STOCHASTIC WEBS WITH FOURFOLD ROTATION SYMMETRY

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1 INTRODUCTION

Weak chaos in nonlinear Hamilton systems has been studied extensively over the last two decades. A lot of attention was paid to the two-dimensional Chirikov-Taylor symplectic (i.e. area-preserving) standard map, modelling (part of) the motion of a charged particle in an inhomogeneous kicking electric field. This map displays weak chaos if it is perturbed from integrability (Chirikov, 1979; Lichtenberg and Lieberman, 1983). In the standard map, the chaotic motion is trapped between KAM-curves if the perturbation from integrability (here the strength of the electric field) is smaller than some threshold value. This causes a separation of chaotic regions in so-called stochastic layers. Only if the perturbation exceeds the threshold value, the chaotic motion is not trapped and the stochastic layers connect to form a two-dimensional infinite net of chaotic saddle connections, a so-called stochastic web, along which chaotic motion proceeds. The latter process is called stochastic diffusion. In the standard map, the threshold for stochastic diffusion is related to the break-up of the last KAM-curve.

This well established mechanism was observed to fail if one considers the same problem but with an additional homogeneous magnetic field (Zaslavsky et al., 1991). In case the frequency of the kicks and the cyclotron frequency of the particle satisfy certain resonance conditions, the threshold for stochastic diffusion in the corresponding two-dimensional symplectic map is observed to disappear, i.e. stochastic diffusion occurs independently of the strength of the electric field (as long it is not zero), the width of the stochastic webs becoming very small for small values of the electric field (Afanasiev et al., 1990). Moreover, the stochastic webs appeared to be extremely symmetric.

The symmetries of these stochastic webs were first thought to be understood on the basis of a Hamiltonian flow approximation (Chernikov et al., 1988; Zaslavsky et al., 1991). However, only recently, in case the resonance satisfies a crystallographic condition, true symmetries of stochastic webs were revealed via an analysis of the reversing symmetry group (of the kth iterate, for some $k \in \mathbb{N}$,) of the symplectic map (recently identified as a reversing k-symmetry group (Lamb and Quispel, 1993)). This analysis showed that the stochastic web is left invariant by a crystallographic symmetry group (Lamb, 1993a; Hoveijn, 1992). This crystallographic group was shown to be a true subgroup of the symmetry group of the pattern of saddle connections in the Hamiltonian flow approximation (the so-called web's skeleton (Zaslavsky et al., 1991)).

Furthermore, in the case of a stochastic web with fourfold rotation symmetry, in Lamb (1993a) the disappearance of the threshold for stochastic diffusion was understood on the basis of a symmetry argument.

In this paper we will study the relation between the occurrence of a stochastic web with fourfold rotation symmetry, its threshold for stochastic diffusion and the underlying (reversing) symmetry group possessed by (the kth iterate, for some $k \in \mathbb{N}$, of) a two-dimensional measure preserving map. [1]

First, we will show that reversing symmetry groups possessing a non-reversing mirror symmetry do not allow for the occurrence of a stochastic web. Then, from the observed crystallographic symmetry group of a stochastic web, the determination of possible underlying crystallographic reversing symmetry groups is not very difficult. In fact, this group can be determined precisely by identifying the reversing or non-reversing nature of the various point symmetries of the stochastic web.

Moreover, we conjecture that if (the kth iterate, for some $k \in \mathbb{N}$, of) a two-dimensional measure preserving map possesses a reversing symmetry group conjugate to a crystallographic reversing symmetry group with a fourfold (reversing) rotation, but no non-reversing mirror, this map generically displays a stochastic web implying the absence of a threshold for stochastic diffusion.

In the last section we present some examples area-preserving maps displaying stochastic webs with an underlying fourfold reversing rotation symmetry.

2 SYMMETRIES AND REVERSING SYMMETRIES

In the symmetry analysis of maps, we distinguish between symmetries and reversing symmetries. We say that a map M is a symmetry of a map L if

$$M \circ L \circ M^{-1} = L, \tag{2.1}$$

and that L has a reversing symmetry S, if

$$S \circ L \circ S^{-1} = L^{-1}. \tag{2.2}$$

Note that if the order of the reversing symmetry S, i.e. the smallest integer n such that $S^n = Id$, is odd then automatically $L^2 = Id$.

If L has a reversing symmetry then (historically) the map is called *weakly reversible* in general, and *reversible* if the reversing symmetry is an involution, i.e. its own inverse (Sevryuk, 1986; Roberts and Quispel, 1992).

The symmetries of a map, together with its reversing symmetries form a group under composition.[2] We call this group the *reversing symmetry group* of the map. The composition of a symmetry and a reversing symmetry is a reversing symmetry and

[1]The class of measure preserving maps is the class of area-preserving maps and maps conjugate to them, see e.g. Roberts and Quispel (1992).

[2]Here it is assumed that the map and its (reversing) symmetries are invertible.

the composition of two (reversing) symmetries is a symmetry. For a more detailed discussion on the structure of reversing symmetry groups, see Lamb (1992).

In case a map U is not a (reversing) symmetry of L but only of L^k (and not of L^l for some $0 < l < k$), U is called a (reversing) k-symmetry of L. For a more detailed discussion on the concepts and applications of reversing k-symmetries, see Lamb and Quispel (1993), and Quispel and Lamb (1993).

If a map L displays a stochastic web, this web is left invariant by all (reversing) k-symmetries of L for every value of $k \in \mathbb{N}$. In this paper, it is the linear (reversing) k-symmetries we are interested in, determining the crystallographic symmetry group of the stochastic web. Since in describing the symmetry properties of a stochastic web it is of no use to distinguish between symmetries and reversing symmetries, the symmetry group of the web is induced from the reversing k-symmetry group by neglecting differences between symmetries and reversing symmetries.

3 CRYSTALLOGRAPHIC SYMMETRIES OF STOCHASTIC WEBS WITH FOURFOLD ROTATION SYMMETRY

In a two-dimensional measure preserving map L, weak chaos may arise as a consequence of intersecting stable and unstable manifolds of hyperbolic periodic points (i.e. fixed points of L^n for some value of n). A stochastic web stems from transversal intersections of the stable manifold of one hyperbolic fixed point and the unstable manifold of another hyperbolic fixed point, so-called heteroclinic points. These intersections create a type of saddle connections along which the chaotic motion proceeds. The microscopic structure of these type of saddle connections is very complicated.

The (sudden) ocurrence of a stochastic web, i.e. an infinite net of chaotic saddle connections, seems non-generic unless translation symmetries ensure the existence of heteroclinic connections between saddle points over the entire phase plane. Hence, we are led to investigate stochastic webs in the context of two-dimensional crystallographic space groups.

Crystallographic space groups are generated by translations, spanning a lattice, and linear point transformations (i.e. tranformations leaving at least one point invariant), preserving this lattice. The point transformations in two-dimensional crystallographic space groups are well known to be mirrors and q-fold rotations with $q \in \{2, 3, 4, 6\}$. For a detailed discussion of the possible combinations, see e.g. Loeb (1971).

In dynamical systems we distinguish between reversing symmetries and symmetries, likewise in magnetic crystals spin-reversing symmetries are distinguished from ordinary symmetries (see e.g. Ludwig and Falter (1988)). Because both concepts of reversing symmetries are very similar, we may borrow the Shubnikov-Belov notation from magnetic groups to identify the crystallographic reversing symmetry groups in dynamical systems. In this notation reversing generators are labeled with a prime. It would lead to far to discuss this notation in detail. For a clear exposition, we refer the reader again to Loeb (1971).

In case a two-dimensional crystallographic reversing symmetry group contains a rotation over an angle $\pi/2$, various consequences are seen to follow. First of all, the translation symmetries give rise to a square unit cell. This unit cell can be regarded to be decorated with several additional mirrors and rotocenters. In this particular case, it follows that we have fourfold but also twofold rotocenters in the unit cell. Also mirrors may occur, but three- or sixfold rotocenters cannot arise. A complete analysis leads to 13 different possible crystallographic reversing symmetry groups with a fourfold (reversing) rotation symmetry.

The following proposition reduces this number of possibilities considerably in the case of a stochastic web.

Proposition *Let $L : I\!R^2 \mapsto I\!R^2$ and measure preserving. Then if for some value of $k \in I\!N$, L^k possesses a reversing symmetry group that contains a non-reversing mirror symmetry, then the dynamics of L cannot display a stochastic web.*

Proof.
Let M be a mirror symmetry of L and $W^u(\mathbf{x}_0)$ and $W^s(\mathbf{x}_0)$ be the unstable, resp. stable manifold of a hyperbolic fixed point $\mathbf{x}_0$ of L^n, for some value of n, i.e

$$\begin{aligned} W^s(\mathbf{x}_0) &:= \{\mathbf{x} \in \mathrm{I\!R}^2 | \lim_{m\to\infty} L^{n\cdot m}\mathbf{x} = \mathbf{x}_0\}, \\ W^u(\mathbf{x}_0) &:= \{\mathbf{x} \in \mathrm{I\!R}^2 | \lim_{m\to-\infty} L^{n\cdot m}\mathbf{x} = \mathbf{x}_0\}. \end{aligned}$$

Moreover, let $\mathrm{Fix}(M)$ denote the mirror line. Then it is easily verified that

(i) If $\mathbf{x}_0 \in \mathrm{Fix}(M)$ then either $W^u(\mathbf{x}_0) \subseteq \mathrm{Fix}(M)$ and $W^s(\mathbf{x}_0) \cap \mathrm{Fix}(M) = \emptyset$, or $W^s(\mathbf{x}_0) \subseteq \mathrm{Fix}(M)$ and $W^u(\mathbf{x}_0) \cap \mathrm{Fix}(M) = \emptyset$.

(ii) If $\mathbf{x}_0 \notin \mathrm{Fix}(M)$ then $W^u(\mathbf{x}_0) \cap \mathrm{Fix}(M) = \emptyset$ and $W^s(\mathbf{x}_0) \cap \mathrm{Fix}(M) = \emptyset$.

Hence we find that stable and unstable manifolds cannot (transversally) intersect the mirror line. So if there is a net of saddle connections, there must be hyperbolic fixed points on the mirror. However, these hyperbolic points have a stable or unstable manifold on the mirror line on which no heteroclinic points shared with hyperbolic points on either side of the mirror can be found. This implies that transport through weak chaos stops at a mirror line since saddle points on different sides of the line cannot make a heteroclinic connection. As the mirror line divides the plane, it acts as a blockade for stochastic diffusion.□

Taking the above proposition into account, we find only 6 crystallographic reversing symmetry groups with a fourfold (reversing) rotation symmetry allowing for a stochastic web. In table 1 these groups are presented together with the crystallographic symmetry groups of the stochastic web, as found by neglecting the difference between symmetries and reversing symmetries. For all three (ordinary) crystallographic groups with a fourfold rotation, we find possible underlying crystallographic reversing symmetry groups. In the case of $p4mm$ symmetry we observe that there is only possible underlying crystallographic reversing symmetry group, whereas in the case of $p4$ and $p4gm$ there are more possibilities. Note that of the 6 crystallographic reversing symmetry groups mentioned in table 1, the groups with a $4'$ possess fourfold rotations only as reversing symmetries, and the groups having a 4 possess fourfold rotations only as non-reversing symmetries. Hence, the twofold rotations in these groups are never reversing symmetries.

To identify from the phase portrait of the map the crystallographic reversing symmetry group of L^k for some value of k, the dynamical implications on the fixed sets of (reversing) point groups, being subgroups of the crystallographic reversing symmetry groups, can be used. In tabel 2 we list the (reversing) point groups occurring as subgroups of the crystallographic symmetry groups with a fourfold (reversing) rotation

Table 1: The crystallographic symmetry group of a stochastic web (SG web) of a two-dimensional symplectic map L and the various possible underlying crystallographic reversing symmetry groups (RSG) of L^k, for some $k \in \mathbb{N}$. In this table only crystallographic groups with a fourfold rotation are considered.

SG web	RSG
$p4$	$p4$, $p4'$,p'_c4
$p4mm$	$p4m'm'$
$p4gm$	$p4g'm'$, $p4'gm'$

Table 2: Dynamics of the fixed sets of (reversing) point groups occurring as subgroups of the crystallographic reversing symmetry groups, mentioned in table 1.

Point Group	Fixed Set	Dynamics
2	twofold rotocenter	fixed point, generically hyperbolic or elliptic
$2m'$	twofold rotocenter on reversing mirror	fixed point, generically hyperbolic or elliptic
m'	reversing mirror	generically not fixed point (may be hyp. or ellip.)
4	fourfold rotocenter	fixed point, generically elliptic
$4'$	fourfold reversing rotocenter	fixed point, generically hyperbolic

that allow for stochastic diffusion. In particular, fourfold reversing rotocenters can be easily distinguished from fourfold rotocenters, since the former cannot be elliptic and the latter cannot be hyperbolic fixed points. For a discussion of local bifurcations in the presence of a reversing point group symmetry see Lamb and Capel (1993).

To conclude this section, we state the following conjecture

Conjecture *Let $L : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be measure preserving and L^k, for some $k \in \mathbb{N}$, possess a reversing symmetry group conjugate to a crystallographic symmetry group with a fourfold (reversing) rotation symmetry, but no non-reversing mirror, then generically L displays a stochastic web.*

Above conjecture implies the absence of a threshold for stochastic diffusion. It is based on the following observations. Because of the fourfold (reversing) rotocenters, the twofold rotocenters, and the repetition of unit cells, generically, hyperbolic points will occur. Then, because of the fourfold (reversing) symmetry and measure preservation, there will be stable and unstable manifolds running from one unit cell into another. Moreover, transversal intersections with other stable or unstable manifolds seem generically unavoidable. Because of the fourfold rotation and translation symmetries these heteroclinic connections lead to a stochastic web.

Note that integrable maps are nongeneric counterexamples to the above conjecture.

4 EXAMPLES

As an illustration of the above discussion, in this section we will present examples of two-dimensional area-preserving maps possessing a crystallographic reversing crystallographic symmetry group with a fourfold reversing rotation. From the analysis in section 3, it follows that there are only two such crystallographic reversing symmetry groups allowing for a stochastic web, namely $p4'$ and $p4'gm'$. In these cases the stochastic webs have symmetry $p4$ and $p4gm$ respectively. For both groups we will present example maps.

To construct the examples, we make use of a convenient decomposition property, derived in Lamb (1992). Let $R_{\pi/2}$ be a rotation over an angle $\pi/2$ around the origin, i.e. $R_{\pi/2}(x,y)=(y,-x)$. Then, $R_{\pi/2}$ is a reversing symmetry of L if and only if

$$L = R_{\pi/2} \circ A, \tag{4.1}$$

such that $A^2 = R^2_{\pi/2}$ ($\circ$ denotes composition). The construction of such a map A is easy. Just let A be conjugate to $R_{\pi/2}$, i.e. $A = C \circ R_{\pi/2} \circ C^{-1}$, where C commutes with $R^2_{\pi/2}$ but not with $R_{\pi/2}$ (to avoid L being conjugate to $R^2_{\pi/2}$). If we take $C(x,y) = (x+f(y), -y)$ with f being odd, i.e. $f(y) = -f(-y)$, then

$$A : \left\{ \begin{array}{lcl} x' & = & -y - f(y'), \\ y' & = & x + f(y), \end{array} \right. \tag{4.2}$$

satisfies $A^2 = R^2_{\pi/2}$.

With this choice of A, we obtain

$$L : \left\{ \begin{array}{lcl} x' & = & x + f(y), \\ y' & = & y + f(x'). \end{array} \right. \tag{4.3}$$

In case f is not only odd but also periodic, L possesses translation symmetries, and hence a crystallographic reversing symmetry group. The groups $p4'$ and $p4'gm'$ occur with different properties of f:

$$\begin{array}{rcl} p4' & : & f(y) = f(y+a) = -f(-y), \\ p4'gm' & : & f(y) = -f(y+a/2) = -f(-y), \end{array}$$

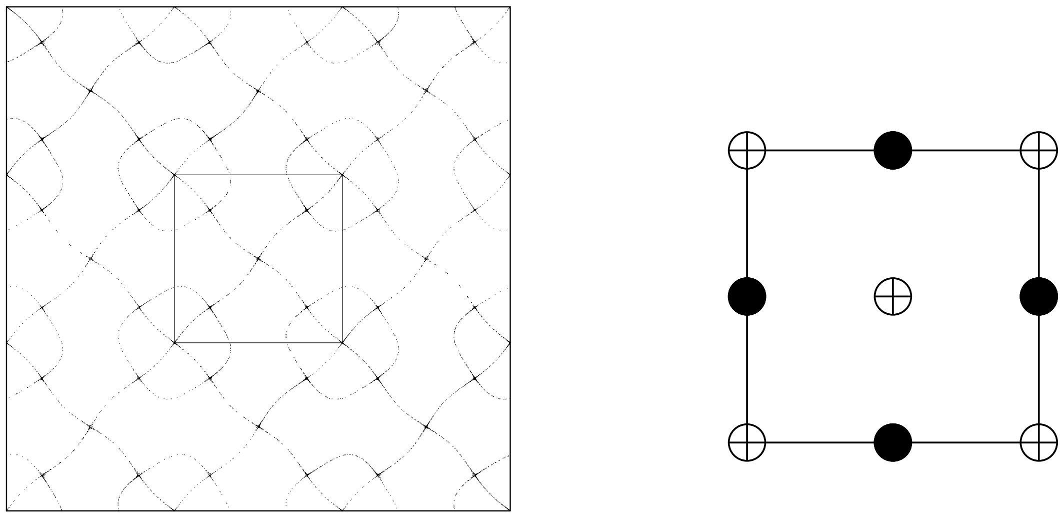
for some nonzero $a \in \mathbb{R}$. For convenience, in explicit examples we will take $a=1$.

In figure 4, part of the stochastic webs generated by (4.3) for different choices of f are depicted, one with $p4'$ and one with $p4'gm'$ symmetry. One additional square has been drawn, indicating the size of the unit cell. In the same figure for both cases the symmetry decoration of these square unit cells is given. The reader can easily verify the dynamical implications of the (reversing) point group symmetries, as discussed in the previous section.

In both above examples no threshold for stochastic diffusion is found, i.e. for all nonzero κ and λ a stochastic web is found. This supports the conjecture with respect to the disappearance of the threshold for stochastic diffusion, as stated at the end of the previous section.

Because C was chosen to be an involution, the map L (4.3) can also be written as $L = M^2$ with $M(x,y) = (y, x+f(y))$. In fact, $p4'$ resp. $p4'gm'$ are reversing 2-symmetry groups of M.

(a)



(b)

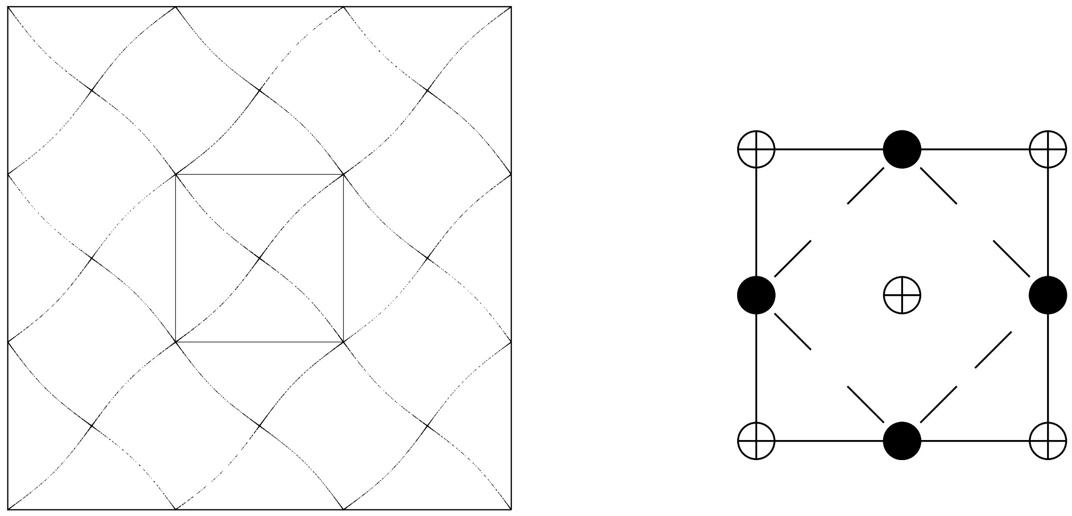


Figure 1: Part of the stochastic web of (4.3) with $f(y) = \kappa \sin(2\pi y) + \lambda \sin(4\pi y)$ on $[-1.5, 1.5] \times [-1.5, 1.5]$ with (a) $\kappa = 0.03$ and $\lambda = 0.06$, and (b) $\kappa = 0.1$ and $\lambda = 0$. The stochastic web in (a) has $p4$ and (b) has $p4gm$ symmetry. Also a square unit cell is drawn. The symmetry decoration of the square unit cells is indicated in accompanying pictures. The crystallographic reversing symmetry groups of the maps are (a) $p4'$, (b) $p4'gm'$. Dashed lines denote reversing mirrors, $\oplus$'s fourfold reversing rotocenters and $\odot$'s twofold rotocenters.

Besides a crystallographic reversing symmetry group L (4.3) also possesses other (reversing) symmetries. For instance, it possesses the nonlinear reversing mirror symmetry $S(x, y) = (x, -y + f(x))$ (Roberts and Quispel, 1992). This mirror runs through the origin but it does not imply the existence of a non-reversing mirror (preventing stochastic diffusion), as a linear reversing mirror would (because of the fact that the origin is a fourfold reversing rotocenter). This observation illustrates the importance of considering the occurrence of (reversing) symmetries in the light of the structure of the reversing symmetry group.

5 CONCLUDING REMARKS

In this paper we discussed the occurrence of stochastic webs in measure preserving maps possessing a reversing symmetry group that is conjugate to a crystallographic group with a fourfold (reversing) rotation. In particular, we conjecture that in these maps generically stochastic webs occur without a threshold for stochastic diffusion.

In the near future we plan to report in detail on stochastic webs with other crystallographic symmetries as well.

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