Fixed points of elements of linear groups

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Abstract

We prove that for any finite group G such that $|G/R(G)| > 120$ (where $R(G)$ is the soluble radical of G), and any finite-dimensional vector space V on which G acts, there is a non-identity element of G with fixed point space of dimension at least $\frac{1}{6}\dim V$. This bound is best possible.

1 Introduction

Let G be a finite group, K a field and V a finite-dimensional KG-module. For $g \in G$, let $C_V(g)$ denote the space of fixed points of g on V. The dimensions of these fixed point spaces have been studied in several papers. Upper bounds in the case where V is irreducible were obtained in [4, 5], culminating in [2, 1.3], where it is shown that $\dim C_V(g) \leq \frac{1}{3}\dim V$ for some $g \in G$. In this paper we prove a counterpart concerning lower bounds. In our result the module V is arbitrary (not necessarily irreducible), but we make some necessary assumptions on the structure of G. Denote by $R(G)$ the soluble radical of G – that is, the largest soluble normal subgroup of G.

Theorem 1 *Let G be a finite group satisfying $|G/R(G)| > 120$. Then for any field K and any KG-module V, there exists a non-identity element $g \in G$ such that*

$$\dim C_V(g) \geq \frac{1}{6}\dim V.$$

This result is best possible in several ways. First, the assumption that $|G/R(G)| > 120$ is necessary. Indeed, if G is a Frobenius complement then

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2000 *Mathematics Subject Classification: 20C20* it has a module V such that $C_V(g) = 0$ for all non-identity elements $g \in G$, and there are Frobenius complements G satisfying $|G/R(G)| = 120$ (e.g. $SL_2(5).2$). Also, the constant $\frac{1}{6}$ in the theorem is best possible, as will be shown below (see the Remark after Proposition 2.2).

Note that the assumption $|G/R(G)| > 120$ simply asserts that G is not soluble and $|G/R(G)|$ is not A_5 or S_5.

Theorem 1 can be viewed in the context of *fixity*. For a KG-module V, define the fixity $\mathrm{fix}(V)$ to be the maximal dimension of $C_V(g)$ for $1 \neq g \in G$. Thus Theorem 1 asserts that $\mathrm{fix}(V) \geq \frac{1}{6} \dim V$ under the given hypotheses. This concept was introduced in [6], where the structure of finite groups having a module of bounded fixity in characteristic 0 is studied. Further results were obtained in [7, 8]; in [7, 2.1], the conclusion of Theorem 1 is obtained in the case where $K = \mathbb{C}$, and the non-modular case (i.e. the case where $\mathrm{char}(K)$ is 0 or coprime to $|G|$) follows from this. Thus our contribution here is to deal with modular representations.

2 Proof of Theorem 1

For the proof of Theorem 1 we need several preliminary results.

Throughout, let G be a finite group and K a field of characteristic l. Since the theorem has been proved in [7] in the non-modular case, we assume that l is a prime dividing $|G|$. Also, extending the field does not affect the fixity of a module, so we assume that K is algebraically closed.

Lemma 2.1 *We have* $\mathrm{fix}(V) \geq \frac{1}{l} \dim V$ *for any KG-module V.*

Proof. Let $g \in G$ be an element of order l. Then $(g-1)^l V = (g^l - 1)V = 0$, which implies the conclusion. ■

Proposition 2.2 *Let $G = SL_2(p)$, where $p \geq 7$ is prime, and assume that $l \geq 7$. Then* $\mathrm{fix}(V) \geq \frac{1}{6} \dim V$ *for any KG-module V.*

Proof. First suppose that l divides $p^2 - 1$. Let $g \in G$ have order 3. It is shown in [8, 5.5(ii)] that for any irreducible KG-module V we have $\dim V \leq 3 \dim C_V(g) + 2r$, where $r \in \{1, 2\}$ and $p \equiv r \bmod 3$. Write $f = \dim C_V(g)$ and assume by contradiction that $\dim V > 6f$. As $\dim V \leq 3f + 2r$ this implies that $2r > 3f$, so $\dim V < 4r \leq 8$. It is well known that $\dim V \geq (p-1)/2$, so this forces $p = 7, 11$ or 13 and $r = 1, 2$ or 1 respectively. The only possibility is $p = 11$, $f = 1$ and $\dim V = 7$; but $SL_2(11)$ has no irreducible module of dimension 7. This proves that $\dim C_V(g) \geq \frac{1}{6} \dim V$ for irreducible KG-modules V.

Now let V be an arbitrary KG-module, and let V_i $(i = 1, \ldots, k)$ be its composition factors (possibly with repetitions). Since the characteristic l is not 3, V and $\oplus_{i=1}^{k} V_i$ are isomorphic as $K\langle g\rangle$-modules, so

$$\dim C_V(g) = \sum_{i=1}^{k} \dim C_{V_i}(g) \geq \sum_{i=1}^{k} \frac{1}{6} \dim V_i = \frac{1}{6} \dim V.$$

Now suppose that $l = p$. Here we use the structure of the irreducible and indecomposable KG-modules, which can be found for example in [1]. For each $1 \leq i \leq p$ there is an irreducible KG-module V_i of dimension i. Here V_2 is the natural $SL_2(p)$-module, and $V_i = S^{i-1}(V_2)$, the $i-1^{th}$ symmetric power of V_2. Note that for i odd, the central involution z of G acts trivially on V_i and so $\mathrm{fix}(V_i) = \dim V_i$.

Let $g \in G$ be an element of order 3. Then g acts as the diagonal matrix $\mathrm{diag}(\omega^{i-1}, \omega^{i-3}, \ldots, \omega^{-(i-3)}, \omega^{-(i-1)})$, where ω is a cube root of 1. It follows that

$$\dim C_{V_i}(g) \geq i/4 \quad \text{for } i \neq 2, 5, \tag{1}$$

while $\dim C_{V_2}(g) = 0$ and $\dim C_{V_5}(g) = 1$.

For $a \in \mathbb{F}_p^*, b \in \mathbb{F}_p$, define

$$g_{a,b} = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in G,$$

and let N be the set of all such elements $g_{a,b}$. Then $N = N_G(P)$ with $P \in \mathrm{Syl}_p(G)$. For an integer j, let S_j be the 1-dimensional KN-module in which $g_{a,b}$ acts as multiplication by a^j. The S_j are the irreducible KN-modules. Every indecomposable KN-module U is uniserial and has a composition series with successive factors $S_j, S_{j-2}, S_{j-4}, \ldots$ for some j, the length depending only on $\dim U$, which is at most p. Those of dimension p are the projective indecomposable KN-modules.

Now let W be an indecomposable KG-module. Then $W \downarrow N = U \oplus Q$, where U is an indecomposable and Q a projective KN-module (see [1, Theorem 1, p.71]). We have $\dim W = k + mp$, where $k = \dim U$ and $\dim Q = mp$. Let $u \in P$ be an element of order p. Then $\dim C_W(u) = 1+m$. Also the multiplicity of any S_j as a composition factor of $W \downarrow N$ is at most $1 + m$. It follows that $\dim C_W(u)$ is at least the multiplicity of any V_i as a composition factor of W. We shall use this lower bound for $i = 2$ and 5.

Now consider an arbitrary KG-module V. Let A (resp. B) be the sum of all indecomposable summands of V which have V_2 (resp. V_5) as a composition factor, and let C be the sum of all the other indecomposable summands. Observe that V_2, V_5 cannot both occur in an indecomposable W, since $C_W(z)$ is a direct summand involving just the composition factors V_i of W with i odd. Hence $A \cap B = 0$ and we have $V = A \oplus B \oplus C$.

Let $a = \dim A$, $b = \dim B$, $c = \dim C$, and write n_2 (resp. n_5) for the multiplicity of V_2 (resp. V_5) as a composition factor of V.

By (1) we have

$$\dim C_V(g) \geq \frac{1}{4}(a - 2n_2) + n_5 + \frac{1}{4}(b - 5n_5) + \frac{1}{4}c. \tag{2}$$

Clearly $\dim C_W(u) \geq \frac{1}{p}\dim W$ for any KG-module W (as $(g-1)^pW = 0$). Combining this with the previous lower bound on $\dim C_W(u)$ for W indecomposable we obtain

$$\dim C_V(u) \geq n_2 + n_5 + c/p. \tag{3}$$

Let $f = \mathrm{fix}(V)$, so that $f \geq \dim C_V(g), \dim C_V(u)$. Assume that $f \leq \frac{1}{6}\dim V = \frac{1}{6}(a+b+c)$. Then (2) gives

$$a + b + c \leq 6n_2 + 3n_5,$$

while (3) gives

$$a + b + c(1 - 6/p) \geq 6n_2 + 6n_5.$$

Thus $6n_2 + 3n_5 \geq a + b + c \geq a + b + c(1 - 6/p) \geq 6n_2 + 6n_5$. It follows that equality holds throughout, and that $c = n_5 = 0$, hence also $b = 0$ and $f = \frac{1}{6}\dim V$. This proves the result (and also helps identifying all possibilities where $\mathrm{fix}(V) = \frac{1}{6}\dim V$ – see the Remark below). ■

Remark Pursuing the final remark in the proof, we claim that the equality $\mathrm{fix}(V) = \frac{1}{6}\dim V$ holds in the $l = p$ case if and only if $p = 11$ and $V = (V_2 \oplus W)^d$, where W is an indecomposable of dimension 10 with composition factors V_2 and V_8. Indeed, if $f = \frac{1}{6}\dim V$ the above proof shows that $V = A$, $\dim V = a = 6n_2$ and $\dim C_V(g) = \dim C_V(u) = n_2 = \frac{1}{4}(a - 2n_2)$. The only V_i satisfying $\dim C_V(g) = i/4$ is V_8, and there is an indecomposable with composition factors V_2 and V_8 if and only if $p = 11$ (see [1, pp.48-49]). Hence $V = (V_2 + W)^d$ as claimed, and for this module we have $\dim C_V(x) \leq \frac{1}{6}\dim V$ for all $x \in G = SL_2(11)$, with equality holding for $x = g$ or u.

Lemma 2.3 *Let L be an l-group, and suppose L has an automorphism u of order 3. Let H be the semidirect product $L\langle u\rangle$. Then $\mathrm{fix}(V) \geq \frac{1}{3}\dim V$ for any KH-module V.*

Proof. Recall that $l \neq 3$, so we can triangularise H and write

$$u = \mathrm{diag}(I_r, \omega I_s, \omega^2 I_t),$$

where $\omega \in K$ is a cube root of unity and $\dim V = r + s + t$. Pick $v \in L$ with $v^u \neq v$, and write

$$v = \begin{pmatrix} A & D & E \\ 0 & B & F \\ 0 & 0 & C \end{pmatrix}$$

where A is $r \times r$, B is $s \times s$ and C is $t \times t$. Then

$$v^{-1}v^u = \begin{pmatrix} I_r & * & * \\ 0 & I_s & * \\ 0 & 0 & I_t \end{pmatrix},$$

and so $\dim C_V(v^{-1}v^u) \geq \max(r, s, t) \geq \frac{1}{3}\dim V$. ■

Proof of Theorem 1

We now complete the proof of the theorem. Let G be a finite group such that $|G/R(G)| > 120$, let K be a field of characteristic l, and let V be a KG-module. As the non-modular case is covered by [7, 2.1], we assume that l is a prime dividing $|G|$. We also assume that $l \geq 7$ in view of Lemma 2.1.

By [7, 2.3], the assumption $|G/R(G)| > 120$ implies that G has a subgroup H which is isomorphic to one of the following groups:

(1) $C_2 \times C_2$

(2) $C_3 \times C_3$

(3) $SL_2(q)$, where q is a power of a prime p and $q \geq 7$, $p \geq 5$

(4) $P\langle u \rangle$, a semidirect product of a nontrivial p-group P by a group $\langle u \rangle$ of order 3 acting nontrivially on P, where p is a prime.

We shall show, for each group H as above, that the fixity of V as a KH-module is at least $\frac{1}{6}\dim V$, and this implies the required conclusion for G.

If $|H|$ is coprime to l, this is obtained in [7, 2.1], so assume that l divides $|H|$. This rules out cases (1) and (2).

Now consider case (3). If $p \geq 7$ then H has a subgroup $SL_2(p)$ to which we can apply Proposition 2.2. And if $p = 5$ then $q \geq 25$, so H has a subgroup $C_5 \times C_5$, for which $\text{fix}(V) \geq \frac{1}{6}\dim V$ (see [6], Lemma 2.1).

Finally, in case (4) our assumption that l divides $|H|$ implies that $p = l$, and so the result follows from Lemma 2.3.

This completes the proof. ■

Variations and applications of our main result will be discussed in a subsequent paper [3].

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