GAUGE THEORY AND COMPLEX DIFFERENTIAL GEOMETRY

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Contents

Introduction and Acknowlegements

This dissertation is principally concerned with the extension of the Seiberg-Witten equations and other gauge theories to six dimensional Kähler (and symplectic) manifolds, though where possible I have treated subjects in greater generality. Solutions of the equations represent special metrics on line bundles, and distinguished representatives of their cohomology groups, and so may have useful applications, but it is really more as an exercise in the techniques of gauge theory that the dissertation was written.

There was not space to include details of every background result or relevant subject, so I have chosen to write up those results that are either not well explained elsewhere, or that I had most trouble with and gained most from explaining. Therefore Smale-Sard theory and the Kuranishi description of the zeros of a Fredholm map, explained well in [2], is used without proof, while the (more elementary) subject of spin^c structures and writing down the Seiberg-Witten equations on a 4-manifold occupy the whole of Section 1. Section 2 deals with some of the standard results of Seiberg-Witten theory that pass over from four dimensions and sets up the problems and conjectures for six dimensions. It then describes the similarities and links with other gauge theories, and how standard techniques do and do not apply here. While, for instance, the use of symplectic formalism and gradient flows turns out to be of limited help in our situation, this is something else I wanted to include for my own benefit.

Section 3 is the core of the dissertation, dealing with the linearisation and solution space of the equations. Finally Section 4 looks at some generalisations to symplectic manifolds.

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1. Spin^c Structures and Dirac Operators

Clifford Algebras. We begin by briefly recalling some of the basic properties of Clifford algebras.

Proposition (1.1). *$Cl^n_{\mathbb{C}} = Cliff(\mathbb{R}^n) \otimes \mathbb{C}$ is (non-naturally) isomorphic to*

$$\begin{cases} M_{2^{n/2}}(\mathbb{C}) & n \text{ even}, \\ M_{2^m}(\mathbb{C}) \oplus M_{2^m}(\mathbb{C}) & n = (2m+1) \text{ odd}. \end{cases}$$

This isomorphism can be taken to intertwine the conjugate-linear involution $^-$ on $Cl^n_{\mathbb{C}}$ (given by $\overline{e_1 e_2 \dots e_k} = (-1)^k e_k \dots e_1$ in standard notation) and the adjoint operator * on $M_r(\mathbb{C})$. Therefore $\mathbb{R}^n \subset Cl^n_{\mathbb{C}}$ maps to skew adjoint matrices.

Proof. Cliff$(\mathbb{R}) \cong \mathbb{C}$ as real algebras, by taking the basis $\{1, e_1\}$ to $\{1, i\}$. So $Cl^1_{\mathbb{C}} \cong \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$. Similarly the maps $Cl^2_{\mathbb{C}} \to \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \to M_2(\mathbb{C})$, $\{1, e_1, e_2, e_1 e_2\} \mapsto \{1, i, j, k\} \mapsto \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \right\}$ are algebra isomorphisms, so the proposition is true for $n = 1, 2$ (a direct check verifies that $^-$ corresponds to taking adjoints in the matrix groups).

But $Cl^{n+2}_{\mathbb{C}} \cong Cl^n_{\mathbb{C}} \otimes_{\mathbb{C}} Cl^2_{\mathbb{C}}$ by taking bases $\{e_i\}_1^n$ and $\{E_1, E_2\}$ for $\mathbb{R}^n$ and $\mathbb{R}^2$ and mapping

$$\begin{aligned} e_i &\mapsto e_i \otimes i E_1 E_2 & i \le n \\ e_{n+i} &\mapsto 1 \otimes E_i & i = 1, 2, \end{aligned}$$

which satisfies the Clifford relations so defines an algebra isomorphism. Since $Cl^2_{\mathbb{C}} \cong M_2(\mathbb{C})$ and $M_r(\mathbb{C}) \otimes M_2(\mathbb{C}) \cong M_{2r}(\mathbb{C})$ the required isomorphism follows inductively. We also find, inductively, that $^-$ and * are intertwined since the isomorphisms above intertwine $^-$ on $Cl^{n+2}_{\mathbb{C}}$ with $^- \otimes ^-$ on $Cl^n_{\mathbb{C}} \otimes_{\mathbb{C}} Cl^2_{\mathbb{C}}$ with $^* \otimes {}^*$ on $M_r(\mathbb{C}) \otimes M_2(\mathbb{C})$ with * on $M_{2r}(\mathbb{C})$. □

The non-naturality of the isomorphisms leads to choices and obstructions in repeating the construction on bundles, except in the complex case, which we turn to now. Hence we shall be mainly concerned with the $n = 2m$ even dimensional case.

Definition. *A Clifford bundle on X^n ($n = 2m$) is a complex vector bundle W such that $End W \cong Cl_{\mathbb{C}}(TX)$.*

Clifford algebras are geometric objects, essentially the exterior algebra of a vector space with product the sum of the exterior and interior products. From this both the wedge and interior product of two vectors can be recovered as the symmetric and antisymmetric parts of the Clifford product. This exhibits $\Lambda\, \mathbb{R}^n \otimes \mathbb{C}$ ($\Lambda = \bigoplus_i \Lambda^i$) as a $Cl^n_{\mathbb{C}}$ module, but it is highly reducible. When, however, $n = 2m$ and $\mathbb{R}^n$ is identified with $\mathbb{C}^m$, we can pair off dimensions in $\mathbb{R}^n$ making $\Lambda\, \mathbb{C}^m$ a $Cl^n_{\mathbb{C}}$ module of lower dimension. In fact

Proposition (1.3). *End* $(\Lambda\,\mathbb{C}^m)$ *is naturally ismomorphic to Cliff* $(\mathbb{C}^m)\otimes_{\mathbb{R}}\mathbb{C}$.

Proof. For $v\in\mathbb{C}^m$ define $\varepsilon(v)w = v\wedge w$ for $w\in\Lambda\,\mathbb{C}^m$, and let $\iota(v)=\varepsilon(v)^* = v\lrcorner$ be its adjoint with respect to the standard Hermitian metric on $\Lambda\,\mathbb{C}^m$. (This is only $\mathbb{R}$-, not $\mathbb{C}$-linear, but we are treating $\mathbb{C}^m$ as a real vector space). Then if $\rho=\varepsilon-\iota$, $\rho(v)^2 w = -v\wedge(v\lrcorner w) - v\lrcorner(v\wedge w) = -\|v\|^2 w$ (e.g., check this in a unitary basis $\{\frac{v}{\|v\|^2},\dots\}$) so that ρ satisfies the Clifford relations and defines a representation $\rho:\mathrm{Cliff}\,(\mathbb{C}^m)\otimes_{\mathbb{R}}\mathbb{C}\to\mathrm{End}\,(\Lambda\,\mathbb{C}^m)$. Now a dimension count, and a check that ρ is injective, gives the result. □

Corollary (1.4). *Any almost complex manifold has canonical Clifford bundles* $\Lambda\,TX$ *and* $\Lambda\,T^*X\cong\Lambda^{0,*}X=\bigoplus_i\Lambda^{0,i}X$.

Proof. The naturality with respect to $\mathbb{C}$-linear maps of the above construction shows that $\Lambda\,TX$ is a Clifford bundle, and the $\mathbb{R}$-linear isomorphism $\mathbb{C}\to\mathbb{C}^*$ induced by the metric takes care of $\Lambda\,T^*X$. □

From now on all manifolds will be compact, connected and oriented. The existence of Clifford bundles is tied with the existence of $Spin^c$ lifts of the oriented frame bundle, and the complex case above with the lift of $U(m)$ to $Spin^c(2m)$, which we now look at. We assume (see [1]) the definitions of $Spin(n)$ (as a subgroup of $\mathrm{Cliff}^{\,n}_{\mathbb{R}}$ and double cover of $SO(n)$) and $Spin^c(n)$ (as a subgroup of $\mathrm{Cliff}^{\,n}_{\mathbb{C}}$ and as $Spin(n)\times_{\mathbb{Z}_2}U(1)$). Thus we have the exact sequences

$$1\to\mathbb{Z}_2\to Spin^c(n)\to SO(n)\times U(1)\to 1,$$
$$1\to U(1)\to Spin^c(n)\to SO(n)\to 1.$$

Proposition (1.5). *The inclusion* $i:U(m)\hookrightarrow SO(2m)$ *does not lift to* $Spin(2m)$, *but the natural map* $j=i\times det:U(m)\to SO(2m)\times U(1)$ *does lift to a map* l *to the double cover* $Spin^c(2m)$.

Proof. The exact homotopy sequence of the fibration $SO(k)\hookrightarrow SO(k+1)\to S^k$ shows that $\pi_1(SO(2m))$ is generated by the loop $\theta\mapsto\begin{pmatrix}\cos\theta & -\sin\theta & \\ \sin\theta & \cos\theta & \\ & & I_{2m-2}\end{pmatrix}$ for $m\geq 1$, which is in the image of i_* (it is $i\begin{pmatrix}e^{i\theta} & \\ & I_{m-1}\end{pmatrix}$), so $i_*\neq 0$. Then since $Spin$ is simply connected the map cannot lift.

Similarly $\pi_1(U(m))$ is generated by the loop $\theta\mapsto\begin{pmatrix}e^{i\theta} & \\ & I_{m-1}\end{pmatrix}$ whose image under $j_*=i_*\times\det{}_*$ is $(1,1)\in\pi_1(SO(2m))\times\pi_1(U(1))$ (where the 1's represent the generators).

Under $Spin(2m)\times U(1)\xrightarrow{p_1}Spin(2m)\times_{\mathbb{Z}_2}U(1)\xrightarrow{p_2}SO(2m)\times U(1)$ the path $[0,2\pi)\ni\theta\mapsto(\cos(\theta/2)+\sin(\theta/2)e_1e_2, e^{i\theta/2})$ in $Spin(2m)\times U(1)$ projects to a loop in $Spin^c(2m)$ (the endpoints (1,1),(-1,-1) are identified under p_1) which projects to the (1,1) loop $\left(\begin{pmatrix}\cos\theta & -\sin\theta & \\ \sin\theta & \cos\theta & \\ & & I_{2m-2}\end{pmatrix}, e^{i\theta}\right)$ in $SO(2m)\times U(1)$. Thus the image of j_* is contained in the image of (the induced map on π_1 of) the double cover, so j lifts to $l : U(m) \to Spin^c(2m)$. $\square$

In fact the proof shows us what the lift is; that is ([1]) take a unitary basis $\{f_j\}_1^m$ such that T is diagonal, $T = \mathrm{diag}\,\{e^{i\theta_j}\}_1^m$, and let $\{e_i\}_1^{2m}$ be the corresponding basis of $\mathbb{R}^{2m}$, i.e. $e_{2j-1} = f_j,\ e_{2j} = if_j$. Then the lift of T is given by

$$l(T) = \prod_{j=1}^{n} \left[(\cos(\theta_j/2) + \sin(\theta_j/2)e_{2j-1}e_{2j})e^{i\theta_j/2}\right].$$

By direct calculation using this formula for l, we see that the following diagram commutes

$$\begin{array}{ccc} U(m) & \xrightarrow{\ l\ } & Spin^c(2m) \\ \downarrow & & \downarrow\rho \\ \mathrm{End}\,(\mathbb{C}^m) & \xrightarrow{\ \Lambda\ } & \mathrm{End}\,(\Lambda\,\mathbb{C}^m)\,, \end{array} \tag{1.6}$$

where ρ is the restriction of the Clifford representation. More generally, if $\psi \in Spin^c(2m)$ projects under p_2 to $\phi \in SO(2m)$ then their induced actions on a fixed Clifford space W_0 (i.e. $Cl_{\mathbb{C}}^{2m} \cong \mathrm{End}\,W_0$) commute:

$$\begin{array}{ccc} \mathbb{R}^{2m} & \xrightarrow{\ \mathrm{Cliff}\ } & \mathrm{End}\,W_0 \\ \phi\downarrow & & \downarrow\mathrm{Ad}\,\psi \\ \mathbb{R}^{2m} & \xrightarrow{\ \mathrm{Cliff}\ } & \mathrm{End}\,W_0\ . \end{array} \tag{1.7}$$

(This is the content of (1.6) restricted to $\phi \in U(m) \subset SO(2m)$ and $\psi = l(\phi)$). By Schur's lemma any other ψ' making (1.7) commute must be a scalar multiple of ψ so (fixing a Hermitian metric on W_0 such that the action of $Spin^c$ is unitary) the set of unitary ψ's making (1.7) commute is precisely the $U(1)$ fibre in $Spin^c(2m)$ covering ϕ. Thus $Spin^c(2m)$ is the set of pairs

$$Spin^c(2m) \cong \{(\phi,\psi) \in SO(2m)\times U(W_0)\,\big|\,(1.7)\text{ commutes}\,\} \tag{1.8}$$

(This is used as the definition of $Spin^c$ in [5]). This description gives us an equivalent definition of the existence of a Clifford bundle (atleast on an even dimensional manifold):

Definition (1.9). *A spin^c structure on X^n is a lift of the $SO(n)$ frame bundle to a principal $Spin^c$ bundle P which looks fibrewise like $1 \to S^1 \to Spin^c(n) \to SO(n) \to 1$.*

From any spin^c structure we get a Clifford bundle $W = P\times_\rho W_0$, and, given W, P is the principal bundle with fibre at $x \in X$

$$P_x = \Big\{(\phi,\psi)\,\Big|\,\phi \in F_xTX,\ \psi : W_0 \to W_x \text{ unitary s.t.} \quad \begin{array}{ccc} \mathbb{R}^{2m} & \xrightarrow{\ \mathrm{Cliff}\ } & \mathrm{End}\,W_0 \\ \phi\downarrow & & \downarrow\mathrm{Ad}\,\psi \\ T_xX & \xrightarrow{\ \mathrm{Cliff}\ } & \mathrm{End}\,W_x \end{array} \quad \text{commutes}\Big\}. \tag{1.10}$$

It is then clear from (1.7) that these constructions are mutual inverses and that $Cl_{\mathbb{C}}(TX) \cong \mathrm{End}\,(W)$.

Now (1.6) shows again why (Corollary (1.4)) any almost complex manifold has a canonical spin^c structure (or Clifford bundle) - the unitary frame bundle lifts to a canonical $Spin^c$ bundle. Similarly the natural map $Spin(n) \to Spin(n) \times U(1) \xrightarrow{p_1} Spin^c(n)$ gives any spin manifold a canonical spin^c structure.

Definition (1.11). *The associated line bundle L of a spin^c structure is the $U(1)$ bundle arising from the projection $Spin^c(n) \to SO(n) \times U(1) \to U(1)$.*

Now if W is a Clifford bundle (with associated line bundle L), then so is $W \otimes \mu$ for any line bundle μ (with action Cliff$\otimes 1$). Let $(\tau_{\alpha\beta}, \sigma_{\alpha\beta}) \in (SO(n) \times U(1))/\mathbb{Z}_2$ and $g_{\alpha\beta} \in U(1)$ be the transition functions of W and μ on $U_\alpha \cap U_\beta$. Then since the map $Spin^c(n) \to U(1)$ is squaring on the $U(1)$ factor, the transition functions for the associated line bundles are $(\sigma^2_{\alpha\beta})$ and $(\sigma^2_{\alpha\beta} g^2_{\alpha\beta})$. So the associated line bundle of $W \otimes \mu$ is $L \otimes \mu^2$.

In the almost complex case (1.6), l lifts $i\times \det$ so the projection $Spin^c(2m) \to U(1)$ corresponds to $U(m) \xrightarrow{\det} U(1)$ on the unitary frame bundle, so the associated line bundle of the canonical spin^c structure is $L = \Lambda^n TX = K^*_X$.

Proposition (1.12). *If a spin^c structure exists on X, the set of spin^c structures is affine, modelled on the set of line bundles $H^2(X;\mathbb{Z})$ via $W \mapsto W \otimes \mu$. If $H^2(X;\mathbb{Z})$ has no 2-torsion then a spin^c structure is uniquely determined by its associated line bundle.*

Proof. The exact sequence $1 \to S^1 \xrightarrow{i} Spin^c(n) \to SO(n) \to 1$ gives the exact sequence of Cech cohomology groups

$$H^1(X; C^\infty_{S^1}) \xrightarrow{i_*} H^1(X; C^\infty_{Spin^c(n)}) \to H^1(X; C^\infty_{SO(n)}) \xrightarrow{\delta} H^2(X; C^\infty_{S^1}).$$

But the exact sequence $1 \to \mathbb{Z} \to C^\infty(\mathbb{R}) \to C^\infty(S^1) \to 1$ and the fineness of the sheaf $C^\infty(\mathbb{R})$ imply that $H^1(X; C^\infty(S^1)) \cong H^2(X;\mathbb{Z})$ and $H^2(X; C^\infty(S^1)) \cong H^3(X;\mathbb{Z})$. Thus the obstruction $\delta(FTX)$ to finding a $Spin^c(n)$ lift of the $SO(n)$ frame bundle can be taken to lie in $H^3(X;\mathbb{Z})$, and the space of spin^c structures is $H^2(X;\mathbb{Z})$ (since i_* is injective) with addition corresponding to tensoring by line bundles. So fixing a Clifford bundle W, the Clifford bundles are $W \otimes \mu$, with associated line bundles $L \otimes \mu^2$. The line bundle determines μ if there is no 2-torsion in $H^2(X;\mathbb{Z})$ since $c_1(L \otimes \mu^2) = c_1(L) + 2c_1(\mu)$ then determines $c_1(\mu)$. $\square$

(The definition of a spin^c structure is often given as a pair (P, L) where P double covers $FTX \times U(L)$, looking like $1 \to \mathbb{Z}_2 \to Spin^c(n) \to SO(n) \times U(1) \to 1$ fibrewise. This then corresponds to our definition by quotienting $H^2(X;\mathbb{Z}_2)$ by its 2-torsion).

Theorem (1.13). *A principal $SO(n)$ bundle $P \to X$ has a $Spin(n)$ lift Q if and only if $w_2(P) = 0$. Then the set of $Spin(n)$ lifts is parameterized by $H^1(X;\mathbb{Z}_2)$.*

Proof. Double covers of a space Z are in 1-1 correspondence with $H^1(Z;\mathbb{Z}_2)$. So we are looking for an element of $H^1(P;\mathbb{Z}_2)$ which restricts fibrewise to the nonzero element of $H^1(SO(n);\mathbb{Z}_2)\cong\mathbb{Z}_2$; this corresponds to the cover being the non-trivial double cover $Spin(n)\to SO(n)$ fibrewise.

We now use the Serre spectral sequence (for $\mathbb{Z}_2$ cohomology) of the fibration $SO(n)\to P\to X$ ($\pi_1 X$ has trivial action on the fibres to $\mathbb{Z}_2$ coefficients). The composition quotients for $H^1(P;\mathbb{Z}_2)$, $E_r^{0,1}$ and $E_r^{1,0}$, converge to $E_\infty^{0,1}$ and $E_\infty^{1,0}$ (in fact they are constant for $r\geq 3$), from which H^1 can be recovered by the exact sequence

$$0\to E_\infty^{1,0}\to H^1(P;\mathbb{Z}_2)\to E_\infty^{0,1}\to 0. \tag{1.14}$$

The diagram at the $E_2^{p,q}$ stage shows there can only be one differential involving these groups, call it $\delta: E_2^{0,1}\to E_2^{2,0}$. Thus $E_\infty^{1,0}=E_2^{1,0}$ and $E_\infty^{0,1}=E_3^{0,1}$, and, since the E_3's are the cohomology of the complex defined by the differentials on the E_2's, we get an exact sequence

$$0\to E_\infty^{0,1}\to E_2^{0,1}\xrightarrow{\delta} E_2^{2,0}\to E_\infty^{2,0}\to 0. \tag{1.15}$$

Splicing together (1.14) and (1.15), and using $E_2^{p,q}=H^p(X;H^q(SO(n);\mathbb{Z}_2))$, gives

$$0\to H^1(X;\mathbb{Z}_2)\to H^1(P;\mathbb{Z}_2)\to H^1(SO(n);\mathbb{Z}_2)\xrightarrow{\delta} H^2(X;\mathbb{Z}_2), \tag{1.16}$$

since $H^0(SO(n);\mathbb{Z}_2)\cong\mathbb{Z}_2\cong H^0(X;\mathbb{Z}_2)\cong H^1(SO(n);\mathbb{Z}_2)$. As the third arrow is the restriction map, the lift exists if and only if the generator $1\in H^1(SO(n);\mathbb{Z}_2)$ is in its image, which occurs if and only if $\delta(1)=0$. So it is sufficient to show that $\delta(1)=w_2(P)$.

This is perhaps most easily seen by looking at the associated homology sequence, dual to (1.16) in that the following pairings commute:

$$\begin{array}{ccc} H^1(SO(n);\mathbb{Z}_2) & \xrightarrow{\delta} & H^2(X;\mathbb{Z}_2) \\ \otimes & & \otimes \\ H_1(SO(n);\mathbb{Z}_2) & \xleftarrow{d} & H_2(X;\mathbb{Z}_2) \\ \downarrow & & \downarrow \\ \mathbb{Z}_2 & & \mathbb{Z}_2 \end{array}.$$

Given $s\in H^2(X;\mathbb{Z}_2)$, we can describe $\langle w_2(P),s\rangle$ as follows (see the obstruction theory definition of w_2 in [6]): Express s as a sum of simplices $s=\sum_i\sigma_i$ in some triangulation of X, so that $d(s)=\sum_i d\sigma_i$. $SO(n)$ is connected so we can lift the 1-skeleton of X to P. P is trivial over each (contractible) σ_i so the lift of $\partial\sigma_i$ defines a loop in $SO(n)$, giving it a value 0 or 1 in $\pi_1(SO(n))\cong\mathbb{Z}_2$. Then $\langle w_2(P),s\rangle$ is the sum (mod 2) of these values, which is also equal to $1(ds)$ (1 is the generator of $H^1(SO(n);\mathbb{Z}_2)$) since, from the homology spectral sequence, ds is precisely the obstruction in $H^1(SO(n);\mathbb{Z}_2)$ of lifting s to P.

Thus $w_2(P)=d^*(1)=\delta(1)$ since they take the same values on H_2, and H^2 is the $\mathbb{Z}_2$-vector space dual of H_2. (1.16) also shows that the set of all spin^c structures, when $w_2=0$, is parameterised by $H^1(X;\mathbb{Z}_2)$. □

Proposition (1.17). *X has a spin^c structure if and only if $w_2(X)$ has an integral lift to $H^2(X;\mathbb{Z})$, i.e. if and only if there is a line bundle L such that $w_2(X) = c_1(L)$ mod 2.*

Proof. As noted after the proof of (1.12), the existence of a spin^c structure is equivalent to the existence of a non trivial double cover of $FTX \times U(L)$ for some line bundle L. Elementary topology of covering spaces shows the restriction of the double cover $Spin(n+2) \to SO(n+2)$ to $SO(n) \times U(1) \subset SO(n+2)$ is isomorphic to $Spin^c(n) \to SO(n) \times U(1)$. Thus a $Spin^c$ lift P of $FTX \times U(L)$ is equivalent to a $Spin(n+2)$ lift Q of $F(T \oplus L)$: P induces Q via $Spin^c(n) \hookrightarrow Spin(n+2)$ and restricting Q to the double cover of $FTX \times U(L) \subset F(TX \oplus L)$ gives P.

Thus a lift exists if and only if, for some L, $w_2(TX \oplus L) = 0 = w_2(X) + w_2(L)$ (since $w_1(L) = 0$), i.e. if and only if $w_2(X) = c_1(L)$ mod 2. □

Of course on an almost complex manifold K_X^*, K_X provide just such lifts, giving the spin^c structures of (1.4).

Dirac operators. From now on X will be even ($n = 2m$) dimensional with a spin^c structure (P, W, L), and we shall often omit mention of the representation $\rho : Cl_{\mathbb{C}}(TX) \to \operatorname{End} W$, simply denoting Clifford multiplication by a dot. Picking an orthonormal basis $\{e_i\}_1^n$ for T_xX, the element $\tau = i^{\frac{n(n+1)}{2}} e_1 \dots e_n$ of $Cl_{\mathbb{C}}(T_xX)$ commutes with even elements of $Cl_{\mathbb{C}}(T_xX)$, anticommutes with $T_xX \otimes \mathbb{C}$, and has square $+1$, so splits $W_x \cong W_x^+ \oplus W_x^-$ into ± 1 eigenspaces. The splitting (and indeed τ) is independent of basis so W splits globally into two spinor bundles W^+ and W^-, both coming from representations (in fact irreducible) of $Spin^c(n)$ (even elements of $Cl_{\mathbb{C}}$), and interchanged by Clifford multiplication by TX. In the almost complex case, $W = \Lambda^{0,*}X \otimes \mu$, $W^+ = \Lambda^{0,\mathrm{ev}}X \otimes \mu$, $W^- = \Lambda^{0,\mathrm{odd}}X \otimes \mu$.

Using the double cover $P \to FTX \times U(L)$ a connection on $FTX \times U(L)$ pulls back to one on P since the Lie algebras are the same. Conversely, we can lift tangent vectors to $FTX \times U(L)$ to one of two tangent vectors to P, and a connection form's value will be the same on both due to its equivariance. So we have a form on $FTX \times U(L)$ which, with a bit of checking, defines a connection, and the two procedures are mutual inverses. Similarly a connection on $FTX \times U(L)$ is equivalent to connections on each of FTX and $U(L)$, so we can make the following definition.

Definition (1.18). *A connection in W is compatible if and only if the associated FTX connection is the Levi-Civita connection. (Thus compatible connections exist and are in 1-1 correspondence with unitary connections A on L).*

Proposition (1.19). *A connection B in W is compatible if and only if ρ is parallel.*

Proof. Given a path γ in X, lift to a B-horizontal path in ψ in P, and project this to a B-horizontal path ϕ in FTX. Also take a ∇_{LC}-parallel vector field v down γ. We want to show that $\rho(v)\psi$ is horizontal if and only if ϕ is ∇_{LC}-parallel, i.e. if and only if $\phi^{-1}(v) \in \mathbb{R}^n$ is a constant, for all such v.

Considering, as in (1.10), ϕ and ψ to be compatible frames at each point, $\phi_x : \mathbb{R}^n \to T_xX$, $\psi_x : W_0 \to W_x$ intertwining the two Clifford actions, then the action of $\rho(v)$ on the frame ψ is given by $\rho(v).\psi = \psi \circ \rho(\phi^{-1}(v)) : W_0 \to W_x$. This is horizontal if and only if $\rho(\phi^{-1}(v))$ is a constant endomorphism of W_0, i.e. if and only if $\phi^{-1}(v)$ is constant. □

Definition (1.20). *The Dirac operator D_A associated to a connection A on L is given by the composition*

$$\Gamma(W) \xrightarrow{\nabla_B} \Gamma(T^*X \otimes W) \xrightarrow{\text{metric}} \Gamma(TX \otimes W) \xrightarrow{\rho} \Gamma(W).$$

Since $\rho\big|_{TX}$ switches $W^\pm$, so does $D_A : \Gamma(W^\pm) \to \Gamma(W^\mp)$.

Proposition (1.21). *D_A is self adjoint.*

Proof. As in (1.10), any frame $P_x \ni \psi : W_0 \to W_x$ preserves the metric, and, by (1.1) the action of $\mathbb{R}^n$ on W_0 is skew adjoint, so the action of TX on W is skew adjoint. Working in a synchronous frame about $x \in X$, we have an orthonormal frame field $\{e_i\}_1^n$ satisfying $\nabla_i e_j = 0$, $[e_i, e_j] = \nabla_i e_j - \nabla_j e_i = 0$, $\forall i, j$, at x. So, at x,

$$\begin{aligned}
\langle D_A s, t\rangle &= \sum_i \langle e_i.\nabla_i s, t\rangle \\
&= \sum_i \langle \nabla_i e_i.s, t\rangle && \text{since } \rho \text{ is parallel and } \nabla_i e_i = 0, \\
&= \sum_i \nabla_i \langle e_i.s, t\rangle - \sum_i \langle e_i.s, \nabla_i t\rangle \\
&= \sum_i \nabla_i \omega(e_i) + \sum_i \langle s, e_i.\nabla_i t\rangle && \text{where } \omega \text{ is the 1-form } \omega(X) = \langle X.s, t\rangle, \\
&= -d^*\omega + \langle s, D_A t\rangle.
\end{aligned}$$

Integrating over X now gives the result. □

Theorem (1.22). *For X Kähler , with Clifford bundle $\Lambda^{0,*}X \otimes \mu$, and connection A in $L = K_X^* \otimes \mu^2$ induced by the Levi-Civita connection and B in μ, the Dirac operator is $D_A = \sqrt{2}\,(\bar\partial_B + \bar\partial_B^*)$.*

(Note: From now on the above notation for the spin^c structure and connections on a Kähler manifold will be standard. In particular, any connection A will be a unitary connection in $L = K_X^* \otimes \mu^2$, associated to a connection B in μ via ∇_{LC}).

Proof. For simplicity we work with $\Lambda^{0,*}X$; coupling everything to μ makes little difference.

Recall from (1.13),(1.14) the canonical spin^c structure arises from the action of $\mathbb{C}^m$ on $\Lambda\,\mathbb{C}^m$ given by $\rho = \varepsilon - \iota$, where ε is exterior multiplication $\wedge$ and ι is its adjoint $\lrcorner$, given by $\iota(v) = \bar{*}\varepsilon(v)\bar{*}$. This gives us an action of $T^{0,1}X$ on $\Lambda^{0,*}X$, so by the isometry

$$TX \xrightarrow{\text{metric}} T^*X \hookrightarrow T^*X \otimes \mathbb{C} \xrightarrow{\sqrt{2}\,\pi^{0,1}} T^{0,1}X,$$

we have the action of $v \in TX \cong T^*X$ given by $w \mapsto \sqrt{2}\,\pi^{0,*}(v \wedge w - v \lrcorner w)$ for $w \in \Lambda^{0,*}X$. Thus our representation is given by $\rho = \sqrt{2}\,\pi^{0,*}\,(\varepsilon - \iota)$.

For any torsion free connection ∇ on TX, the induced connection ∇ on the bundle $\Lambda = \Lambda T^*X$ has the property that the composition $\Lambda \xrightarrow{\nabla} T^*X \otimes \Lambda \xrightarrow{\wedge} \Lambda$ is the exterior derivative d. So $\pi^{0,*} \circ \varepsilon \circ \nabla = \pi^{0,*} \circ d = \bar{\partial}$, and similarly $\pi^{0,*} \circ \iota \circ \nabla = \pi^{0,*}\,\bar{*}\varepsilon\bar{*}\nabla = \bar{*}\pi^{n,*}\,\varepsilon\nabla\bar{*}$ (since $\bar{*}$ is parallel), and this equals $\bar{*}\pi^{n,*}\,d\bar{*} = \bar{*}\bar{\partial}\bar{*} = -\bar{\partial}^*$.

From these two formulae, $\rho \circ \nabla = \sqrt{2}\,(\bar{\partial} + \bar{\partial}^*)$, and taking ∇ to be the (torsion free) Kähler connection proves the theorem. □

Recalling the vector space isomorphism $\Lambda V \cong Cl(V),\ v \wedge w \mapsto \frac{1}{2}(v.w - w.v)$, it is natural to extend ρ to all of ΛV, and in particular to 2-forms, by $\rho(v \wedge w) = \frac{1}{2}(\rho(v)\rho(w) - \rho(w)\rho(v))$. This defines $\rho(F_A)$ for a unitary connection A on L as $F_A \in \Omega^2(i\mathbb{R})$, and we can finally give the Seiberg-Witten equations *on a 4-manifold.*

The Seiberg-Witten equations. The equations, for a section $\Phi \in \Gamma(W^+)$ and a unitary connection A on L, are

$$\begin{cases} D_A\Phi = 0, \\ \rho(F_A^+) = (\Phi\Phi^*)_0. \end{cases} \tag{1.23}$$

Here $(\Phi\Phi^*)_0 \in \operatorname{End} W$ denotes the trace free part of the endomorphism $\psi \mapsto \langle\psi, \Phi\rangle\Phi$.

As with the ASD equations, the Seiberg-Witten equations can be derived as the minima of a functional. There is no space to go into it here, but with some Chern-Weil theory and a Weitzenböck formula, the functional reduces to $\frac{1}{2}\|\rho(F_A^+) - (\Phi\Phi^*)_0\|^2 + 2\|D_A\Phi\|^2$+const, and we do use a functional of this form later.

On a Kähler manifold we write $\Phi = (\alpha, \beta) \in \Omega^0(\mu) \times \Omega^{0,2}(\mu) = \Gamma(W^+)$. Theorem (1.22) then implies that the first equation of (1.23) is $\bar{\partial}_B\alpha + \bar{\partial}_B^*\beta = 0$, and a simple calculation shows the second becomes $F_A^{0,2} = \bar{\alpha}\beta,\ i\Lambda F_A = -\frac{1}{2}(|\alpha|^2 - |\beta|^2)$.

2. The Equations

In three complex dimensions we again consider the Clifford bundle $W^+ = \Lambda^{0,\mathrm{ev}} \otimes \mu$, and the equation $D_A\Phi = 0$ for $\Phi \in \Gamma(W^+)$. Writing $\Phi = (\alpha, \beta),\ \alpha \in \Omega^0(\mu),\ \beta \in \Omega^{0,2}(\mu)$, this becomes $\bar{\partial}_B\alpha + \bar{\partial}_B^*\beta = 0,\ \bar{\partial}_B\beta = 0$; notice there is now an additional equation in this higher dimensional case.

So the equations we consider on a Kähler 3-manifold are for a unitary connection A on $L = K_X^* \otimes \mu^2$ corresponding to B on μ, and sections $\alpha \in \Omega^0(\mu),\ \beta \in \Omega^{0,2}(\mu)$:

$$\begin{aligned} &\text{(i)} && \bar\partial_B \alpha + \bar\partial_B^* \beta = 0 \\ &\text{(ii)} && \bar\partial_B \beta = 0 \\ &\text{(iii)} && F_A^{0,2} = \bar\alpha\beta \\ &\text{(iv)} && i\Lambda F_A = -\frac{1}{2}(|\alpha|^2 - |\beta|^2) \end{aligned} \qquad (SW)$$

Notice these equations make sense on a Kähler surface too, where they reduce to (1.23), and many of the results in this section apply to both two and three complex dimensions.

The Gauge Group. The natural symmetry group of the pair (W, L) is the gauge group $\mathcal{G} = \Gamma(U(1) \times X)$, the smooth $U(1)$-valued functions acting on W by multiplication and on L with weight two, for the reasons noted above. On a Kähler manifold $L = K_X^* \otimes \mu^2$, and $\mathcal{G}$ acts on μ with weight one. Therefore $g \in \mathcal{G}$ acts by

$$\alpha \mapsto g.\alpha,\ \ \beta \mapsto g.\beta,\ \ B \mapsto B - g^{-1}dg,\ \ A \mapsto A - 2g^{-1}dg,$$

leaving the equations (SW) unaltered. We also let $\mathcal{G}^c$ denote the complexification of $\mathcal{G}$, $\mathcal{G}^c = \Gamma(\mathbb{C}^* \times X)$, and $\mathcal{M}$ the moduli space $\mathcal{M} = \{\text{Solutions of (SW)}\}/\mathcal{G}$.

We now note some simple consequences of these equations applying equally in two or three dimensions.

Proposition (2.1). *Any solution of (SW) has one of α and β identically zero.*

(a) If $\deg L = 0$, then $\alpha \equiv 0 \equiv \beta$ and
$$\begin{cases} F_A^{0,2} = 0, \\ i\Lambda F_A = 0. \end{cases} \qquad (SW0)$$

(b) If $\deg L < 0$, then $\beta \equiv 0$ and
$$\begin{cases} \bar\partial_B \alpha = 0, \\ F_A^{0,2} = 0, \\ i\Lambda F_A = -\frac{1}{2}|\alpha|^2. \end{cases} \qquad (SW-)$$

(c) If $\deg L > 0$, then $\alpha \equiv 0$ and
$$\begin{cases} \Delta_B \beta = 0, \\ F_A^{0,2} = 0, \\ i\Lambda F_A = \frac{1}{2}|\beta|^2, \end{cases} \qquad (SW+)$$

where Δ_B is the $\bar\partial_B$-Laplacian on $\Omega^{0,2}(\mu)$.

(Note: In each case $F_B^{0,2} = \frac{1}{2}F_A^{0,2} = 0$ so A and B define holomorphic structures on L and μ respectively, compatible via $L = K_X^* \otimes \mu^2$, where K_X^* is holomorphic).

Proof. Applying $\bar\partial_B$ to (SW)(i) gives $F_B^{0,2}\alpha + \bar\partial_B\bar\partial_B^*\beta = 0$, but $F_B^{0,2} = \frac{1}{2}\bar\alpha\beta$ from (iii), so $\frac{1}{2}|\alpha|^2\beta + \bar\partial_B\bar\partial_B^*\beta = 0$. Taking the L^2 inner product with β implies that

$\bar{\alpha}\beta \equiv 0 \equiv \bar{\partial}_B^*\beta$, so (ii) implies that $\bar{\partial}_B\alpha = 0$. If β is not identically zero then α must be zero on some open subset of X, so is zero everywhere by unique analytic continuation ($\bar{\partial}_B\alpha = 0$ and X is connected).

Therefore we are reduced to the equations

$$\begin{aligned} \bar{\partial}_B\alpha &= 0 = \bar{\partial}_B\beta = \bar{\partial}_B^*\beta, \\ F_A^{0,2} &= 0, \\ i\Lambda F_A &= -\frac{1}{2}(|\alpha|^2 - |\beta|^2), \end{aligned}$$

with one of α and β identically zero. But $\deg L = \frac{i}{2\pi}\int F_A \wedge \omega^{m-1} = \frac{i}{2\pi}\int F_A . \omega \frac{\omega^m}{m} = -\frac{1}{4\pi m}\int (|\alpha|^2 - |\beta|^2)\omega^m$, where $|\omega|^2 = m = \dim_{\mathbb{C}} X = 2$ or 3.

Thus the sign of $\deg L$ determines which of α or β vanishes, giving the three cases above. □

Theorem (2.2). *If $\deg L = 0$ the moduli space of solutions $\mathcal{M}$ of (SW) can be identified with the set of distinct holomorphic structures on L, i.e. the Jacobian $H^1(X;\mathbb{R})/H^1(X;\mathbb{Z})$.*

Proof. This is a very easy form of the existence and uniqueness of Hermitian Yang-Mills connections on stable holomorphic bundles ([2,7]). Given a holomorphic structure on L (a unitary connection A with $F_A^{0,2} = 0$, say), the $\mathcal{G}^c$ action of $g = e^f$ on A gives an isomorphic holomorphic structure $\bar{\partial}_{g(A)} = g \circ \bar{\partial}_A \circ g^{-1}$ with $\partial_{g(A)}$ chosen to make $g(A)$ unitary. Its curvature is

$$F_{g(A)} = F_A + \partial\bar{\partial}f - \bar{\partial}(\overline{\bar{\partial}f}) = F_A + 2\partial\bar{\partial}(\mathrm{Re}\, f)$$

so that

$$i\Lambda F_{g(A)} = i\Lambda F_A - \Delta u, \quad u = 2\mathrm{Re}\, f.$$

Thus we want to solve $\Delta u = i\Lambda F_A$ for real-valued smooth u, but $i\Lambda F_A$ is real as A is unitary, and orthogonal to the kernel of Δ since $\int i\Lambda F_A \, d\mu = \mathrm{const.} \deg L = 0$. The Fredholm alternative now proves existence.

Since the set of holomorphic structures on L isomorphic to $\bar{\partial}_A$ is the $\mathcal{G}^c$ orbit of A, the uniqueness result we want is that is A and $g(A)$ both satisfy (SW0) then they are $\mathcal{G}$-related, i.e. $\exists h \in \mathcal{G}$ such that $g(A) = h(A)$.

The trivial bundle $\mathrm{End}\, L \cong L^* \otimes L$ has a connection $C = A^* \otimes g(A)$ (in the obvious notation) defining a holomorphic structure. Its curvature satisfies $F_C\phi = F_{g(A)} \circ \phi - \phi \circ F_A$ so $i\Lambda F_A = 0$ implies $i\Lambda F_C = 0$. So by the Kähler identities (see [3] or Section 4 below) the $\bar{\partial}_C$-Laplacian equals half the ∇_C-Laplacian. Thus the holomorphic section $g \in \mathrm{End}\, L$ satisfies $\nabla_C g = 0$ (a priori we only knew $\bar{\partial}_C g = 0$ since only $\bar{\partial}_{g(A)}$, and not $\partial_{g(A)}$, was formed by conjugating A with g). Letting $h = (\bar{g}g)^{-\frac{1}{2}}g$, h is unitary and $\nabla_C h = 0$ so h provides a gauge transformation between A and $g(A)$, as required. □

A note on notation. The above proof used the fact that a $\mathcal{G}^c$ orbit of a holomorphic connection is the set of all equivalent holomorphic structures on a unitary bundle. Equivalently we could consider only the $\mathcal{G}$ orbit of a $\bar{\partial}$-operator and vary the metric on L; the result amounts to the fact that a holomorphic structure and Hermitian metric uniquely determine a connection. These three different points of view are entirely equivalent for the full Seiberg-Witten equations (SW), under the correspondence

$$\left\{\begin{array}{l}\text{Distinct}\\ \text{Holomorphic}\\ \text{Structures}\end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l}\mathcal{G}^c \text{ orbits}\\ (g\alpha, g\beta, g(A))\\ \text{Fixed metric } |\,.\,|\end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l}\mathcal{G} \text{ orbits } \forall \text{ metrics}\\ g\in\mathcal{G}^c \ \sim \begin{array}{l}(h\alpha, h\beta, h(A))\\ \text{Metric } (\bar{g}g)^{\frac{1}{2}}|\,.\,|\end{array}\\ \text{where } h=(\bar{g}g)^{-\frac{1}{2}}g\in\mathcal{G}.\end{array}\right\}$$

In what follows we shall feel free to vary the metric to find solutions, and interpret the results in terms of isomorphic holomorphic structures and the fixed metric used in the equations.

The $\deg L < 0$ case. While the equations (SW0) suggest the set of holomorphic structures on L or μ, the equations (SW$-$) suggest pairs (holomorphic structure $\bar{\partial}_B,\ \alpha \in H^0_B(\mu)$) and (SW+) suggests pairs $(\bar{\partial}_B, \beta \in H^2_B(\mu))$. We consider first the (SW$-$) case $\deg L < 0$.

So we fix a holomorphic structure $\bar{\partial}_B$ on μ and a holomorphic section $\alpha \in H^0_B(\mu)$. We solve for the metric $|\,.\,| = e^u|\,.\,|_0$ so that the induced unitary connection $B = B_0 + 2\partial u$ (where B_0 is the connection induced by $|\,.\,|_0$) satisfies (SW$-$).

$F_B = F_{B_0} + 2\bar{\partial}\partial u$, and (SW$-$) reduces to solving

$$i\Lambda F_A = i\Lambda F_{A_0} + 4i\Lambda\bar{\partial}\partial u = -\frac{1}{2}|\alpha|^2 = -\frac{1}{2}e^{2u}|\alpha|_0^2,$$

that is,

$$2i\Lambda\bar{\partial}\partial u + \frac{1}{4}|\alpha|_0^2 e^{2u} = -\frac{1}{2}i\Lambda F_{A_0},$$

or,

$$\Delta u + be^{2u} = a. \tag{2.3}$$

This is the vortex equation with $b = \frac{1}{4}|\alpha|_0^2$ nonnegative, and strictly postive somewhere since $\deg L \neq 0$; and $\int a\, d\mu > 0$. Here Δ is the standard d-Laplacian.

Theorem (2.4). *The vortex equation (2.3) has a unique C^∞ solution u.*

Proof. We fill out the sketch given in [5]. Letting $\bar{a}$ be the average of a, we can solve $\Delta v = a - \bar{a}$. Then (2.3) is the equation $\Delta\tilde{u} + Be^{2\tilde{u}} = \bar{a}$ for $\tilde{u} = u - v$, where $B = be^{2v}$ satisfies the same conditions as b. So we may assume that *a is a positive constant* in (2.3).

Lemma (2.5). *There exist functions $u_\pm$ such that $\Delta u_\pm + be^{2u_\pm} \gtrless a$, and any solution u of (2.3) satisfies $u_- < u < u_+$. We can replace $<$ by $\leq$ and $>$ by $\geq$ throughout.*

Proof. We do the strict inequality case, the other is identical. For u_- we may take a sufficiently negative constant. Take a function f strictly positive where $b = 0$ such that $\int f = 0$, and solve $\Delta v = f$. Then let $u_+ = Av + B$ for constants A and B chosen such that $\Delta u_+ + be^{2u_+} = Af + be^{2(Av+B)} \geq a$.

For any solution u of (2.3) set $w = u_+ - u$. Then $\Delta w + be^{2u}(e^{2w} - 1) > 0$ so at a minimum of w, where $\Delta w \leq 0$, $(e^{2w} - 1) > 0$ so $w > 0$ and $u < u_+$. Similarly $u > u_-$. □

Given a solution u, taking non-strict inequalities in (2.5) we may let $u_+ = u = u_-$, proving uniqueness.

To prove existence we use a continuity method and the a priori bounds given by the strict inequality case of (2.5). Take a C^∞ family of functions a_t, $t \in [0,1]$, with $a_0 = b$, $a_1 = a$ and $a_t > 0\ \forall t$. We show the set

$$\mathcal{S} = \left\{ t \in [0,1] \,\middle|\, \exists u_t \in C^\infty \text{ s.t. } \Delta u_t + be^{2u_t} = a_t \right\}$$

is both open and closed. Then it contains $t = 0$, corresponding to $u_0 = 0$, so it contains $t = 1$ giving a solution $u = u_1$ to (2.3).

Closed: Suppose $\mathcal{S} \ni t_i \to t \neq 0$. Choosing $u_\pm$ to satisfy $\Delta u_\pm + be^{2u_\pm} \gtrless a_t$ then for sufficiently large i the inequalities hold for $a_i = a_{t_i}$ too, so $u_- < u_i = u_{t_i} < u_+$. Thus, taking a subsequence if necessary, by the Arzela-Ascoli theorem there is a C^0 limit u_∞.

Therefore $\Delta u_i = a_i - be^{2u_i}$ is also C^0 convergent to a function which is L^2 orthogonal to the constants, so may be written $\Delta \tilde{u}_\infty = a_t - be^{2u_\infty}$, $\tilde{u}_\infty \in L^2_2$.

Now $\Delta u_i \xrightarrow{C^0} \Delta \tilde{u}_\infty \Rightarrow \Delta u_i \xrightarrow{L_2} \Delta \tilde{u}_\infty$, but $\Delta : L^2_2 \to L_2$ is invertible on $L^2 / \{\text{constants}\}$ so $u_i \xrightarrow{L^2_2} \tilde{u}_\infty$ modulo constants. But u_i tends to u_∞ in C^0, and so in L^2, so $\tilde{u}_\infty = u_\infty$ (by adding a constant to $\tilde{u}_\infty$ if necessary).

Hence we have found $u_\infty \in L^2_2 \cap C^0$ such that $\Delta u_\infty = a_t - be^{2u_\infty}$. Inductively suppose that $u_\infty \in L^2_k$, then we want to show $e^{2u_\infty} \in L^2_k$ as well. But it is sufficient to show that $de^{2u_\infty} = e^{2u_\infty} du_\infty \in L^2_{k-1}$, which it is since e^{2u_∞} is continuous. Therefore $\Delta u_\infty \in L^2_k$ and $u_\infty \in L^2_{k+2}$. Inductively, then, $u_\infty \in \bigcap_k L^2_k = C^\infty$.

Open: For k sufficiently large that $L^2_k \hookrightarrow C^0$, the map $u \mapsto \Delta u + be^{2u}$ is bounded $L^2_k \to L^2_{k-2}$ (we effectively showed above that $u \mapsto e^{2u}$ is continuous $L^2_k \to L^2_k$), and in fact differentiable with derivative $v \mapsto \Delta v + 2be^{2u}v$. This is elliptic, self adjoint, and has trivial kernel: if $v \in \ker$ then taking the inner product with v shows $\|dv\|^2 + 2\int be^{2u}|v|^2\, d\mu = 0$ so $v \equiv 0$ as b is nonnegative and not identically zero. Thus the derivative is also onto and the solution set is open by the inverse function theorem. (Any solution is in L^2_k for all sufficiently large k, so it is C^∞). □

Interpreted in terms of the Seiberg-Witten equations for a fixed metric, we have a solution, unique modulo $\mathcal{G}$, for a holomorphic structure isomorphic to $\bar{\partial}_B$, and a unique section which corresponds to a multiple of α under the isomorphism taking the holomorphic structure to $\bar{\partial}_B$. Thus we have proved

Theorem (2.6). *If* $\deg L < 0$ *the moduli space of solutions of (SW) can be indentified with* $\bigcup_B \mathbb{P}(H^0_B(\mu))$*, where* B *runs over the set of all distinct isomorphism classes of holomorphic structures* $\bar{\partial}_B$ *on* μ*.*

Of course $\mathbb{P}(H^0_B(\mu))$ is the set of divisors representing $(\mu, \bar{\partial}_B)$ as two holomorphic sections have the same zero set if and only if they are constant multiples of each other, so we have

Theorem (2.7). *If* $\deg L < 0$ *the moduli space of solutions of (SW) is isomorphic to the set of analytic hypersurfaces in* X *in the homology class Poincaré dual to* $c_1(\mu)$*.*

This is a remarkable result on a Kähler surface since the original Seiberg-Witten equations on a 4-manifold make no reference to complex structure, yet contain information about the holomorphic curves in homology classes.

The $\deg L > 0$ case, and symplectic quotients. From now on we shall be concerned with the equations (SW+)

$$\begin{aligned} \Delta_B \beta &= 0, \\ F^{0,2}_A &= 0, \\ i\Lambda F_A &= \frac{1}{2}|\beta|^2. \end{aligned}$$

So we have a natural generalisation of the $\deg L < 0$ case, looking for distinguished representatives of a cohomology group (or, equivalently, special metrics on a line bundle). The natural conjecture, given Theorem (2.6), is that we will find $\mathcal{M} \equiv \bigcup_B \mathbb{P}(H^2_B(\mu))$. In fact on a Kähler surface $\Lambda^{0,2} \otimes \mu$ is a line bundle, 'Serre dual' to $\mu^* \otimes K_X$, so it is easy to convert the equations into $\deg L < 0$ form (see [5]) and prove the conjecture true, since by Serre duality, $H^0_B(\mu^* \otimes K_X) \cong H^2_B(\mu)^*$. On a Kähler 3-manifold, however, $\Lambda^{0,2}$ is a rank 3 bundle, making the equations far harder to solve. Even singling out an element of cohomology requires the equation $\Delta_B \beta = 0$ instead of just $\bar{\partial}_B \beta = 0$.

This situation is, however, a familiar one in gauge theory. In Yang-Mills theory we do not have a section, but look for special metrics with ASD curvature (atleast on holomorphic bundles); while equations such as those of Hitchin [4] are similar to ours except that the holomorphic section is a section of an endomorphism bundle. In all of these cases we can expect an existence and uniqueness result, atleast for 'generic' bundles and sections - those satisfying a stability condition. That there was no such condition for (SW−) corresponds to the fact that stability conditions are usually trivially satisfied by line bundles.

The stability conditions arise most naturally when the moduli problem is expressed in terms of (infinite dimensional) symplectic geometry. Stability is then what we need to form a good quotient by the gauge group, in analogy with the finite dimensional case, which we describe briefly now.

We consider a symplectic manifold (M, ω) with compatible metric and almost complex structure I (so $\omega(X, Y) = \langle X, IY\rangle$) with a Lie group G acting on M preserving ω. Under certain mild conditions there exists a unique moment map $m : M \to \mathfrak{g}^*$ to the dual of the Lie algebra of G, satisfying

(i) m is equivariant under the action of G on M and the coadjoint action on $\mathfrak{g}^*$,

(ii) $dm(\xi) = v_\xi \lrcorner \omega, \quad \forall \xi \in \mathfrak{g}$,

$$(2.8)$$

where v_ξ is the vector field induced on M by the action of G along ξ. [The notation comes from mechanics, which also gives a large supply of cases where m exists. Here $M = T^*N$ is the phase space of a configuration space N, with G acting on N. There is a natural symplectic form $\omega = d\theta$ on M, where θ is the tautological 1-form on T^*N. By naturality, then, G preserves ω. Then the moment map at $x \in M$ is the pullback of θ, under G's action on x, to $e \in G$. Thus its value on $\xi \in \mathfrak{g}$ is the generalised momentum $\theta(v_\xi)$ of ξ's action at a point of phase space M.]

The stabilizer G_ζ of $\zeta \in \mathfrak{g}^*$ then acts on $m^{-1}(\zeta)$ and in the generic case the quotient is a manifold. It is in fact symplectic as ω is G-invariant so passes down to a form on $m^{-1}(\zeta)\big/G_\zeta$ which is nondegenerate as ω's degeneracy on $m^{-1}(\zeta)$ is precisely along G_ζ orbits, by (2.8)(ii). In particular we can hope that $m^{-1}(0)\big/G$ is a symplectic manifold.

The action of G extends to G^c by $v_{i\xi} = Iv_\xi, \ \forall i\xi \in i\mathfrak{g}$. The gradient flow of $\|m\|^2$ is contained in a G^c orbit: using an Ad-invariant inner product $(.\,,.)$ on $\mathfrak{g}$ to identify $\mathfrak{g}$ with $\mathfrak{g}^*$, we have

$$\langle v, \mathrm{grad}_x \|m\|^2\rangle = 2(dm_x(v), m_x) = 2\omega(v_{m(x)}, v)$$

from (2.8)(ii). But this equals $\langle 2Iv_{m(x)}, v\rangle$ so grad $\|m\|^2(x) = 2v_{im(x)}$. In fact, for a 'semistable' point x of M, the gradient flow of $\|m\|^2$ converges inside $G^c x$ to a point of $m^{-1}(0)$, and Gx contains all the points of $m^{-1}(0)$ in $G^c x$. Thus we can identify $\{\text{Semistable points}\}\big/G^c$ with the symplectic manifold $m^{-1}(0)/G$.

The above description has been deliberately vague because it is infinite dimensions that interests us where precise theorems are not available, except in specific instances such as the Yang-Mills equations. Here, the space $\mathcal{A}^{1,1}$ of unitary (1,1) connections on a bundle E on a Kähler n-manifold has a natural symplectic (in fact Kähler) structure given by $\Omega(a, b) = \int a \wedge b \wedge \omega^{n-1}$, where $a, b \in \Omega^1(\mathfrak{g}_E)$ are tangent to $\mathcal{A}^{1,1}$, ω is the Kähler form, and we are using an Ad-invariant inner product to contract a and b. Ω is invariant under the guage group $\mathcal{G} = \Gamma(\mathcal{G}_E)$, and there is a moment map $m(A) = F_A \wedge \omega^{n-1}$ (where $\Omega^{2n}(\mathfrak{g}_E)$ lies in $\mathfrak{g}^*$ by contraction and integration). Thus $m^{-1}(0)\big/\mathcal{G}$ is the moduli space of isomorphism classes of Yang-Mills connections satisfying $F_A^{0,2} = 0 = \Lambda F_A^{1,1}$. Here the correct notion of stability is the usual one of stability of a holomorphic bundle so we might expect the stable quotient $(\mathcal{A}^{1,1})^s\big/\mathcal{G}^c$ to equal $m^{-1}(0)\big/\mathcal{G}$. As a $\mathcal{G}^c$ orbit of a holomorphic connection gives all isomorphic connections corresponding to different metrics, this amounts to saying that every stable holomorphic bundle admits a metric of ASD curvature, unique modulo $\mathcal{G}$. This is proved true in [2] for $SU(n)$ bundles precisely by solving for the gradient flow of $\|m\|^2$ - on stable orbits it converges to an ASD connection.

If $c_1(E) \neq 0$ clearly $m^{-1}(0) = \emptyset$, but we can consider $m^{-1}(\lambda I \mathrm{vol})$ (preserved by $\mathcal{G}$) and get a corresponding result for Hermitian Yang-Mills connections on any Kähler manifold, as proved in [7].

Similarly Hitchin's equations can be cast into moment map form, as can most of (SW). We have the symplectic forms:

(i) On $\mathcal{A}^{1,1}(L)$, $\Omega_{\mathcal{A}}(a,b) = \int a \wedge b \wedge \omega^{n-1}$ for $a, b \in \Omega^1(\mathfrak{g}_L) = \Omega^1(i\mathbb{R})$, with moment map $m(A) = 2F_A \wedge \omega^{n-1}$, the 2 arising from $\mathcal{G}$'s weight 2 action on L.

(ii) On $\Omega^0(\mu)$, $\Omega_0(s,t) = \int \mathrm{Re}\langle s, it\rangle \omega^n$ for $s, t \in \Omega^0(\mu)$ with moment map $m(s) = \frac{i}{2}|s|^2\omega^n$.

(iii) On $\Omega^{0,2}(\mu)$, $\Omega_2(s,t) = \int \mathrm{Re}\langle s, it\rangle \omega^n$ with moment map $m(s) = \frac{i}{2}|s|^2\omega^n$.

These are $\mathcal{G}$-invariant and indeed symplectic (in fact Kähler, and $\mathcal{A}^{1,1}$ is a Kähler submanifold of the affine space $\mathcal{A}$). Therefore the symplectic form $\Omega = \frac{1}{2n}\Omega_{\mathcal{A}} \oplus -\Omega_0 \oplus \Omega_2$ on $\mathcal{A}^{1,1} \times \Omega^0(\mu) \times \Omega^{0,2}(\mu)$ has moment map

$$m(A,\alpha,\beta) = -i\left(i\Lambda F_A + \frac{1}{2}\left(|\alpha|^2 - |\beta|^2\right)\right)\omega^n,$$

which is what we are trying to make zero. We can restrict to the symplectic submanifold $\left\{(A,\alpha,\beta) \,\middle|\, \bar\partial_B \alpha = 0 = \bar\partial_B \beta\right\}$ so, in the $\deg L < 0$ (and stable) case, we can expect that given a holomorphic structure and section α, we can find a unique metric satisfying (SW$-$). This is what we have proved in (2.4), showing that all points are stable.

For $\deg L > 0$, however, we need to incoorporate the $\bar\partial_B^*\beta = 0$ condition which is destroyed by the $\mathcal{G}^c$ action, as $\left\{(A,\alpha,\beta) \,\middle|\, \bar\partial_B^* = 0\right\}$ is not a symplectic submanifold. We can try to derive $\bar\partial_B^* = 0$ from a moment map. $G = \Omega^{0,1}(\mu)$ acts on $\Omega^{0,2}(\mu)$ via $\gamma : \beta \mapsto \beta - \bar\partial_B\gamma$ preserving Ω_2. But the 'moment map' $\beta \mapsto \bar\partial_B^*\beta = LG$ is not $\Omega^{0,1}(\mu)$-equivariant so the G action does not descend to $m^{-1}(0)$. Thus this first attempt at extending the symplectic quotient method, so successful for H^0, to H^2, fails, and I made no more progress in this direction.

Gradient flows. As mentioned above, an equivalent procedure to forming the symplectic quotient is to minimise $\|m\|^2$ down convergent gradient flow lines. In fact for the ASD equations for a holomorphic bundle on a Kähler surface $\|m\|^2 = \|F_A^+\|^2 = \frac{1}{2}\|F_A\|^2 - 4\pi^2 c_2(E)$ is essentially the Yang-Mills functional, and $\|m\|^2$ provides a generalisation of this to higher dimensions. Similarly the Seiberg-Witten functional on a Kähler surface essentially reduces to $\|m\|^2$ (where m is as above) for $\beta = 0$, and this can be generalised to three dimensions. We first consider $\deg L < 0$, to show how things might work for $\deg L > 0$.

Let $\mathcal{F} = \|i\Lambda F_A + \frac{1}{2}|\alpha|^2\|^2$ on $\mathcal{A}^{1,1}(\mu) \times \Omega^0(\mu)$, then for $(a,s) \in T_{(B,\alpha)}(\mathcal{A}^{1,1}(\mu) \times \Omega^0(\mu)) = \Omega(i\mathbb{R}) \oplus \Omega^0(\mu)$,

$$\begin{aligned}\langle \mathrm{grad}\,\mathcal{F}, (a,s)\rangle &= 2\,\mathrm{Re}\left\langle i\Lambda F_A + \frac{1}{2}|\alpha|^2, 2i\Lambda da + \mathrm{Re}\langle \alpha, s\rangle\right\rangle \\ &= \mathrm{Re}\left\langle -id^*\left[(i\Lambda F_A + \frac{1}{2}|\alpha|^2)\omega\right], a\right\rangle + 2\,\mathrm{Re}\left\langle (i\Lambda F_A + \frac{1}{2}|\alpha|^2)\alpha, s\right\rangle.\end{aligned}$$

So, taking the metric on $\mathcal{A}^{1,1}(\mu) \times \Omega^0(\mu)$ to be (the real part of) that on $\mathcal{A}^{1,1}(\mu)$ plus ε times that on $\Omega^0(\mu)$, we have, by the Kähler identities,

$$\operatorname{grad} \mathcal{F} = 4(\partial - \bar{\partial}) m(A, \alpha) \oplus 2\varepsilon^{-1} m(A, \alpha)\alpha, \tag{2.9}$$

where $m(A, \alpha) = i\Lambda F_A + \frac{1}{2}|\alpha|^2$. Hence stationary points with $\alpha \not\equiv 0$ are solutions of $i\Lambda F_A + \frac{1}{2}|\alpha|^2 = 0$. But $\xi \in \mathfrak{g}^c = \Omega^0(\mathbb{C})$ induces the vector field

$$(\partial\bar{\xi} - \bar{\partial}\xi) \oplus \xi\alpha$$

on $\mathcal{A}^{1,1}(\mu) \times \Omega^0(\mu)$ at (B, α), so $\operatorname{grad} \mathcal{F}$ is in the image of $4m(A, \alpha) \in \mathfrak{g}^c$ (for $\varepsilon = 1/2$). Therefore, starting with a holomorphic structure $\bar{\partial}_B$ on μ and $\alpha \in H^0_B(\mu)$ and solving the flow equations

$$\begin{aligned} \frac{d}{dt} A_t &= -4(\partial - \bar{\partial})(i\Lambda F_{A_t} + \frac{1}{2}|\alpha_t|^2), \quad & A_0 = A, \\ \frac{d}{dt} \alpha_t &= -4(i\Lambda F_{A_t} + \frac{1}{2}|\alpha_t|^2), & \alpha_0 = \alpha, \end{aligned}$$

we get a path converging to an isomorphic holomorphic structure and holomorphic section satisfying (SW−), i.e. $m(A, \alpha) = 0$. In fact solving instead for the path in $\mathcal{G}^c$ is essentially what we did in the continuity method of (2.4) involving the path a_t.

For $\deg L > 0$ we can consider, say, $\mathcal{F} = \|i\Lambda F_A - \frac{1}{2}|\beta|^2\|^2 + \|\bar{\partial}_B\|^2 + \|\bar{\partial}^*_B\|^2$ on $\mathcal{A}^{1,1}(\mu) \times \Omega^{0,2}(\mu)$, again with metric $\operatorname{Re}\langle\ ,\ \rangle \oplus \frac{1}{2}\operatorname{Re}\langle\ ,\ \rangle$. Then for $(a, s) \in \Omega^1(i\mathbb{R}) \oplus \Omega^{0,2}(\mu)$, a messy calculation shows that

$$\begin{aligned} \operatorname{grad} \mathcal{F} = &\left[4(\partial - \bar{\partial}) \left(i\Lambda F_A - \frac{1}{2}|\beta|^2 \right) + \left(\beta \lrcorner \bar{\partial}_B \beta - \overline{\beta \lrcorner \bar{\partial}_B \beta} \right) + \left(\bar{\partial}^*_B \beta \lrcorner \beta - \overline{\bar{\partial}^*_B \beta \lrcorner \beta} \right) \right] \\ &\oplus \left[-4 \left(i\Lambda F_A - \frac{1}{2}|\beta|^2 \right) \beta + 4\Delta_B \beta \right]. \end{aligned} \tag{2.10}$$

This gives us a corresponding set of gradient flow equations whose convergence to solutions of (SW+) (stationary points of $\operatorname{grad} \mathcal{F}$) could be analysed. However, closely related to the fact that (SW−) comes from a moment map and (SW+) does not, (2.9) was along a $\mathcal{G}^c$ orbit but (2.10) is not - the equation $\Delta_B \beta = 0$ is destroyed by the $\mathcal{G}^c$ action. So we may find a solution, but we will have moved away from our original holomorphic structure and element of $H^2_B(\mu)$, so we are not finding special metrics or sections.

3. Linearisation and Perturbation

In this section we look at the linearisation of the equations $(SW+)$ on a Kähler 3-manifold. There are many equivalent sets of equations we could look at (for instance we could include α's, since these must vanish for $\deg L > 0$), most of which would be over- or under-determined. Ideally we would like the set to be elliptic; the best we can do here is the elliptic complex below (equivalently, as we shall see, we could add an extra variable u and get an elliptic system).

Let $\mathcal{A}(\mu)$ be the set of unitary connections on μ, and let $\mathcal{C} = \mathcal{A}(\mu) \times \Omega^{0,2}(\mu)$. Then we are looking for the zero set of

$$\begin{aligned} \Phi : \quad \mathcal{C} \quad &\to \Omega^{0,2} \oplus \Omega^0(\mathbb{R}) \oplus \Omega^{0,2}(\mu) \\ (B,\beta) &\mapsto \ (F_B^{0,2}, i\Lambda F_A - \frac{1}{2}|\beta|^2, \Delta_B \beta), \end{aligned}$$

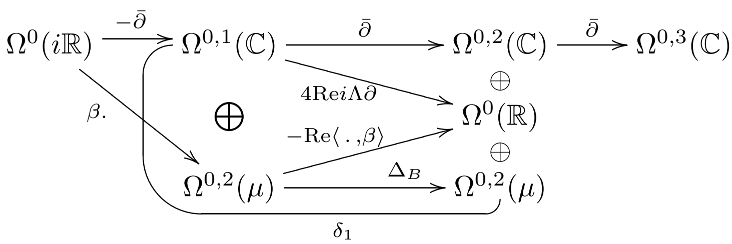
where A is the induced connection on L, as before.

We also have the action of the gauge group $\mathcal{G} = \Gamma(S^1 \times X)$ on $\mathcal{C}$, given by $g(B,\beta) = (B - g^{-1}dg, g\beta)$. Together the derivatives of these maps at a solution (B,β) give, in the usual way, a complex,

$$\Omega^0(i\mathbb{R}) = T_e\mathcal{G} \ \to \ T_{(B,\beta)}\mathcal{A}_\mu \oplus \Omega^{0,2}(\mu) \ \to \ \Omega^{0,2} \oplus \Omega^0(\mathbb{R}) \oplus \Omega^{0,2}(\mu).$$

Unfortunately this is not elliptic, and we must either modify $F_B^{0,2}$ to $F_B^{0,2} + \bar\partial^* u$, for $u \in \Omega^{0,3}$ (leaving the zero set of Φ unaltered, by the Bianchi identity: if $F_B^{0,2} + \bar\partial^* u = 0$ then $0 = \bar\partial F_B^{0,2} = -\bar\partial\bar\partial^* u$ so taking the inner product with u shows that $\bar\partial^* u = 0$), or equivalently, extend the complex to the right, as below.

So, identifying $\Omega^{0,1}$ with $T_B\mathcal{A}(\mu)$ by $a \mapsto a - \bar a$, a straightforward calculation shows the linearisation $D\Phi$ fits into the complex



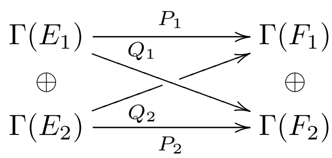
(3.1)

where $\delta_1(a) = \bar\partial_B(*\bar a \wedge (*\beta)) + \bar\partial_B^*(a \wedge \beta)$.

We must now check this is elliptic in an appropriate sense, the problem being that we have a complex of mixed order. This could have been avoided by using the first order elliptic operator $\bar\partial_B + \bar\partial_B^* : \ \Omega^{0,\mathrm{even}}(\mu) \to \Omega^{0,\mathrm{odd}}(\mu)$ but this would have meant extending the complex to the left and right. Using Δ_B is simpler, and the elliptic theory is easily modified, as follows.

Proposition (3.2). *If $P_i : \Gamma(E_i) \to \Gamma(F_i)$ $(i = 1, 2)$ are elliptic differential operators of order p_i, and $Q_1 : \Gamma(E_1) \to \Gamma(F_2)$, $Q_2 : \Gamma(E_2) \to \Gamma(F_1)$ are differential operators of order $q_i < p_i$, then the operator*

$D = P_1 + P_2 + Q_1 + Q_2 :$



is Fredholm with index $\operatorname{ind} D = \operatorname{ind} P_1 + \operatorname{ind} P_2$.

(Note: The indexes refer to spaces of smooth sections, not Sobolev spaces.)

Proof. Completing the spaces of sections in appropriate Sobolev norms, the P_i, Q_i extend to give a bounded operator

$$D:\quad \begin{array}{ccc} L^2_{k+p_1}(E_1) & \xrightarrow{\quad P_1 \quad} & L^2_k(F_1) \\ \oplus & {\scriptstyle Q_1\,\searrow\ \nearrow\,Q_2} & \oplus \\ L^2_{k+p_2}(E_2) & \xrightarrow[\quad P_2 \quad]{} & L^2_k(F_2). \end{array}$$

Since $q_i < p_i$, this is a compact perturbation of the direct sum of the two operators

$$\begin{aligned} P_1 :\ & L^2_{k+p_1}(E_1) \to L^2_k(F_1), \\ P_2 :\ & L^2_{k+p_2}(E_2) \to L^2_k(F_2). \end{aligned}$$

By the ellipticity of the P_i's, these are Fredholm maps with combined index $\operatorname{ind} P_1 + \operatorname{ind} P_2$, so D is Fredholm with this index too. Now we need only show the same holds for smooth sections.

If $(u_1, u_2) \in (L^2_{k+p_1} \oplus L^2_{k+p_2}) \cap \ker D$ then

$$P_1 u_1 + Q_2 u_2 = 0 = P_2 u_2 + Q_1 u_1. \tag{3.3}$$

The P_i have parametrices G_i, pseudodifferential operators of order $-p_i$ such that $G_i P_i - I$ is an operator of order -1, K_i say.

Applying G_i to (3.3) gives $u_1 + K_1 u_1 + G_1 Q_2 u_2 = 0 = u_2 + K_2 u_2 + G_2 Q_1 u_1$. Now, $u_i \in L^2_k$ so $K_i u_i \in L^2_{k+1}$, and as $p_i > q_i$, $G_1 Q_2 u_2 \in L^2_{k+p_2+p_1-q_2} \subset L^2_{k+p_1+1}$ and $G_2 Q_1 u_1 \in L^2_{k+p_2+1}$. Thus $u_i \in L^2_{k+p_i+1}$ and ker D is independent of k, and lies in $\bigcap_k L^2_k = C^\infty$.

Similarly, forming the adjoints of all our operators (with respect to the L^2 inner product) and using the pairing (by integration) between L^2_r and L^2_{-r} we can identify the cokernel with the kernel of the adjoint operator

$$D^*:\quad \begin{array}{ccc} L^2_{-k-p_1}(E_1) & \xleftarrow{\quad P_1^* \quad} & L^2_{-k}(F_1) \\ \oplus & {\scriptstyle Q_1^*\,\nwarrow\ \swarrow\,Q_2^*} & \oplus \\ L^2_{-k-p_2}(E_2) & \xleftarrow[\quad P_2^* \quad]{} & L^2_{-k}(F_2), \end{array}$$

which, by the same argument, consists of smooth sections. □

Now, recall the way a complex (V^i, D) is converted into the form of a single operator $D + D^* : \bigoplus_{\text{even}} V^i \to \bigoplus_{\text{odd}} V^i$, which is elliptic if and only if the complex is elliptic, with index the Euler characteristic of the original complex.

In this form our complex (3.1) is a perturbation, in the sense of Proposition (3.2), of the direct sum of the complexes

$$\begin{array}{ccccccc} \Omega^0(i\mathbb{R}) & \xrightarrow{-\bar\partial} & \Omega^{0,1}(\mathbb{C}) & \xrightarrow{\bar\partial} & \Omega^{0,2}(\mathbb{C}) & \xrightarrow{\bar\partial} & \Omega^{0,3}(\mathbb{C}) \\ & & & \searrow^{4\mathrm{Re}\, i\Lambda\partial} & \oplus & & \\ & & & & \Omega^0(\mathbb{R}) & & \end{array} \tag{3.4}$$

$$\Omega^{0,2}(\mu) \xrightarrow{\Delta_B} \Omega^{0,2}(\mu). \tag{3.5}$$

So if we can show (3.4) is elliptic, then (3.1) has finite dimensional cohomology groups with Euler characterisitic the same as (3.4) ((3.5) is clearly elliptic with zero index).

Lemma (3.6). *(3.4) is elliptic.*

Proof. The (real) symbol sequence of (3.4) at a point $v \in T^*_x X\backslash\{0\}$ is

$$\begin{array}{ccccccc} \Omega^0(i\mathbb{R}) & \xrightarrow{-v^{0,1}\wedge} & \Omega^{0,1}(\mathbb{C}) & \xrightarrow{v^{0,1}\wedge} & \Omega^{0,2}(\mathbb{C}) & \xrightarrow{v^{0,1}\wedge} & \Omega^{0,3}(\mathbb{C}) \\ & & & \searrow^{4\mathrm{Re}\, i\Lambda v^{1,0}\wedge} & \oplus & & \\ & & & & \Omega^0(\mathbb{R}) & & \end{array}$$

where $v^{0,1}$ is the image of v under the composite $T^*X \hookrightarrow T^*X \otimes \mathbb{C} \to T^{0,1}X$, and similarly for $v^{1,0}$.

That the composites of two symbols is zero follows from the fact that (3.4) is a complex, or by direct check. At the $\Omega^{0,1}$ stage we have, for $f \in \Omega^0(i\mathbb{R})$,

$$-\mathrm{Re}\, i\Lambda v^{1,0} \wedge fv^{0,1} = -if\Lambda(v^{1,0} \wedge v^{0,1} + \overline{v^{1,0} \wedge v^{0,1}}) = -if\Lambda(v^{1,0} \wedge v^{0,1} + v^{0,1} \wedge v^{1,0})$$

which is zero. Conversely, given $a \in \Omega^{0,1}$ such that $-v^{0,1} \wedge a = 0 = \mathrm{Re}\,(i\Lambda v^{1,0} \wedge a)$, we must have $a = fv^{0,1}$ for some $f \in \Omega^0(\mathbb{C})$, and then

$$i\Lambda(fv^{1,0} \wedge v^{0,1} - \bar f v^{0,1} \wedge v^{1,0}) = i(f + \bar f)\Lambda(v^{1,0} \wedge v^{0,1})$$

vanishes if and only if $f + \bar f = 0$, i.e. $f \in \Omega^0(i\mathbb{R})$.

The symbol sequence is also onto at $\Omega^0(\mathbb{R})$, and elsewhere is well known to be exact. □

So far we have proved

Theorem (3.7). *(3.1) has finite dimensional cohomology groups H^i $(i = 0, 1, 2, 3)$, with Euler characteristic equal to that of (3.4).*

We now need to identify the real dimensions h^i of the cohomology groups of (3.4). Clearly $h^0 = 1$ and $h^3 = 2h^{0,3}$. Let H', H'' be the first and second cohomology groups of (3.4).

Theorem (3.8). $H' \cong H^{0,1}$.

Proof. We have a well defined map $H' \to H^{0,1}$ induced by the identity on the chain groups. To show it is surjective, we must show that give $a \in \Omega^{0,1}$ with $\bar\partial a = 0$, $\exists\, b \in \Omega^0(\mathbb{C})$ such that $a + \bar\partial b \in \ker(\operatorname{Re} i\Lambda\partial)$. So we want $\operatorname{Re}(i\Lambda\partial a + i\Lambda\partial\bar\partial b) = 0$, that is $\Delta(\operatorname{Re} b) = \operatorname{Re}(i\Lambda\partial a)$ which we can solve if and only if $\operatorname{Re}\int i\Lambda\partial a\, d\mu = 0$, by the Fredholm alternative. But

$$\int \Lambda\partial a\, d\mu = c\int \partial a\wedge\omega^2 = c\int da\wedge\omega^2 = c\int d(a\wedge\omega^2) = 0.$$

To show injectivity, suppose a and $a+\bar\partial b$ represent elements of H', i.e. $\bar\partial a = 0$ and $\operatorname{Re} i\Lambda\partial(a+\bar\partial b) = 0 = \operatorname{Re} i\Lambda\partial a$. Then as above, $\Delta(\operatorname{Re} b) = 0$ so $\operatorname{Re} b$ =const and $\bar\partial b \in \bar\partial\,\Omega^0(i\mathbb{R})$, which shows $[a] = [a+\bar\partial b]$ in H'. $\square$

Theorem (3.9). $H'' \cong H^{0,2}\oplus\mathbb{R}$.

Proof. The cocycles are $Z^{0,2}(\mathbb{C})\oplus\Omega^0(\mathbb{R})$ and the proof of the theorem above shows that the image of $\bar\partial\oplus 4\operatorname{Re} i\Lambda\partial$ is $B^{0,2}(\mathbb{C})\oplus\{f\in\Omega^0(\mathbb{R}) \mid \int f\,d\mu = 0\}$. $\square$

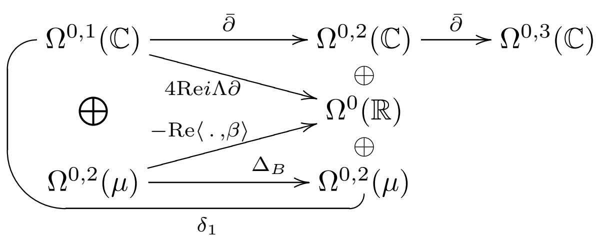
Theorem (3.10). *The (real) Euler characterisitic of (3.1) is* $1 - 2h^{0,1} + 2h^{0,2} + 1 - 2h^{0,3} = 2\chi(\mathcal{O})$, *which, by the Hirzebruch-Riemann-Roch theorem for the trivial holomorphic line bundle* $\mathcal{O}$, *equals* $\frac{1}{12}\langle c_1(X)\smile c_2(X), [X]\rangle$.

We now need to identify the cohomology groups H^i of (3.1) at a solution (B,β). $H^0 = 0$ since $\beta\not\equiv 0$, H^1 is our naïve linearised model for the moduli space of solutions, and $H^3 = H^{0,3}$, which leaves

Theorem (3.11). $H^2 \cong H^{0,2}\oplus H^{0,2}_B(\mu)$.

Proof. We compute H^2 from (3.1),

$$D = \bar\partial + \delta_1 + \delta_2 + \delta_3 + \Delta_B :$$



The cocycles are $Z^{0,2}\oplus\Omega^0(\mathbb{R})\oplus\Omega^{0,2}(\mu)$, and we can identify H^2 with the (real) L^2 orthogonal complement of $\operatorname{im} D$ within this space, by Proposition (3.2). So suppose that $(b,f,s)\in Z^{0,2}\oplus\Omega^0(\mathbb{R})\oplus\Omega^{0,2}(\mu)$ is $\perp_{\mathbb{R}} \operatorname{im} D$, then, for all $a\in\Omega^{0,1}$,

$$0 = \langle D(a,0),(b,f,s)\rangle = \operatorname{Re}\langle\bar\partial a, b\rangle + 4\operatorname{Re} i\int\Lambda\partial a f\,du + \operatorname{Re}\int\langle\delta_1(a),s\rangle d\mu. \quad (3.12)$$

Setting $a = \bar\partial u$, the first term vanishes, and

$$4\mathrm{Re}\int \Delta u\, f\, d\mu = 4\mathrm{Re}\int u\Delta f\, d\mu = \mathrm{Re}\int \langle u, \bar\partial^* \delta_1^*(s)\rangle d\mu,$$

where Δ is the $\bar\partial$-Laplacian. Since this holds for all $u \in \Omega^0(\mathbb{C})$, we have

$$4\Delta f = \bar\partial^* \delta_1^*(s).$$

Then a messy calculation, using the relation $\bar\partial^*(\tau \lrcorner \sigma) = (-1)^{|\tau|}(\tau \lrcorner \bar\partial^* \sigma - \bar\partial \tau \lrcorner \sigma)$ (derived easily from the Leibnitz rule), shows that the right hand side equals

$$4\Delta f = -\langle \Delta_B s, \beta\rangle. \tag{3.13}$$

We also have

$$0 = \langle D(0,\gamma),(b,f,s)\rangle = -\mathrm{Re}\int \langle \beta,\gamma\rangle f\, d\mu + \mathrm{Re}\langle \Delta_B\gamma, s\rangle, \quad \forall \gamma \in \Omega^{0,2}(\mu),$$

implying that $\Delta_B s = f\beta$, which, in (3.13), gives $(4\Delta + |\beta|^2)f = 0$. But $(4\Delta + |\beta|^2)$ is a strictly positive operator since $\beta \not\equiv 0$, so $f \equiv 0$. Hence $\Delta_B s = 0$ too, forcing $\bar\partial_B s = 0 = \bar\partial_B^* s = \delta_1(s)$, so (3.12) decouples to give $\langle \bar\partial a, b\rangle = 0 \quad \forall a \in \Omega^{0,1}$, i.e. $\bar\partial^* b = 0$.

Therefore we have proved that $(\operatorname{im} D)^{\perp_{\mathbb{R}}}$ is contained in $\mathcal{H}^{0,2} \oplus \{0\} \oplus \mathcal{H}_B^{0,2}(\mu)$ (where $\mathcal{H}$ denotes the harmonic space $\ker\bar\partial \cap \ker\bar\partial^* = \ker\Delta$). However, $(\operatorname{im} D)^\perp$ also contains this space - the only thing left to show is that it is L^2 orthogonal to $\operatorname{im}\delta_1$, but if $s \in \mathcal{H}_B^{0,2}(\mu)$ then $\bar\partial_B s = 0 = \bar\partial_B^* s$, so that $\langle \delta_1(a), s\rangle = \langle -\bar\partial_B(*\bar a \wedge (*\beta)) + \bar\partial_B^*(a\wedge\beta), s\rangle = 0$. Thus $H^2 \cong H^{0,2} \oplus H_B^{0,2}(\mu)$ as claimed. □

Theorem (3.14). *A neighbourhood of a solution (B,β) in the moduli space of solutions of (SW+)/$\mathcal{G}$ is the zero set of a smooth nonlinear map $f : H^1 \to H_B^2(\mu)$, where H^1 has (real) dimension $b^1 - 2 + 2\dim_{\mathbb{C}} H_B^2(\mu)$.*

Proof. Comparing the Euler characteristics of (3.1) and (3.4),

$$0 - \dim H^1 + (2h^{0,2} + 2\dim_{\mathbb{C}} H_B^2(\mu)) - 2h^{0,3} = \frac{1}{12}c_1.c_2 = 2 - 2h^{0,1} + 2h^{0,2} - 2h^{0,3}$$

gives the dimension of H^1, as $b^1 = 2h^{0,1}$. Since the action of $\mathcal{G}$ on solutions is free ($\beta \not\equiv 0$) we can appeal to the standard theory (described in [2]) of the Kuranishi model of the moduli space, ignoring the $H^{0,2}$ summand of $H^2 \cong H^{0,2} \oplus H_B^2(\mu)$, and the higher cohomology $H^{0,3}$, because the $F_B^{0,2} = 0$ condition of the zero set $\{\Phi = 0\}$ is already linear; the linearised model $\bar\partial a = 0$ for $a \in \Omega^{0,1}$ is the precise condition for $b + a - \bar a$ to define a holomorphic structure on the line bundle μ. □

It is tempting to call $\dim H^1 = b^1 - 2 + 2\dim_{\mathbb{C}} H_B^2(\mu)$ the virtual dimension of the moduli space, and it is, notice, the dimension of what we hoped the moduli space might be, i.e. $\bigcup_B \mathbb{P}(H_B^2(\mu))$ (where B runs over the set of isomorphism classes of holomorphic structures on μ), perhaps with some unstable points thrown out (e.g.

those at which $\dim H^2_B(\mu)$ jumps). However, this is misleading, since we have no hope of (B,β) being a regular value of the map f, and so of the moduli space being (locally) a manifold of this dimension. In other similar moduli problems we usually include only those solutions at which the higher cohomology group H^2 vanishes, but in our case the vanishing of H^2 would mean that $H^2_B(\mu)=\emptyset$, so that (SW+) would have no solutions at all. Even if we gauge-invariantly perturb the equations to $i\Lambda F_A - \frac{1}{2}|\beta|^2 = g$, for $g\in\Omega^0(\mathbb{R})$, we still do not get a regular point, at the origin, of the map

$$\begin{aligned}\widetilde{\Phi}:\ \mathcal{C}\times\Omega^0(\mathbb{R}) &\to \Omega^{0,2}\oplus\Omega^0(\mathbb{R})\oplus\Omega^{0,2}(\mu)\\ (B,\beta,g) &\mapsto (F_B^{0,2}, i\Lambda F_A - \frac{1}{2}|\beta|^2 - g, \Delta_B\beta),\end{aligned}$$

i.e., $D\widetilde{\Phi}$ is not onto $B^{0,2}\oplus\Omega^0(\mathbb{R})\oplus\Omega^{0,2}(\mu)$ (in fact, the working in (3.11) shows $\operatorname{im} D\widetilde{\Phi} = B^{0,2}\oplus\Omega^0(\mathbb{R})\oplus(\mathcal{H}_B^{0,2}(\mu))^\perp$ as before). To get a regular perturbation we must alter the $\Delta_B\beta=0$ equation to $\Delta_B\beta=\eta,\quad \eta\in\Omega^{0,2}(\mu)$. (Since, for $g\in\mathcal{G},\ \Delta_{gB}(g\beta)=g\Delta_B\beta$ we can maintain gauge invariance by acting $\mathcal{G}$ on η by $g:\eta\mapsto g\eta$). Then, for generic η (see [2] for the relevant Sard-Smale theory) $H^2\cong H^{0,2}$ and we get a local model about (B,β) for the moduli space, of dimension b^1-2. But notice we have lost our special representative of cohomology $H^2_B(\mu)$. Again, this does seem to demonstrate that a dimension $b^1-2+2\dim_{\mathbb{C}}H^2_B(\mu)$ for the moduli space is unrealistic.

In fact, since the original Φ *is* nonlinear, the map f is not zero, and therefore its zero set cannot have dimension $\dim H^1$, and so we have proved

Theorem (3.15). *For a given holomorphic structure, there is an arbitrarily close holomorphic structure $\bar\partial_B$ on μ and $\beta\in H^2_B(\mu)$ such that there does not exist a metric on μ making (B,β) a solution of (SW+). (B is the connection induced from $\bar\partial_B$ and the metric).*

This does not rule out there being a subset of 'stable' pairs for which the moduli space is what we hoped it would be. So we need to look at the space of solutions of (SW+) at a fixed holomorphic structure $\bar\partial_B$, modulo $\mathcal{G}$. Considering a $\mathcal{G}^c$ orbit of B, any $g\in\mathcal{G}^c$ is gauge equivalent to a $(0,\infty)$-valued map, so letting $\mathcal{G}^r=\Gamma((0,\infty)\times X)$, we look for solutions $(g(B),\gamma)$ of (SW+), $g\in\mathcal{G}^r$. First we would like to know which of these are gauge equivalent.

If $g(B)=h(B),\ g,h\in\mathcal{G}^r$, then $\bar\partial\log g=\bar\partial\log h\ \Rightarrow g/h=$const, so we work modulo the constants, on $\mathcal{G}^r/\mathbb{R}_+$. Then if $(g_1(B),\gamma_1)=h((g_2(B),\gamma_2))$ for $h\in\mathcal{G}$, $g_1=g_2$ and h is a constant in $U(1)$. Therefore we are looking for the zero set of

$$\begin{aligned}\Psi:\ \mathcal{G}^r/\mathbb{R}_+\times\Omega^{0,2}(\mu) &\to \Omega^0(\mathbb{R})\oplus\Omega^{0,2}(\mu)\\ (g,\gamma) &\mapsto (i\Lambda F_{g(A)}-\frac{1}{2}|\gamma|^2, \Delta_{g(B)}\gamma),\end{aligned}\tag{3.16}$$

quotiented by $U(1)$. The derivative at the solution (B, β) is given by

$$D\Psi : \quad \begin{array}{ccc} \Omega^0(\mathbb{R})/\mathbb{R} & \xrightarrow{\quad 4\Delta \quad} & \Omega^0(\mathbb{R}) \\ \oplus & \delta_1 \searrow \quad \nearrow -\mathrm{Re}\langle\,.\,,\beta\rangle & \oplus \\ \Omega^{0,2}(\mu) & \xrightarrow[\quad \Delta_B \quad]{} & \Omega^{0,2}(\mu), \end{array}$$

where, much as in (3.1), $\delta_1(u) = -\bar{\partial}_B(\bar{\partial}u \lrcorner \beta) + \bar{\partial}_B^*(\bar{\partial}u \wedge \beta)$. This, by Proposition (3.2), has index equal to the sum of those of

$$\Omega^0(\mathbb{R})/\mathbb{R} \xrightarrow{4\Delta} \Omega^0(\mathbb{R}),$$
$$\Omega^{0,2}(\mu) \xrightarrow{\Delta_B} \Omega^{0,2}(\mu),$$

i.e. -1. (This immediately shows we cannot have a regular point, of course). We compute the dimension of the cokernel of $D\Psi$, taking (f, s) in $(\operatorname{im} D)^{\perp_{\mathbb{R}}}$. Then $\langle D(u, \gamma), (f, s)\rangle = 0$, which, just as in (3.11), tells us that

$$4\Delta f = -\langle \Delta_B s, \beta\rangle$$

on putting $\gamma = 0$, and

$$\Delta_B s = \beta f$$

on putting $u = 0$. Thus, as in (3.11), $f \equiv 0$, $\Delta_B s = 0$, and the cokernel of $D\Psi$ is isomorphic to $H^2_B(\mu)$. Thus, the dimension of the kernel of $D\Psi$ is $2\dim_{\mathbb{C}} H^2_B(\mu) - 1$, and we have

Theorem (3.17). *The local moduli space of solutions of (3.16), about a solution (B, β) of (SW+), is the zero set of a smooth nonlinear map from a vector space of dimension $(2\, dim_{\mathbb{C}} H^2_B(\mu) - 1)$ to $H^2_B(\mu)$, quotiented by the free action of $U(1)$.*

Proof. The $U(1)$ action is free because $\beta \not\equiv 0$, and the rest is the Kuranishi model mentioned earlier ([2]).

So, even if the moduli space were a manifold, it would have dimension less than $(2\dim_{\mathbb{C}} H^2_B(\mu) - 1)$ since the $U(1)$ action is free, and the map in (3.17) is *not* identically zero (Ψ *is* nonlinear). Thus we have proved the following stronger version of Theorem (3.15).

Theorem (3.18). *Given a holomorphic structure $\bar{\partial}_B$ on μ, the moduli space of solutions of (SW+) with an isomorphic holomorphic structure is not $\mathbb{P}(H^2_B(\mu))$, that is there exists $\beta \in H^2_B(\mu)$ such that (B, β) satisfies (SW+) for no metric on μ.*

We end this section by noting that we have not actually found a single solution of (SW+). To do this we could try solving an equation for the metric similar to the vortex equation (2.3) for $\deg L < 0$, but involving a nasty non-local term:

$$\Delta u + \frac{1}{4}|P_{\mathcal{H}_u}\beta|_0^2 e^{2u} = a. \tag{3.19}$$

Here $P_{\mathcal{H}_u}$ denotes the projection onto the harmonic space of the $\bar{\partial}_B$-Laplacian, using the metric $e^u|\,.\,|_0$. We could try to iterate, starting with harmonic β, and solving (3.19) with $P_{\mathcal{H}_u}\beta$ replaced by β, giving $u^{(1)}$ say. Then resolve with $P_{\mathcal{H}_{u^{(1)}}}\beta$, etc. Or a continuity method, similar to that of (2.4), would require the linearisation of $P_{\mathcal{H}_u}$ with respect to u, and, more seriously, a priori bounds as in (2.5). Of course both these methods are attempts to find a metric to solve (SW+) for a given $(B,\beta), \beta \in H^2_B(\mu))$, and Theorem (3.18) shows they are doomed to failure for some choices of β. Perhaps, therefore, the best method of finding a solution, would be to solve for the gradient flow of (2.10).

4. Gauge Theories on Symplectic Manifolds

We begin by sketching some of the basic facts about the moduli space of holomorphic bundles, topologically equivalent to a fixed $SU(r)$ bundle E, over a Kähler manifold. On a Kähler surface the ASD equations $F^+_A = 0$ become the integrability condition for the pair (E, A) to define a holomorphic bundle, $F^{0,2}_A = 0 = F^{2,0}_A$, and the equation $i\Lambda F^{1,1}_A = 0$ which, naïvely speaking, singles out a special connection (equivalently, metric) for the given holomorphic structure. In fact, it is shown in [2,7], that each $\mathcal{G}^c$ orbit of a *stable* holomorphic $SU(r)$ bundle contains precisely one orbit of ASD connections. Thus we can form the moduli space of stable holomorphic bundles isomorphic to E, which equals the moduli space of ASD connections on E. Roughly speaking {holomorphic structures}/ $\mathcal{G}^c$ ≡{ASD connections}/ $\mathcal{G}$, as in our discussion of moment maps in Section 2. While the ASD equations do not generalise to higher dimensions, on a Kähler n-manifold we can write down the (more general) Hermitian-Yang-Mills equations for a $U(r)$ bundle,

$$F^{0,2}_A = 0, \quad i\Lambda F^{1,1}_A = \lambda I. \tag{4.1}$$

(λ is a constant). Then (see [7]) we have the result that for any stable bundle there exists a metric satisfying (4.1), for $\lambda = 2\pi \left(\int c_1(E) \wedge \omega^{n-1}\right) / \left(\int \omega^n\right)$.

As in Section 3, these equations are not elliptic, but fit into an elliptic complex. Equivalently, on a Kähler 3-manifold, we can modify the first equation to $F^{0,2}_A = \bar{\partial}^*_A u,\ u \in \Omega^{0,3}(\mathfrak{g}_E)$, giving the same result, by the Bianchi identity. This shows what the analogue of the moduli space of holomorphic bundles, or Hermitian Yang-Mills connections, should be on a symplectic 6-manifold; that is solutions of the elliptic system

$$\begin{aligned} F^{0,2}_A &= \bar{\partial}^*_A u, \\ i\Lambda F_A &= \lambda I, \end{aligned}$$

modulo the gauge group $\mathcal{G} = \Gamma(\mathcal{G}_E)$. Thus we can extend some Yang-Mills invariants to a symplectic 6-manifold.

The space of solutions of a *linear* elliptic system is finite dimensional so that an L^2_1 bounded subset, say, will be compact. The nonlinear case is more complicated but an L^2 bound on the curvature F_A is the starting point for crucial (weak) compactness theorems of moduli spaces (see for instance [2] - locally we can take A to be guage equivalent to a connection matrix in Coulomb gauge $d^*A = 0$, which together with $\|F_A\|_{L^2}$ controls $\|(d+d^*)A\|_{L^2}$ and thus $\|A\|_{L^2_1}$). We shall concentrate on obtaining these bounds.

A vector bundle E on a Kähler n-manifold X has the topological invariant

$$-\int |F_A^{0,2}|^2 + |F_A^{2,0}|^2 + |F_\omega^{1,1}|^2 - |F_\perp^{1,1}|^2 \, d\mu = c(n) \int -\mathrm{tr}\,(F_A)^2 \wedge \omega^{n-2}$$
$$= 4\pi^2 c(n) \left(c_2(E).\, \omega^{n-2}\right) = C,$$

using the pointwise splitting of $\Lambda^{1,1}$ into $\langle\omega\rangle$ and its orthogonal complement $\Lambda_\perp^{1,1}$. Therefore,

$$\|F_A\|^2_{L^2} = \int |F_A^{0,2}|^2 + |F_A^{2,0}|^2 + |F_\omega^{1,1}|^2 + |F_\perp^{1,1}|^2 \, d\mu$$
$$= C + 2\int |F_A^{0,2}|^2 + |F_A^{2,0}|^2 + |F_\omega^{1,1}|^2 \, d\mu.$$

Thus, if $F_A^{0,2} = 0 = F_A^{2,0}$, $i\Lambda F_A = \lambda$, we get our desired bound

$$\|F_A\|^2_{L^2} = C + 2\lambda^2 \mathrm{Vol}. \tag{4.2}$$

We can now try to mimic this in the symplectic case with the equations

$$F_A^{0,2} = \bar\partial_A^* u, \; i\Lambda F_A = \lambda. \tag{4.3}$$

The Bianchi identity now becomes $(d_A F_A)^{0,3} = 0 = \bar\partial_A F_A^{0,2} + \mathcal{N}(F_A^{1,1})$, where $\mathcal{N}$ is the Nijenhius tensor: $\Lambda^{1,1} \to \Lambda^{0,2}$. So

$$\int |F_A^{0,2}|^2 \, d\mu = \int |\bar\partial_A^* u|^2 \, d\mu = \int \langle \Delta_A u, u\rangle d\mu \leq \frac{\|\Delta_A u\|^2}{\lambda_1},$$

where λ_1 is the first *nonzero* eigenvalue of $\Delta_A = \bar\partial_A \bar\partial_A^* : \Omega^{0,3}(\mathfrak{g}_E) \to \Omega^{0,3}(\mathfrak{g}_E)$. Therefore, $\Delta_A u = \bar\partial_A \bar\partial_A^* u = \bar\partial_A F_A^{0,2} = -\mathcal{N}(F_A^{1,1})$, yielding

$$\int |F_A^{0,2}|^2 \, d\mu \leq \frac{\|\mathcal{N}\|^2}{\lambda_1} \|F_A^{1,1}\|^2. \tag{4.4}$$

We have, as before, the topological invariant C (ω is still closed), so for a unitary connection satisfying (4.3), (4.4) gives us

$$\|F_A\|^2_{L^2} = C + 2\int |F_A^{0,2}|^2 + |F_A^{2,0}|^2 + |\Lambda F_A^{1,1}|^2 \, d\mu \leq C + 2\lambda^2 \mathrm{Vol} + 4\frac{\|\mathcal{N}\|^2}{\lambda_1}\|F_A^{1,1}\|^2.$$

This yields the obvious bound $\|F_A\|^2_{L^2} \leq (C+2\lambda^2\mathrm{Vol})/(1-4\frac{\|\mathcal{N}\|^2}{\lambda_1})$ for $\lambda_1 > 4\|\mathcal{N}\|^2$, but we can do better. Let $f = \|F_A\|^2_{L^2}$, $k = C + 2\lambda^2\mathrm{Vol}$, $n = 2\frac{\|\mathcal{N}\|^2}{\lambda_1}$, $\alpha = 2\|F_A^{0,2}\|^2$, $\beta = \|F_A^{1,1}\|^2$, then

$$f = \alpha + \beta, \; k = \beta - \alpha, \; \text{and } \alpha \leq n\beta,$$

from (4.4) and the definition k. Thus

$$f = k + 2\alpha = k + \frac{2n}{1+n}\alpha + \frac{2}{1+n}\alpha \leq k + \frac{2n}{1+n}(\alpha+\beta) = k + \frac{2n}{1+n}f.$$

This gives the bound $f \leq k\left(\frac{1+n}{1-n}\right)$ for $n < 1$, i.e.,

$$\|F_A\|^2_{L^2} \leq (C + 2\lambda^2 \mathrm{Vol})\left(\frac{\lambda_1 + 2\|\mathcal{N}\|^2}{\lambda_1 - 2\|\mathcal{N}\|^2}\right), \text{ for } \lambda_1 > 2\|\mathcal{N}\|^2, \tag{4.5}$$

which reduces to (4.2) in the Kähler case $\mathcal{N} \equiv 0$. So we get a compactness result when the first nonzero eigenvalue of Δ_A on $\Omega^{0,3}(\mathfrak{g}_E)$ is greater than $2\|\mathcal{N}\|^2$. We now cast this in more geometric form with a Weitzenböck formula, though it should be noted that we lose something in the process, since we derive a lower bound for the first eigenvalue, not the first nonzero eigenvalue.

Theorem (4.6). *Let E be a unitary vector bundle over the $2n$-dimensional symplectic manifold X, with a unitary connection A satisfying $i\Lambda F_A = \lambda I$. Then the smallest eigenvalue of $\Delta_A : \Omega^{0,n}(\mathfrak{g}_E) \to \Omega^{0,n}(\mathfrak{g}_E)$ is greater than or equal to the minimum of the scalar curvature of X.*

Proof. We follow lectures of Donaldson in setting up the Weitzenböck formula. Given a vector bundle W (later this will be $\mathfrak{g}_E$) with a connection A, we can form, using the Levi-Civita connection on X, the operators

$$\partial_A,\ \bar\partial_A,\ \nabla'_A,\ \nabla''_A \quad \text{on } \Lambda^{p,q} \otimes W,$$

where ∂_A will be the antisymmetrization of ∇'_A, etc.

On $\Lambda^{0,n} \otimes W = K_X \otimes W$, $\partial_A = \nabla'_A$, so their Laplacians are the same, $\Delta^{\partial}_A = \Delta'_A$ say. To relate the other Laplacians we need the Kähler identities (see [3,8]) in the symplectic bundle-valued case.

Lemma (4.7). *Given a bundle W with a unitary connection A over a symplectic manifold X, $\bar\partial^*_A = -i[\Lambda, \partial_A]$, and $\partial^*_A = i[\Lambda, \bar\partial_A]$.*

Proof (For the easy case of $\bar\partial^*_A$ on $\Omega^{0,q}(W)$ only). For $s \in \Omega^{0,q}(W)$, $t \in \Omega^{0,q-1}(W)$, using the Hodge-Riemann bilinear relations ([8]),

$$\begin{aligned}
\langle \bar\partial^*_A s, t\rangle_{L^2} &= \int \langle s, \bar\partial_A t\rangle d\mu = c\int s \wedge (\bar\partial_A t)^* \wedge \omega^{n-q} \\
&= c\int s \wedge \partial_A(t^*) \wedge \omega^{n-q} \\
&= c(-1)^q \int d(s \wedge t^* \wedge \omega^{n-q}) - (\partial_A s) \wedge t^* \wedge \omega^{n-q} \\
&= c(-1)^{q+1} \int \partial_A s \wedge (t \wedge \omega)^* \wedge \omega^{n-q-1} \\
&= -i\langle \partial_A s, t \wedge \omega\rangle_{L^2} = -i\langle \Lambda \partial_A s, t\rangle_{L^2}. \quad \square
\end{aligned}$$

Thus on $\Omega^{0,q}(W)$, $\Delta_A^{\bar\partial} = \bar\partial_A^*\bar\partial_A + \bar\partial_A\bar\partial_A^* = -i(\Lambda\partial_A\bar\partial_A + \bar\partial_A\Lambda\partial_A)$. But $\Delta_A^{\partial} = \partial_A^*\partial_A = i\Lambda\bar\partial_A\partial_A - i\bar\partial_A\Lambda\partial_A = \Delta_A^{\bar\partial} + i\Lambda(\partial_A\bar\partial_A + \bar\partial_A\partial_A)$.

So $\Delta_A^{\bar\partial} - \Delta_A^{\partial} = -i\Lambda\circ F_A^{1,1}$. Note this means the operator 'wedge with $F_A^{1,1}$ then do $-i\Lambda$', rather than wedge with $-i\Lambda F_A^{1,1}$, so it is zero on the top power $\Lambda^{0,n}\otimes W$.

Putting $q=0$ and replacing W by $\Lambda^{0,q}\otimes W$ gives

$$\Delta_A'' - \Delta_A' = -i\Lambda\circ F^{1,1}_{\Lambda^{0,q}\otimes W} = I\otimes -i\Lambda(F^{1,1}_{\Lambda^{0,q}}) - i\Lambda(F_A^{1,1})\otimes I.$$

So, on the top exterior power $q=n$ we have

$$\Delta_A^{\bar\partial} = \Delta_A^{\partial} = \Delta_A' = \Delta_A'' + I\otimes i\Lambda(F^{1,1}_{\Lambda^{0,n}}) + i\Lambda(F_A^{1,1})\otimes I. \tag{4.8}$$

Specialising to the case of $W=\mathfrak{g}_E$, the connection is induced from A on E, and what we have called F_A above is

$$F_{\mathfrak{g}_E} = [F_A,\,.], \quad \text{i.e } F_{\mathfrak{g}_E}\wedge s = F_A\wedge s + (-1)^{p+q+1}s\wedge F_A,$$

for $s\in\Omega^{p,q}(\mathfrak{g}_E)\subset\Omega^{p,q}(\operatorname{End}E)$. Thus $F^{1,1}_{\mathfrak{g}_E} = \operatorname{Ad}(F_A^{1,1}) \Rightarrow i\Lambda(F^{1,1}_{\mathfrak{g}_E}) = i\Lambda(\operatorname{Ad}F_A^{1,1}) = 0$, since by assumption $i\Lambda F_A^{1,1} = \lambda I$.

So the last term of (4.8) drops out leaving

$$\Delta_A^{\bar\partial} = \Delta_A'' - i\Lambda\,\mathrm{tr}R = \Delta_A'' + s,$$

where R is the curvature of the Levi-Civita connection and s is the scalar curvature. Since Δ_A'' is a positive operator the result follows. □

Theorem (4.9). *Let E be a stable unitary bundle over a symplectic 6-manifold whose scalar curvature satisfies min$(s) > 2\|\mathcal{N}\|^2$. Then there is a uniform bound on $\|F_A^{0,2}\|_{L^2}$ for any unitary connection A on E satisfying $F_A^{0,2} = \bar\partial_A^* u$ and $i\Lambda F_A = \lambda I$ for some constant λ.*

Proof. This now follows from equation (4.5) and Theorem (4.6). □

The Seiberg-Witten Equations. We would like to do a similar thing with the Seiberg-Witten equations, generalising them to a symplectic 6-manifold, by writing them as

(i) $\bar\partial_B\alpha = -\bar\partial_B^*\beta,$

(ii) $\bar\partial_B\beta = 0,$

(iii) $F_A^{0,2} = \bar\alpha\beta + \bar\partial^* u,$

(iv) $i\Lambda F_A = -\dfrac{1}{2}(|\alpha|^2 - |\beta|^2).$

As we have seen in Section 3, this works, for instance, if $\alpha = 0$, in that the equations reduce to (SW+) on a Kähler manifold and $\bar\partial^* u = 0$. We now try (and fail) to show what we would like, i.e. that on a Kähler 3-manifold these equations imply $\bar\partial^* u = 0$.

By the Bianchi identity,

$$\int \langle u, \bar\partial F_A^{0,2}\rangle d\mu = 0 \quad \Rightarrow \int \langle \bar\partial^* u, \bar\alpha\beta\rangle + |\bar\partial^* u|^2 d\mu = 0. \tag{4.10}$$

$F_A^{0,2} = 2F_B^{0,2}$, so applying $\bar\partial_B$ to (i) and substituting (iii) gives

$$\frac{1}{2}|\alpha|^2\beta + \frac{1}{2}\alpha(\bar\partial^* u) + \bar\partial_B\bar\partial_B^*\beta = 0 \quad \Rightarrow \int |\alpha|^2|\beta|^2 + \langle\bar\partial^* u, \bar\alpha\beta\rangle + 2|\bar\partial_B^*\beta|^2 d\mu = 0,$$

which, with (4.10), gives

$$\int |\alpha|^2|\beta|^2 - |\bar\partial^* u|^2 + 2|\bar\partial_B^*\beta|^2 d\mu = 0. \tag{4.11}$$

(4.10) and Cauchy-Schwartz give

$$\int |\bar\partial^* u|^2 d\mu = -\int \langle\bar\partial^* u, \bar\alpha\beta\rangle d\mu$$

$$(*) \qquad \le \left(\int |\bar\partial^* u|^2 d\mu\right)^{\frac{1}{2}} \left(\int |\bar\alpha\beta|^2 d\mu\right)^{\frac{1}{2}} \le \frac{1}{2}\left(\int |\bar\partial^* u|^2 d\mu + \int |\bar\alpha\beta|^2 d\mu\right),$$

with equality at (*) if and only if $-\langle\bar\partial^* u, \bar\alpha\beta\rangle \equiv |\bar\partial^* u||\bar\alpha\beta|$. So

$$\int |\bar\partial^* u|^2 d\mu \le \int |\bar\alpha\beta|^2 d\mu, \tag{4.12}$$

which in (4.11) implies that $\bar\partial_B^*\beta \equiv 0$, so (i) decouples to give

$$(\mathrm{i}') \qquad \bar\partial_B\alpha = 0, \quad \bar\partial_B\beta = 0 = \bar\partial_B^*\beta.$$

(4.11) also shows equality must hold in (4.12) and so at (*), so that

$$-\langle\bar\partial^* u, \bar\alpha\beta\rangle \equiv |\bar\partial^* u||\bar\alpha\beta|. \tag{4.13}$$

So $\int \left(|\bar\partial^* u| - |\bar\alpha\beta|\right)^2 d\mu = \int \left(|\bar\alpha\beta|^2 - |\bar\partial^* u|^2\right) d\mu + 2\int \left(-|\bar\partial^* u||\bar\alpha\beta| + |\bar\partial^* u|^2\right) d\mu$

vanishes, since the two terms are (4.11) and (4.10) respectively. Therefore $|\bar\partial^* u| \equiv |\bar\alpha\beta|$, which, with (4.13), shows that

$$(\mathrm{ii}') \qquad \begin{aligned} \bar\partial^* u &= -\bar\alpha\beta, \\ F_A^{0,2} &= 0. \end{aligned}$$

This is very close to what we want, but we still have not shown that $\bar\partial^* u = 0$ (if this is even true).

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