric vector fields. According to theorem 1.1 the vector field may be taken to be symmetric with respect

[6] In this specific case it turns out that the normal form (3.4) is more convenient than the one obtained along the lines of (2.41) proposed in remark 2.2.

[7] Note that although the period-2 orbits (C_-, D_-) and (C_+, D_+) are not symmetric, they are 2-symmetric with respect to M_x and M_y, i.e. for each of these orbits $o(\mathbf{x}_0)$ there exists a point $\mathbf{x}$ such that $M_x(\mathbf{x}) \in o(\mathbf{x}_0)$ resp. $M_y(\mathbf{x}) \in o(\mathbf{x}_0)$. For more detailed discussions of k-symmetric periodic orbits, see [2].

to the group generated by A_s (the semi-simple part of the linear part of L) and Σ. Both when $c=1$ and $c=-1$ we here find this group to be $4m$ (i.e. $\mathbf{D}_4$). Thus the local k-symmetric bifurcations as discussed above are related to generic local steady-state bifurcations of $4m$ symmetric vector. The latter are well-studied, see e.g. [14]. In Fig. 4 we have depicted local phase portraits illustrating these local bifurcations (cf. also [12]). In order to relate them to the local bifurcations of $2m$ 2-symmetric maps one has to take into account the action of A_s which is in case $c=1$ a reflection in the line $x=y$, and in case $c=-1$ a rotation around $\mathbf{0}$ over an angle $\pi/2$. Finally, one has to determine the symmetry properties of the solution branches.

3.2 Reversing 2-symmetry group $2m'$

In this section we will study local bifurcations occurring at a fixed point of a map $L : \mathbb{R}^2 \mapsto \mathbb{R}^2$ possessing the group $\langle M_x, R_\pi \rangle \simeq 2m'$ as a reversing 2-symmetry group. This group differs from the $2m$ group in the previous section in the fact that the mirrors M_x and M_y are now reversing. The ϕ_L-orbits are again given by (3.1).

The point $\mathbf{0}$ is a fixed point (because R_π is a symmetry), and $\langle M_x, R_\pi \rangle$ is an isotropy subgroup of this point.

The ϕ_L-orbit structure (3.1) of the reversing 2-symmetry group puts restrictions to the linear part L_1 of a map. The relation

$$L_1 \circ M_x \circ L_1 = M_y, \tag{3.7}$$

causes the linear part L_1 to be of the form

$$L_1 = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \tag{3.8}$$

with $\det(L_1) = a^2 - b^2 = -1$, where we have used the scaling freedom in x and y to symmetrize L_1. Furthermore, we may choose without loss of generality $b > 0$. Note that the k-symmetry properties cause the fixed point $\mathbf{0}$ generically to be a saddle point (it cannot be an attractor).

Local bifurcations may occur if the eigenvalues of L_1 lie on the unit circle, i.e. when $a = 0$ and $b = 1$. So let us construct a general normal form at the bifurcation point, i.e. with

$$L_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.9}$$

Because L_1 is semi-simple, a k-symmetric normal form may be derived using theorem 2.5. We first obtain a general normal that commutes with L_1:

$$\tilde{L}^{(3)} : \left\{ \begin{array}{rcl} x' & = & y + \mathrm{g}(x,y), \\ y' & = & x + \mathrm{g}(y,x), \end{array} \right. \tag{3.10}$$

where $g(0,0) = \partial_x g(0,0) = \partial_y g(0,0) = 0$. From this general normal form we find a $2m'$ 2-symmetric normal form directly by imposing the k-symmetry properties on the general normal form (3.10). Up to third order this leads to

$$\tilde{L}^{(3)} : \left\{ \begin{array}{rcl} x' & = & y + \mathrm{g}_{12} x y^2 + \mathrm{g}_{30} x^3, \\ y' & = & x + \mathrm{g}_{12} x^2 y + \mathrm{g}_{30} y^3. \end{array} \right. \tag{3.11}$$

The local bifurcations are found from unfolding this normal form, preserving the reversing 2-symmetry group $2m'$:

$$\tilde{L}^{(3)}_{\varepsilon} : \left\{ \begin{array}{lcl} x' & = & \varepsilon x + y + \mathrm{g}_{12} x y^2 + \mathrm{g}_{30} x^3, \\ y' & = & x + \varepsilon y + \mathrm{g}_{12} x^2 y + \mathrm{g}_{30} y^3. \end{array} \right. \tag{3.12}$$

Note that we may unfold in such a way that the formal symmetry is preserved. The unfolded normal form is accurate up to terms of order $|\varepsilon|^2$ and $|x,y|^4$, so that e.g. $\det(D\tilde{L}^{(3)}_{\varepsilon}|_{\mathbf{0}}) = \varepsilon^2 - 1$.

In Table 3 we present the periodic orbits and fixed points that are involved in the local bifurcations occurring in this unfolding. Again we find that there are six types of local bifurcations, according to the regimes of Fig. 1 in which the coefficients g_{12} and g_{30} are. The corresponding local bifurcation diagrams in the regimes (I), (II) and (III) are depicted in Fig. 5.

Following theorem 1.1, we may study the above bifurcations also by studying generic local bifurcations of vector fields possessing a reversing symmetry group generated by Σ and A_s. In this case, the resulting reversing symmetry group is $4'm$. In [26] these bifurcations were studied. Taking into the account the action of A_s (here a reflection in the line $x = y$), the $2m'$ 2-symmetric local bifurcations found are easily related to the local bifurcations depicted in Fig. 6.

4 Discussion

In this last section, we would like to discuss the duality of the presence of reversing k-symmetry groups. It namely turns out that maps which possess a reversing k-symmetry group Σ which is a compact Lie group may possess another reversing k-symmetry group $\tilde{\Sigma}$ which is also a compact Lie group, but which has an action that is not simultaneously linearizable with that of Σ.

For instance, maps L possessing a reversing 2-symmetry group $2m'$ (as discussed in section 3.2), can be decomposed as

$$L = M_x \circ R, \tag{4.1}$$

where $R^2 = R_\pi$ [27]. However, since M_x is a reversing 2-symmetry and L is a symmetry of itself, it follows that R is a reversing 2-symmetry to. Moreover, the group generated by R is a reversing 2-symmetry group of L isomorphic to $4'$ ($\mathbb{Z}_4$ with reversing generator). However, if the action of L is nontrivial, the action of R need to be nonlinear. Because the $4'$ group is a compact Lie group, there exists a choice of local coordinates for which the action of R becomes linear. However, in that coordinate frame the action of M_x will in general have become nonlinear.

This curious phenomenon has its counterpart in the ambiguity in the structure of the group generated by a compact Lie group Σ that is a reversing k-symmetry group of the map L, and the group $\mathcal{L} := \{L^n \ : \ n \in \mathbb{Z}\}$. Namely, in the case of reversing k-symmetry groups and L is nontrivial ($\mathcal{L} \cap \Sigma = \{Id\}$), this group is a group product between Σ and $\mathcal{L}$ that is neither direct nor semi-direct. (Note the analogy with the observation in section 1.2 that neither Σ nor $\mathcal{S}$ is a normal subgroup of Γ.) In that case it may happen that not only $\mathcal{E}_k = \mathcal{L} \cdot \Sigma$ but also $\mathcal{E}_k = \mathcal{L} \cdot \tilde{\Sigma}$ with $\Sigma \not\simeq \tilde{\Sigma}$ and Σ and $\tilde{\Sigma}$ both being compact Lie groups. However, if the action of Σ is linear, then the action of $\tilde{\Sigma}$ will in general be nonlinear.

For instance, maps with reversing 2-symmetry group $2m'$ are always also 2-symmetric with respect to (a group isomorpic to) $4'$. An explicit example of such a mapping has been presented and diccussed in [28]. Just by a change of coordinates, one may either linearize the action of $2m'$ or the action of $4'$. In fact, it appears to be essential that there exists no choice of coordinates in which the action of the reversing 2-symmetry groups isomorphic to $2m'$ and $4'$ will be simultaneously linear in a neighbourhood of $\mathbf{0}$, as this would imply trivial dynamics by (4.1). In this respect it should be noted that the appearance of an isotropy subgroup $4'm$ on the level of the normal forms occurs only on a *formal* level.

It should be noted that the coexistence of not simultaneously linearizable isotropy subgroups only arises in the presence of reversing k-symmetries. For instance, in section 3.1 we found in the case of 2-symmetry group $2m$ that the normal form has formally $4m$ symmetry. However, this does not imply the simultaneous existence of a 2-symmetry group isomorphic to $\mathbb{Z}_4$.

The occurrence of not simultaneously linearizable compact Lie groups as reversing k-symmetry groups is an interesting aspect of k-symmetric dynamical systems, which certainly deserves further attention.

Although k-symmetry has its definition in terms of maps, in [24] it was pointed out that k-symmetry may arise in the context of Poincaré return maps of periodically driven systems admitting combined space-timeshift symmetries. It would be interesting relating the present results on local bifurcations in k-symmetric maps to corresponding bifurcations in the flows of nonautonomous vector fields, or in a local context to bifurcations of periodic solutions with mixed space-time symmetries.

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	$\mathbf{x}=(x,y)$	p	(λ_1,λ_2)	$\Sigma_{\mathbf{x}}$	$\Sigma_{o(\mathbf{x})}$
O	$(0,0)$	1	$(1+\frac{1}{2}\varepsilon,-1-\frac{1}{2}\varepsilon)$	$2m$	$2m$
$A_\pm$	$\pm(\sqrt{\frac{-\varepsilon}{g_{12}+g_{30}}},\sqrt{\frac{-\varepsilon}{g_{12}+g_{30}}})$	1	$(1-\varepsilon,-1-\varepsilon\frac{g_{12}-g_{30}}{g_{12}+g_{30}})$	–	–
$B_\pm$	$\pm(\sqrt{\frac{-\varepsilon}{g_{12}+g_{30}}},-\sqrt{\frac{-\varepsilon}{g_{12}+g_{30}}})$	2	$(1-2\varepsilon,1+2\varepsilon\frac{g_{12}-g_{30}}{g_{12}+g_{30}})$	–	2
$C_\pm$	$\pm(0,\sqrt{-\varepsilon/g_{30}})$	2	$(1-2\varepsilon,1-\varepsilon\frac{g_{12}-g_{30}}{g_{30}})$	m	–
$D_\pm$	$\pm(\sqrt{-\varepsilon/g_{30}},0)$	2	$(1-2\varepsilon,1-\varepsilon\frac{g_{12}-g_{30}}{g_{30}})$	m	–

Table 1: Orbits with period p involved in the local bifurcation in (3.6) with $c=1+\varepsilon$. The points $\mathbf{x}=(x,y)$ and the eigenvalues (λ_1,λ_2) of $D[\tilde{L}_c^{(3)}]^p|_{\mathbf{x}}$ are given up to first order in ε. Furthermore $\Sigma_{\mathbf{x}}$ denotes the (type of) isotropy subgroup of the point $\mathbf{x}$, and $\Sigma_{o(\mathbf{x})}$ the isotropy subgroup of the whole orbit of $\mathbf{0}$. Note that the period-2 orbits are (B_-,B_+), (C_-,D_-), and (C_+,D_+).

	$\mathbf{x}=(x,y)$	p	(λ_1,λ_2)	$\Sigma_{\mathbf{x}}$	$\Sigma_{o(\mathbf{x})}$
O	$(0,0)$	1	$(i(1+\frac{1}{2}\varepsilon),-i(1+\frac{1}{2}\varepsilon))$	$2m$	$2m$
$A_\pm$	$\pm(\sqrt{\frac{\varepsilon}{g_{12}+g_{30}}},\sqrt{\frac{\varepsilon}{g_{12}+g_{30}}})$	4	$(1-4\varepsilon,1+4\varepsilon\frac{g_{12}-g_{30}}{g_{12}+g_{30}})$	–	$2m$
$B_\pm$	$\pm(\sqrt{\frac{\varepsilon}{g_{12}+g_{30}}},-\sqrt{\frac{\varepsilon}{g_{12}+g_{30}}})$	4	$(1-4\varepsilon,1+4\varepsilon\frac{g_{12}-g_{30}}{g_{12}+g_{30}})$	–	$2m$
$C_\pm$	$\pm(0,\sqrt{\varepsilon/g_{30}})$	4	$(1-4\varepsilon,1-2\varepsilon\frac{g_{12}-g_{30}}{g_{30}})$	m	$2m$
$D_\pm$	$\pm(\sqrt{\varepsilon/g_{30}},0)$	4	$(1-4\varepsilon,1-2\varepsilon\frac{g_{12}-g_{30}}{g_{30}})$	m	$2m$

Table 2: Orbits of period p involved in the local bifurcations in (3.6) with $c=-(1+\varepsilon)$. The notation is similar to that of Table 1. Note that the period-4 orbits are (A_+,B_+,A_-,B_-), and (C_+,D_+,C_-,D_-).

	$\mathbf{x}=(x,y)$	p	(λ_1,λ_2)	$\Sigma_{\mathbf{x}}$	$\Sigma_{o(\mathbf{x})}$
O	$(0,0)$	1	$(\varepsilon+1,\varepsilon-1)$	$2m'$	$2m'$
$A_\pm$	$\pm(\sqrt{\frac{-\varepsilon}{g_{12}+g_{30}}},\sqrt{\frac{-\varepsilon}{g_{12}+g_{30}}})$	1	$(1-2\varepsilon,-1+2\varepsilon\frac{g_{12}-g_{30}}{g_{12}+g_{30}})$	–	–
$B_\pm$	$\pm(\sqrt{\frac{-\varepsilon}{g_{12}+g_{30}}},-\sqrt{\frac{-\varepsilon}{g_{12}+g_{30}}})$	2	$(1+4\varepsilon,1-4\varepsilon\frac{g_{12}-g_{30}}{g_{12}+g_{30}})$	–	2
$C_\pm$	$\pm(0,\sqrt{-\varepsilon/g_{30}})$	2	$(1+2\varepsilon\sqrt{-2\frac{g_{12}-g_{30}}{g_{30}}},1-2\varepsilon\sqrt{-2\frac{g_{12}-g_{30}}{g_{30}}})$	m'	–
$D_\pm$	$\pm(\sqrt{-\varepsilon/g_{30}},0)$	2	$(1+2\varepsilon\sqrt{-2\frac{g_{12}-g_{30}}{g_{30}}},1-2\varepsilon\sqrt{-2\frac{g_{12}-g_{30}}{g_{30}}})$	m'	–

Table 3: Orbits of period p involved in the local bifurcations in (3.12). The notation is similar to that of Table 1. Note that the period-2 orbits are (B_-,B_+), (C_-,D_-), and (C_+,D_+).

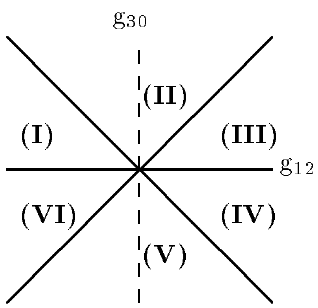

Figure 1: Six regimes in the plane g_{12}-g_{30} for which different types of bifurcations occur.

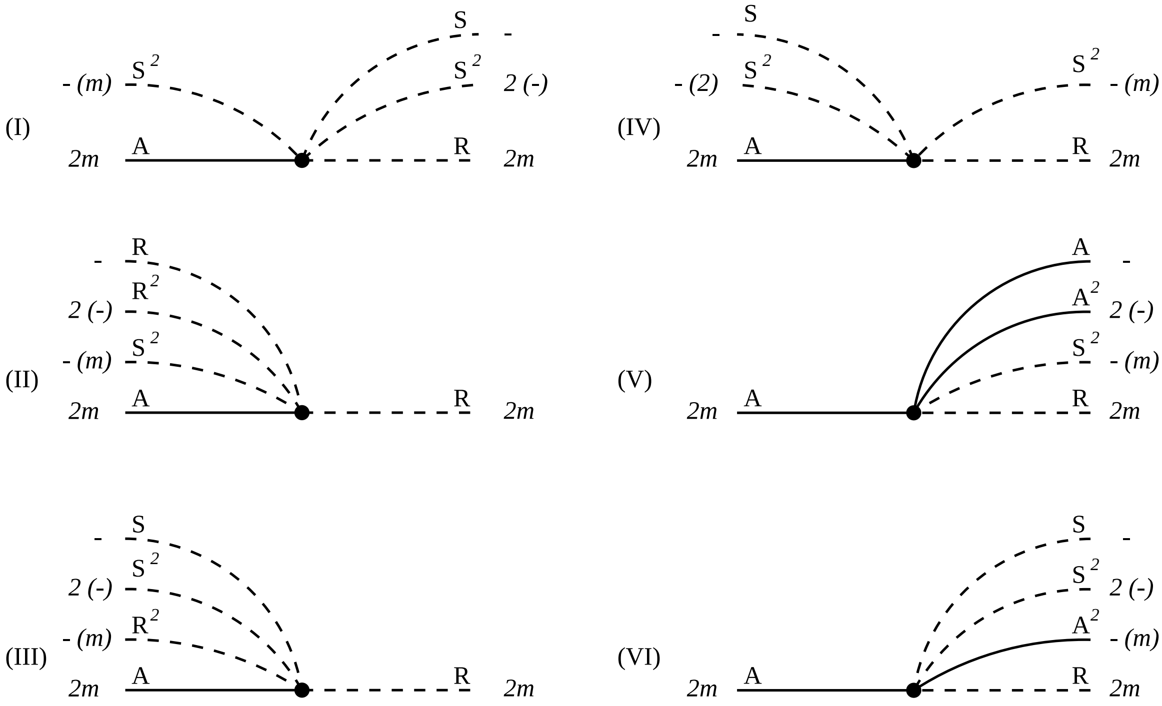

Figure 2: Bifurcation diagrams of the codimension-one bifurcations occurring at $c = 1$ in (3.6) with the parameters g_{12}-g_{30} in the regimes (I)-(VI) as indicated in Fig. 1. The stability types of the orbits are indicated with capitals: A (attractor), R (repeller), S (saddle). Solid lines denote stable branches (modulus of eigenvalues ≤ 1), and dashed lines denote unstable branches. Periods of the periodic orbits represented are indicated in superscript, e.g. A^2 denotes an attracting orbit of period 2. In the case of periodic orbits it may happen that the isotropy of the whole orbit differs from the isotropy of the individual points constituting it. In that case, the isotropy of the individual points is indicated between brackets, e.g. 2 (-) indicates an orbit that is invariant under $-Id$ containing points of trivial isotropy. Following usual conventions (cf. [14]), equivalent branches (i.e. those related by symmetries) are represented only once. For instance in (I), the S branch with trivial symmetry represents two branches of saddle points that are mapped onto each other by the symmetry $-Id$ (which is the only symmetry in the 2-symmetry group $2m$). Note that the bifurcation diagrams in the regimes (IV), (V) and (VI) are the time-reversed versions of (I), (II) and (III) (i.e. to obtain those, one has to change attractors into repellers and vice-versa).

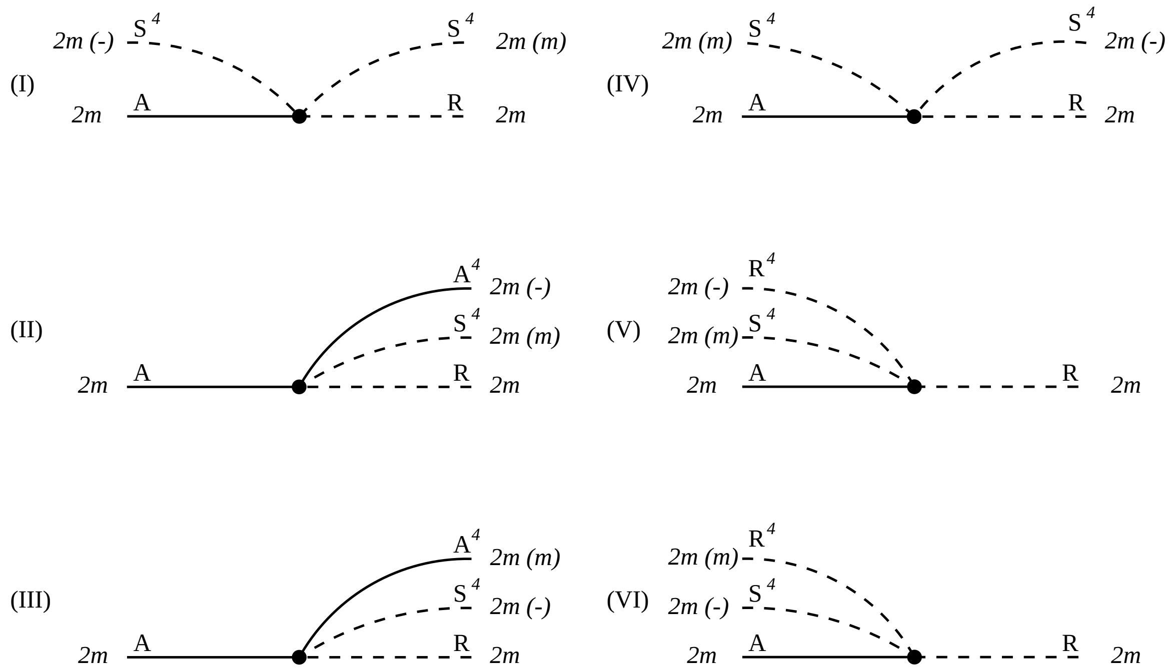

Figure 3: Bifurcation diagrams of the codimension-one bifurcations occurring at $c = -1$ in (3.6) with the parameters g_{12}-g_{30} in the regimes (I), (II) and (III) as indicated in Fig. 1. The notation is similar to that of Fig. 2. The bifurcation diagrams in the regimes (IV), (V) and (VI) are the time-reversed versions of (I), (II) and (III).

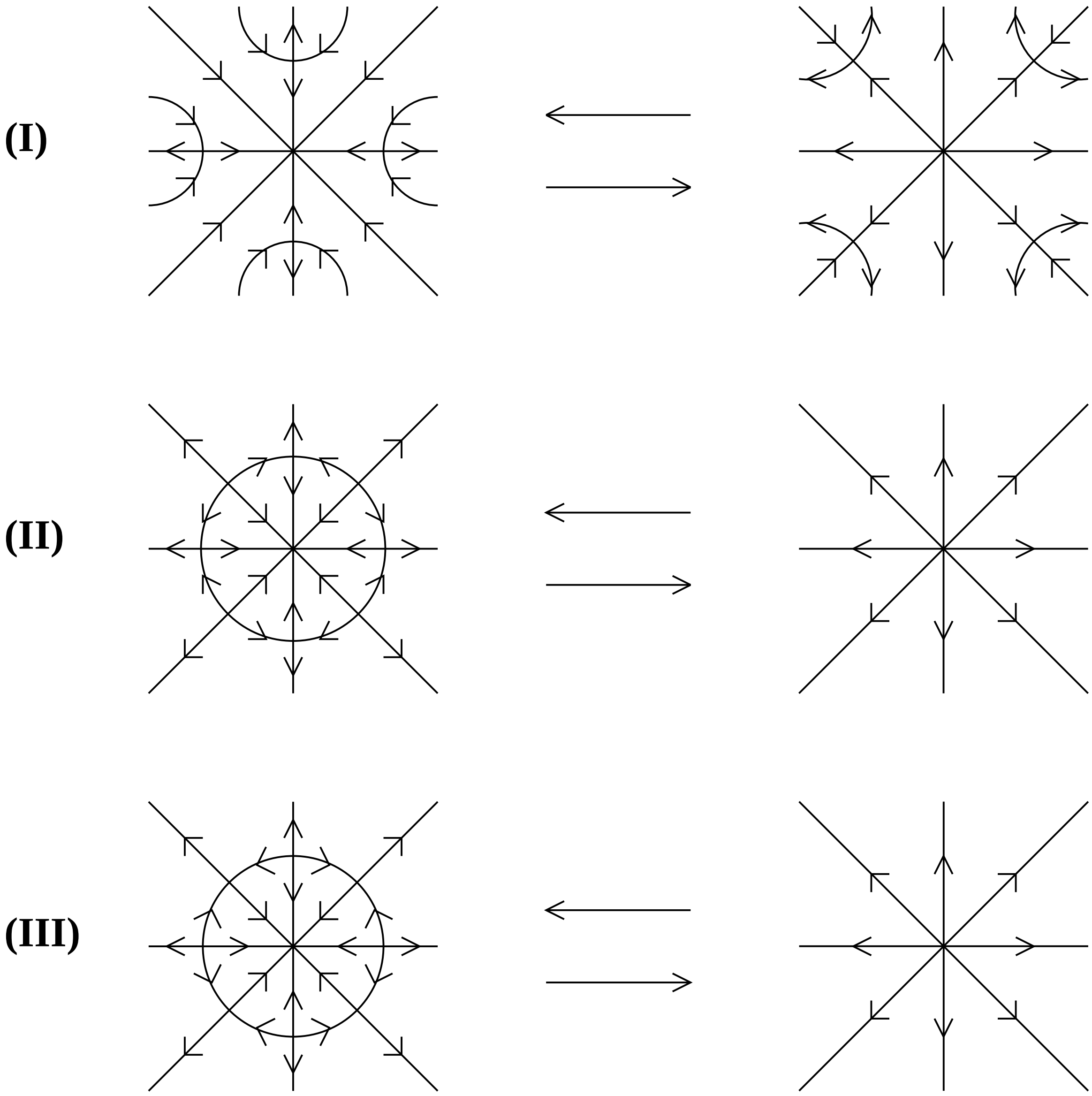

Figure 4: Local codimension-one bifurcations of $4m$ symmetric vector fields on the plane $\mathbb{R}^2$. The $2m$ 2-symmetric bifurcations (I), (II), and (III) of Fig. 2 ($c = 1$) are related to the above bifurcations. The time-reversed versions of the above flows approximate the bifurcations of Fig. 3 ($c = -1$).

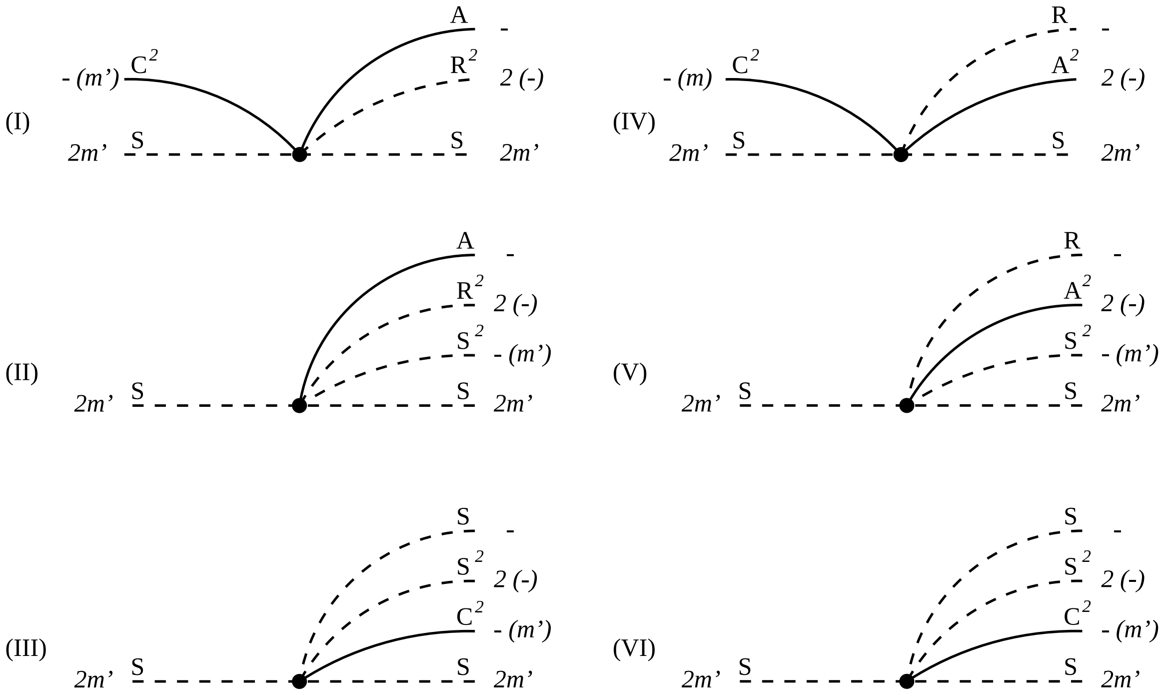

Figure 5: Bifurcation diagrams of the codimension-one bifurcations occurring in (3.12) with the parameters g_{12}-g_{30} in the regimes (I), (II) and (III) as indicated in Fig. 1. Stability type C denotes centre points (complex conjugate eigenvalues on the unit circle). The rest of the notation is similar to that of Fig. 2. The bifurcation diagrams in the regimes (IV), (V) and (VI) are the time-reversed versions of (I), (II) and (III). Note that those of (III) and (VI) are identical.

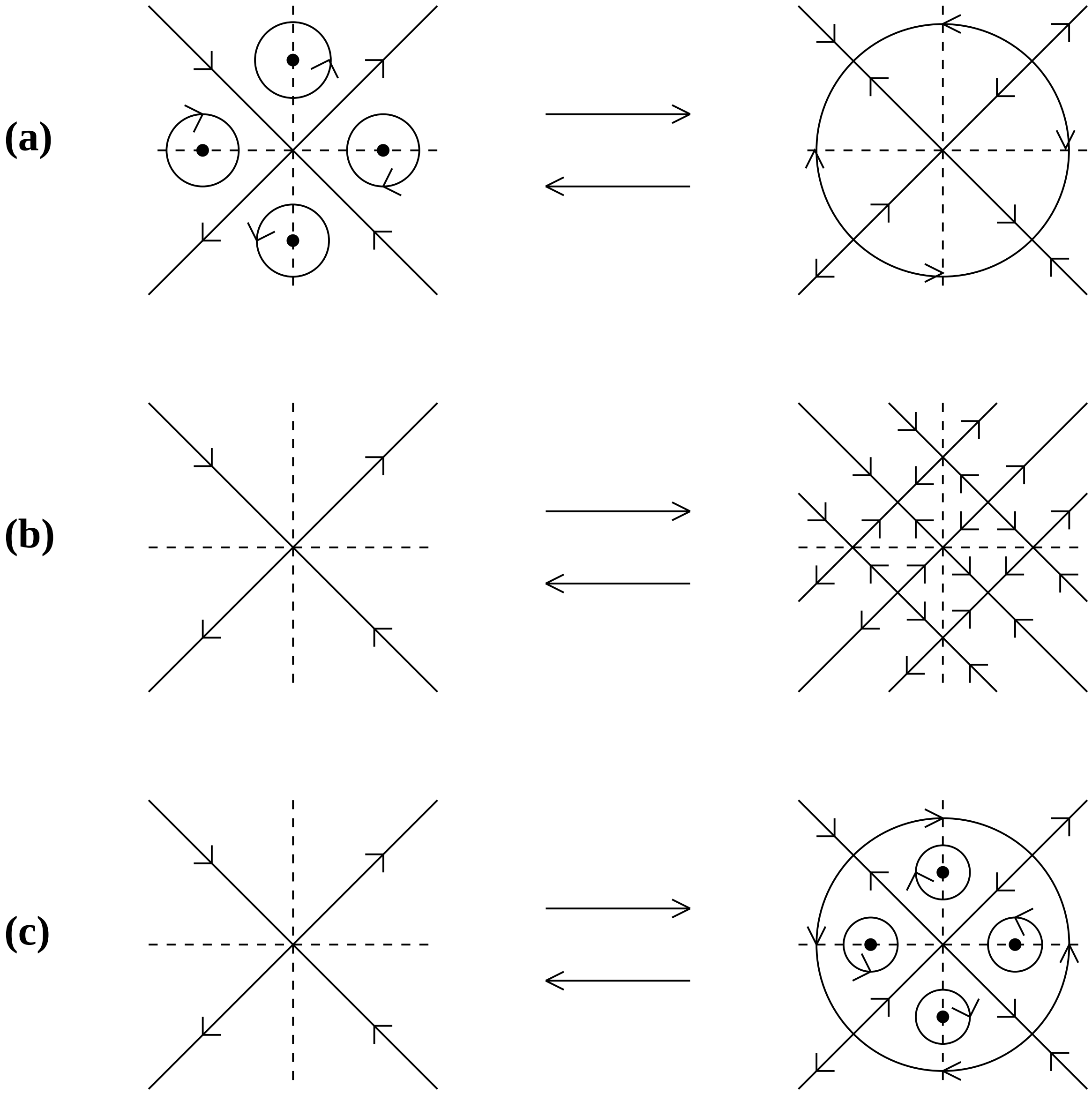

Figure 6: Local codimension-one bifurcations of $4'm$ symmetric vector fields on the plane $\mathbb{R}^2$. The $2m'$ 2-symmetric bifurcations (I), (II), and (III) of Fig. 5 are related to the above bifurcations.

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