Foreword

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1 Introduction

This volume is an integrated collection of papers working out several new directions of research on 3-folds under the unifying theme of *explicit birational geometry*. Section 3 summarises briefly the contents of the individual papers.

Mori theory is a conceptual framework for studying minimal models and the classification of varieties, and has been one of the main areas of progress in algebraic geometry since the 1980s. It offers new points of view and methods of attacking classical problems, both in classification and in birational geometry, and it raises many new problem areas. While birational geometry has inspired the work of many classical and modern mathematicians, such as L. Cremona, G. Fano, Hilda Hudson, Yu. I. Manin, V. A. Iskovskikh and many others, and while their results undoubtedly give us much fascinating experimental material as food for thought, we believe that it is only within Mori theory that this body of knowledge begins to acquire a coherent shape.

At the same time as providing adequate tools for the study of 3-folds, Mori theory enriches the classical world many times over with new examples and constructions. We can now, for example, work and play with hundreds of families of Fano 3-folds. From where we stand, we can see clearly that the classical geometers were only scratching at the surface, with little inkling of the gold mine awaiting discovery.

The theory of minimal models of surfaces works with nonsingular surfaces, and the elementary step it uses is Castelnuovo's criterion, which allows us to contract -1-curves (exceptional curves of the first kind). A chain of such contractions leads us to a minimal surface S, either $\mathbb{P}^2$ or a scroll over a curve, or a surface with K_S numerically nonnegative (now called *nef*, see 2.2 below). These ideas were well understood by Castelnuovo and Enriques a century ago, and are so familiar that most people take them for granted. However, their higher dimensional generalisation was a complete mystery until the late 1970s, and may still be hard to grasp for newcomers to the field. It involves a suitable category of mildly singular projective varieties, and the crucial new ingredient of *extremal ray* introduced by Mori around 1980. As we discuss later in this foreword, extremal rays provide the elementary steps of the minimal model program (the divisorial contractions and flips of the Mori category that generalise Castelnuovo's criterion) and also the definition of Mori fibre space (that generalise $\mathbb{P}^2$ and the scrolls), our primary object of interest.

Higher dimensional geometry, like most other areas of mathematics, is marked by creative tensions between abstract and concrete on the one hand, general and special on the other. Contracting a -1-curve on a surface is a concrete construction, whereas a Mori extremal ray and its contraction is abstract (compare Remark 2.4.1). The "general" tendency in the classification of varieties, exemplified by the work of Iitaka, Mori, Kollár, Kawamata and Shokurov, includes things like Iitaka–Kodaira dimension, cohomological methods, and the minimal model program in substantial generality. The "special" tendency, exemplified by Hudson, Fano, Iskovkikh, Manin, Pukhlikov, Mori and ourselves, includes the study of special cases, for their own sake, and sometimes without hope of ever achieving general status.

By *explicit*, we understand a study that does not rest after obtaining abstract existence results, but that goes on to look for a more concrete study of varieties, say in terms of equations, that can be used to bring out their geometric properties as clearly as possible. For example, the list of Du Val singularities by equations and Dynkin diagrams is much more than just an abstract definition or existence result, and can be use for all kinds of purposes. This book initiates a general program of explicit birational geometry of 3-folds (compare Section 5). On the whole, our activities do not concern themselves with 3-folds in full generality, but work under particular assumptions, for example, with 3-folds that are hypersurfaces, or have only terminal quotient singularities (see below). The advantage is that we can get a long way into current thinking on 3-folds while presupposing little in the way of technical background in Mori theory.

Treating 3-folds and contractions between them in complete generality would lead us of necessity into a number of curious and technically difficult backwaters; these include many research issues of great interest to us, but we leave them to more appropriate future publications (see however Section 5 below). Making the abstract machinery work in dimension ≥ 4 is another important area of current research, but the geometry of 4-folds is presumably intractable in the explicit terms that are our main interest here.

2 The Mori program

This section is a gentle introduction to some of the ingredients of the 3-fold minimal model program, with emphasis on the aspects most relevant to our current discussion. Surveys by Reid and Kollár [R1], [Kol1], [Kol2] also offer introductory discussions and different points of view on Mori theory. At a technically more advanced level, we also recommend a number of excellent (if somewhat less gentle) surveys: Clemens, Kollár and Mori [CKM], Kawamata, Matsuda and Matsuki [KMM], Kollár and Mori [KM], Mori [Mo] and Wilson [W].

2.1 Terminal singularities

It was understood from the outset that minimal models of 3-folds necessarily involve singular varieties (one reason why is explained in 2.5). The *Mori category* consists of projective varieties with *terminal singularities*; the most typical example is the cyclic quotient singularity $\frac{1}{r}(a, r-a, 1)$. Here a is coprime to r, and the notation means the quotient $\mathbb{C}^3/(\mathbb{Z}/r)$, where the cyclic group $\mathbb{Z}/r$ acts by

$$(x, y, z) \mapsto (\varepsilon^a x, \varepsilon^{r-a} y, \varepsilon z),$$

and ε is a primitive rth root of 1. The most common instance is $\frac{1}{2}(1,1,1)$, the cone on the Veronese surface. The effect of saying that this point is terminal is that if we first resolve it by blowing up, then run a minimal model program on the resolution, we will eventually need to contract down everything we've blown up, taking us back to the same singularity.

There are a few other classes of terminal singularities, including isolated hypersurface singularities such as $xy = f(z,t) \subset \mathbb{C}^4$, where $f(z,t) = 0$ is an isolated plane curve singularity, and a combination of hypersurface and quotient singularity, for example, the hyperquotient singularity obtained by dividing the hypersurface singularity $xy = f(z^r, t)$ by the cyclic group $\mathbb{Z}/r$ acting by $\frac{1}{r}(a, r-a, 1, 0)$. At some time you may wish to look through some sections of Reid [YPG] (especially Theorem 4.5) for a more formal treatment. But for most purposes, the cyclic quotient singularity $\frac{1}{r}(a, r-a, 1)$ is the main case for understanding 3-fold geometry, and if you bear this in mind, you will have little trouble understanding this book.

2.2 Theorem on the Cone

The *Mori cone* $\overline{\mathrm{NE}}\,X$ (see Figure 2.2.1) is probably the most profound and revolutionary of Mori's contributions to 3-folds. An n-dimensional projective variety X over $\mathbb{C}$ is a $2n$-dimensional oriented compact topological space, and its second homology group $H_2(X, \mathbb{R})$ is a finite dimensional real vector space. Every algebraic curve $C \subset X$ can be triangulated and viewed as an oriented 2-cycle, and thus has a homology class $[C] \in H_2(X, \mathbb{R})$. Then by definition $\overline{\mathrm{NE}}\,X$ is the closed convex cone in $H_2(X, \mathbb{R})$ generated by the classes

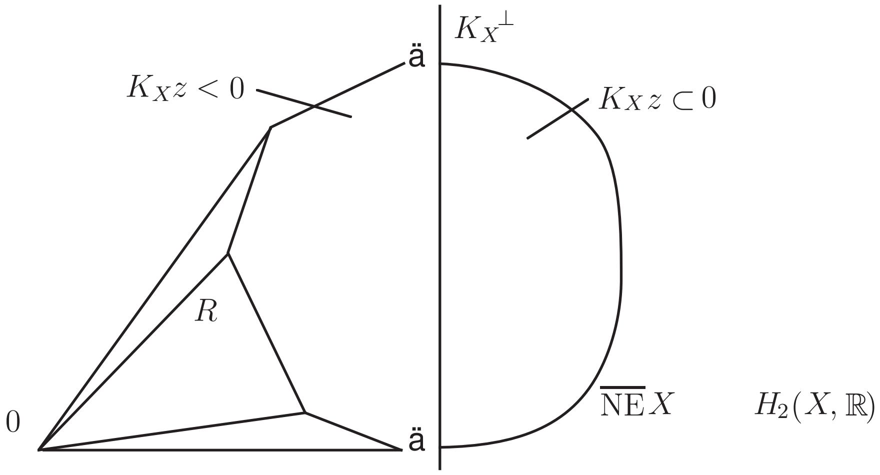

Figure 2.2.1: The Mori cone: $\overline{\mathrm{NE}}\,X$ is locally rational polyhedral in $K_X z < 0$

$[C]$ of algebraic curves $C \subset X$. You can think of this as follows: $H_2(X,\mathbb{R})$ is a property of the topological space X, whereas the structure of X as a projective algebraic variety provides the extra information of the Mori cone $\overline{\mathrm{NE}}\,X \subset H_2(X,\mathbb{R})$.

The shape of $\overline{\mathrm{NE}}\,X$ contains information about linear systems and embeddings $X \hookrightarrow \mathbb{P}^N$. Taking intersection number $D{\cdot}C$ with a divisor, or evaluating $\alpha \cap [C]$ with a cohomology class $\alpha \in H^2(X,\mathbb{R})$ (say, the first Chern class of a line bundle L) defines a linear form on $H_2(X,\mathbb{R})$. We say that D or α is *nef* if this linear form is ≥ 0 on $\overline{\mathrm{NE}}\,X$; that is, a divisor D is nef if $D \cdot C \geq 0$ for every curve $C \subset X$. Under an embedding, every algebraic curve must have positive degree; it is known that, under rather mild assumptions, X is projective if and only if $\overline{\mathrm{NE}}\,X$ is a genuine cone with a point.

To state Mori's theorem, we assume that the canonical divisor class K_X, (or equivalently, the first Chern class of the cotangent bundle) makes sense as a linear form on $H_2(X,\mathbb{R})$. This is a mild extra assumption on X, that certainly holds if X is nonsingular or has at worst quotient singularities. The theorem on the cone then says that $\overline{\mathrm{NE}}\,X$ is a rational polyhedral cone in the half-space of $H_2(X,\mathbb{R})$ on which K_X is negative. This theorem is particularly powerful for Fano varieties, defined by the condition that $-K_X$ is ample: for these, the entire cone $\overline{\mathrm{NE}}\,X$ is contained in $K_X z < 0$, so that $\overline{\mathrm{NE}}\,X$ is a finite rational polyhedral cone.

2.3 Extremal rays and the contraction theorem

Mori theory applies mainly to varieties with K_X not nef. This condition says that $K_X C < 0$ for some curve C, or that the part of the cone $\overline{\mathrm{NE}}\,X$ in the half-space $K_X z < 0$ is nonempty. Since this part of the cone is locally rational polyhedral, it follows that, if K_X is not nef, $\overline{\mathrm{NE}}\,X$ has at least one extremal ray R with $K_X \cdot R < 0$. Here an *extremal ray* is just a half-line $R = \mathbb{R}_+ z \subset \overline{\mathrm{NE}}\,X$ that is extremal in the sense of convex geometry (that is,

$$z_1, z_2 \in \overline{\mathrm{NE}}\,X \text{ and } z_1 + z_2 \in R \implies z_1, z_2 \in R).$$

Let $R \subset \overline{\mathrm{NE}}\,X$ be an extremal ray with $K_X \cdot R < 0$. Then there exists a contraction morphism

$$f_R \colon X \to Y,$$

characterised by the property that a curve $C \subset X$ is mapped to a point if and only if $C \in R$ (more precisely, the class of C). The morphism $f_R \colon X \to Y$ is called a *Mori contraction* or an *extremal contraction*. It is determined by the extremal ray R, and has categorical properties such as $-K_X$ relatively ample and $\rho(X/Y) = 1$ that turn out to be surprisingly strong: for example $-K_X$ ample puts us in a position where vanishing results based on Kodaira vanishing kill almost all the cohomology.

The cone and contraction theorems are proved in Kollár and Mori [KM]; on the whole, we can get by without reference to the technicalities of the proof, and you may prefer to take these results on trust for now.

2.4 Types of extremal rays

The next step is the case division on the dimension of the image Y and of the exceptional locus of the contraction morphism $f_R \colon X \to Y$, called the *classification of extremal rays* (or *rough classification*). The cases when the contraction $f_R \colon X \to Y$ has $\dim Y < \dim X$ lead to the definition of Mori fibre space and Fano varieties discussed in 2.6. In the other cases, we are dealing with birational modifications of X, and, as we see in 2.5, the aim is to proceed inductively towards a minimal model, as in the classical case of surfaces.

Remark 2.4.1 Note the contrast with the classical case: for surfaces, the thing we contract is a geometric locus. We find a -1-curve C and establish that it can be contracted in terms of a neighbourhood of C. In contrast, Mori theory in dimension ≥ 3 works primarily in terms of *categorical definitions* and *existence theorems*: the thing to be contracted is an extremal ray R of $\overline{\mathrm{NE}}\,X$ (the definition of which uses the totality of curves on X). The proof of the general theorems saying that R is contractible by a morphism f_R makes sophisticated use of numerical and cohomology vanishing properties of X.

The geometric nature of the contraction is only studied as a second step; even basic things such as the geometric locus that is contracted or even the dimension of the image cannot be anticipated. f_R may be birational, a proper fibre space, or the constant morphism to a point. This curious inversion of thinking is another of Mori's characteristic contributions to the subject, and the logic still comes as a surprise to anyone knowing a traditional treatment of the classification of surfaces. After all, Castelnuovo and Enriques could scarcely have guessed that (i) contracting a -1-curve, (ii) projecting a geometrically ruled surface to its base curve, and (iii) the constant map of $\mathbb{P}^2$ to a point would find a unified treatment as extremal contractions, and that this idea, however outlandish it might appear at first sight, would lay the foundations of all future work in classification.

2.5 Birational modifications: divisorial contractions, flips and the minimal model program

The extremal contractions that are most similar to contracting a -1-curve on a nonsingular surface (Castelnuovo's criterion) are the *divisorial contractions.* Here the case assumption is that $f_R\colon X \to Y$ is birational, and contracts a divisor of X to a locus of Y of codimension ≥ 2. The categorical properties of f_R then guarantee automatically that the exceptional locus of f_R is an *irreducible* divisor, and that Y has terminal singularities. This is the point at which terminal singularities force themselves on our attention: even if X is nonsingular, Y may be singular. Because Y is still in the Mori category, we can repeat the same game starting from Y.

The other birational case is when f_R is *small*, that is, every component of $\operatorname{Exc} f_R$ has codimension ≥ 2; in this case there cannot be any cohomology class in $H^2(X,\mathbb{R})$ that corresponds to the canonical divisor of Y, so that Y can *never* have terminal singularities. (If such a class existed, its pullback to X would coincide with K_X, which would then be numerically trivial on the fibres of f_R. This contradicts $-K_X$ ample, the defining property of a Mori extremal contraction.)

Because Y is no longer in the Mori category, the minimal model program cannot just continue inductively from Y. The subject was stuck at this point for a few years in the 1980s, before Mori proved the 3-fold flip theorem: there is a *flip*

$$\begin{tikzcd} X \arrow[rr, dashed, "t_R"] \arrow[dr] & & X^+ \arrow[dl] \\ & Y & \end{tikzcd} \tag{2.5.1}$$

where $X^+ \to Y$ is another birational map from a 3-fold X^+, characterised by the property that K_{X^+} is ample over Y. In other words, the birational map $t_R\colon X \dashrightarrow X^+$ cuts out from X a finite number of curves on which K_X is negative, and in their place glues back into X^+ a finite number of curves on which K_{X^+} is positive. The definition of flip may seem somewhat obscure, but many nice attributes of X^+ follow from it; in particular, the morphism $X^+ \to Y$ is also small, and X^+ again has terminal singularities, so is in the Mori category. In dimension ≥ 4, the existence of the flip diagram (2.5.1) is called the *flip conjecture*; this seems to be one of the most intractable problems in the subject.

Divisorial contractions and Mori flips are the elementary steps in the Mori minimal model program. A sequence of these leads after a finite number of steps to a variety X', which is either a *minimal model*, that is, a variety with $K_{X'}$ nef, or a Mori fibre space $f\colon X' \to S$.

2.6 The definition of Mori fibre space

We now discuss the remaining cases in the classification of extremal rays, when the contraction $f_R\colon X \to Y$ maps to a smaller dimensional variety, that is, $\dim Y < \dim X$. Then f_R (or X itself) is called a *Mori fibre space* (Mfs). Note that, following Iitaka and Ueno, we say *fibre space* to mean a morphism $f\colon X \to Y$, often assumed to have connected fibres and Y normal, possibly with varying fibres, singular fibres, even fibres of different dimensions; this is not to be confused with the much stricter notion of fibre bundle.

The cases when Y is a surface and $X \to Y$ is a conic bundle (that is, the general fibre is a conic) or when Y is a curve and $X \to Y$ a fibre space of del Pezzo surfaces are the natural analogues of ruled surfaces. For the logical framework of Mori theory, we include in the definition of Mori fibre space the case that the contraction $f_R\colon X \to Y = \mathrm{pt.}$ is the constant map to a point: then the morphism f_R is trivial, but its categorical properties include the fact that $-K_X$ is ample, and $\operatorname{Pic} X$ has rank 1. In this case X is called a *Fano 3-fold*; in contrast to the classical terminology, we allow X to be singular.

2.7 Biregular geometry versus birational geometry

The dividing line between biregular and birational geometry has changed through the generations, and is possibly still open to debate. The Italian school worked primarily in birational terms, and Zariski and Weil used birational ideas (at least in part) in setting up foundations for biregular geometry. The modern view, with scheme theory firmly established as the foundation, constructs birational geometry within this biregular framework. Thus, while the dichotomy between surfaces having nonvanishing plurigenera and ruled surfaces (or "adjunction terminates") is manifestly birational, we no longer think of it as the primary result of classification, but derive it from biregular results. This new view was instrumental in the success of Mori theory.

When we run a Mori minimal model program on a given 3-fold V, the end product is either a minimal model X with K_X nef, or a Mori fibre space, typically, a Fano 3-fold X or a conic bundle over a surface $X \to S$. The properties that define the 3-fold X are biregular in nature, so that we view X as a biregular construction. From this point of view, the proof of classification should also be considered a biregular activity, since the point is to prove that a given minimal 3-fold X has the right plurigenera and Kodaira dimension. Our conclusion is that birational geometry begins with the question of birational maps between different Mori fibre spaces.

3 What this book contains

This section discusses briefly the papers in this book, and their contribution to the above program of study. The papers are:

(1) K. Altmann: One-parameter families containing three-dimensional toric Gorenstein singularities

(2) J. Kollár: Nonrational covers of $\mathbb{P}^m \times \mathbb{P}^m$

(3) A. V. Pukhlikov: Essentials of the method of maximal singularities

(4) A. R. Iano-Fletcher: Working with weighted complete intersections

(5) A. Corti, A. Pukhlikov and M. Reid: Fano 3-fold hypersurfaces

(6) A. Corti: Singularities of linear systems and 3-fold birational geometry

(7) M. Reid: Twenty five years of 3-folds, an old person's view

Klaus Altmann's paper (1) is a study of the deformation theory of toric Gorenstein 3-fold singularities. It relates to the classification of 3-fold flips as follows: we know that any Mori flip diagram (2.5.1) can be obtained from a $\mathbb{C}^\times$ action on a 4-fold Gorenstein singularity $0 \in A$ by taking the quotient by the $\mathbb{C}^\times$ action in different interpretations – the so-called *variation of geometric invariant theory quotient*, see Dolgachev and Hu [DH], Reid [R2] and Section 5.3 below. Moreover, the general anticanonical divisor $S \in |-K_X|$ (the *general elephant*) is a surface with only Du Val singularities, according to Kollár and Mori [KM1], Theorem 1.7. Its inverse image in A is a $\mathbb{C}^\times$ cover $B \to S$, and is a hyperplane section $B \subset A$, so that A can be viewed as a 1-parameter deformation of B. It frequently happens that S is of type A_n, and then B is toric, so that Altmann's theory applies in many cases to give a classification of 3-fold flips. Altmann's previous work [Al] used the notion of Minkowski decomposition of polytopes to give a complete treatment of the deformation of isolated 3-fold toric Gorenstein singularities; in the present paper, he shows how to modify his method to the case of toric varieties having singularities in codimension 2.

János Kollár's paper (2) provides a new method of proving irrationality, adding to the known collection of rationally connected varieties that are not rational: finite covers of $\mathbb{P}^m \times \mathbb{P}^n$ with ramification divisor of large enough degree in one factor, and hypersurfaces in $\mathbb{P}^m \times \mathbb{P}^n$ of large enough degree. His technique involves reduction to characteristic p, and a rather clever and surprising analysis of the stability of the tangent bundle in characteristic p. In fact, he proves the slightly more general structural property that these varieties are not even ruled. In the case of conic bundles, these results are spectacularly close to the conjectural bound for rationality (compare, for example, paper (2), Remark 1.2.1.1 with Corti's paper (6), 4.10 and 4.11). This provides the strongest confirmation to date of the conjectures on conic bundles, in a numerical range that is inaccessible to all other methods.

The papers (3)–(6) form a connected suite of papers around the subject of *birational rigidity*. The notion, discussed in more detail in Section 4.5 below, originates in the famous result of Iskovskikh and Manin [IM] that a nonsingular quartic 3-fold $X_4 \subset \mathbb{P}^4$ has no birational maps to Fano varieties (other than isomorphisms to itself). Pukhlikov's paper (3) describes his important simplification and elaboration of Iskovskikh and Manin's treatment. This paper is partly based on notes of lectures given at the 1995–96 Warwick algebraic geometry symposium and the preprint, with its clear treatment of the Russian methods, strongly stimulated our collaboration in the joint paper (5). The different approach in Pukhlikov's papers also offers a useful ideological and practical counterweight to the methods of Corti's paper (6).

Our long joint paper (5) is the real heart of this book. In it, we carry out a substantial portion of a program of research on birational rigidity, treating the *famous 95* families of Fano 3-fold weighted hypersurfaces. We refer to Section 4.5 and the introduction to paper (5) for further discussion of birational rigidity.

Anthony Iano-Fletcher's paper (4) is a well written tutorial introduction to weighted projective spaces and their subvarieties. This paper has been available for many years as a Max Planck Institute preprint, and is widely quoted in the literature; it contains many very useful results and methods of calculation, including one derivation of the list of the famous 95 hypersurfaces, and thus forms an essential prerequisite for paper (5).

Corti's paper (6) contains a detailed introduction to the Sarkisov program. It develops and applies powerful new methods to quantify and analyse the singularities of linear systems, clarifying and providing technical alternatives to the methods initiated by Iskovskikh and Manin based on the study of the resolution graph. The new ideas are based on the Shokurov connectedness principle in log birational geometry, and seem to provide the most powerful currently known technique to exclude birational maps between Mori fibre spaces. The results are applied to give rigidity criteria for Mori fibre spaces in a number of cases, and our joint paper (5) also appeals to them for one or two technical points.

Reid's historical paper (7) is a *Heldenleben* that needs no introduction.

4 Mori theory and birational geometry

Following our introductory remarks on Mori theory in Section 2, we now give a brief introduction to our view of birational geometry, including the Sarkisov program and birational rigidity.

4.1 Fano-style projections

Fano based his treatment of the 3-folds $V_{2g-2} \subset \mathbb{P}^{g+1}$ for $g \geq 7$ on the idea of constructing a birational map by projection from a suitably chosen centre. Typically, the double projection of V from a line L involves a diagram

$$\begin{array}{ccc} V' & \dashrightarrow & V'' \\ \downarrow & & \downarrow \\ V & & W, \end{array} \tag{4.1.1}$$

where $V' \to V$ is the blowup of $L \subset V$, the map $V' \dashrightarrow V''$ flops the lines meeting L (in good cases, finitely many lines with normal bundle of type $(-1,-1)$), and $V'' \to W$ contracts the surface $E \subset V''$ swept out by conics meeting L to a curve $\Gamma \subset W$. Fano thought of the map $V \dashrightarrow W$ as the rational map defined by linear projection, and factoring it in biregular terms was not his primary concern.

For us, on the other hand, it is important to view Fano's projection as a general construction in the Mori category: $V' \to V$ is an extremal extraction, $V' \dashrightarrow V''$ a rational map that is an isomorphism in codimension 1 (in good cases, a composite of classical flops), and $V'' \to W$ the contraction of an extremal ray. All 4 of the varieties in (4.1.1) are in the Mori category, and the two morphisms are contractions of extremal rays.

Remark 4.1.1 We take the opportunity to clear up a possible source of confusion that occurs throughout the subject: in Fano's case, the *single* projection from L contracts the flopping lines by a morphism $V' \to \overline{V}$ to a variety $\overline{V}$ having (in good cases) only 3-fold ordinary double points; we think of $\overline{V}$ as the *midpoint* of the construction of the link $V \dashrightarrow W$. It is a Fano variety in some sense, since it has terminal singularities and $-K_{\overline{V}}$ is ample, but it is *not in the Mori category*, because it is not $\mathbb{Q}$-factorial: the exceptional scroll over L maps to a divisor in $\overline{V}$ that is not Cartier and not $\mathbb{Q}$-Cartier at the nodes.

4.2 Sarkisov links

Sarkisov links play the role of "elementary transformations" for birational maps between Mori fibre spaces. The *Sarkisov program* factors an arbitrary birational map $X \dashrightarrow Y$ between the total spaces of Mori fibre spaces $X \to S$ and $Y \to T$ as a chain of such links (see Corti's paper (6) and [Co] for details).

A general *Sarkisov link* is given by a diagram in the Mori category that is a variation on (4.1.1), but with general extremal contractions allowed as the morphisms. As discussed in much more detail in Corti [Co] and paper (6), if we start from a Mori fibre space $X \to S$, any link $(X/S) \dashrightarrow (Y/T)$ is given by one of the following constructions. First, we replace $X \to S$ by a new morphism $X_1 \to S_1$ having rank $N^1(X_1/S_1) = 2$: for this, either

(i) blow up X by an extremal blowup $X_1 \to X$, and let $X_1 \to S_1$ be the composite $X_1 \to X \to S = S_1$; or

(ii) contract the base by an extremal contraction $S \to S_1$, and let $X_1 \to S_1$ be the composite $X_1 = X \to S \to S_1$.

In either case $N^1(X_1/S_1) = \mathbb{R}^2$, and $\overline{\mathrm{NE}}(X_1/S_1)$ has an initial extremal ray corresponding to the given morphism $X_1 \to X$ or $X_1 \to S$. This sets up a restricted type of minimal model program called a 2*-ray game*: because a cone in $\mathbb{R}^2$ is just a "wedge", it has a far side that is a (pseudo-) extremal ray (possibly not of Mori type). The case that leads to a link is when the minimal model program runs to completion in the Mori category: the far ray can be contracted, and possibly after a chain of inverse flips, flops and flips, the minimal model program ends with a divisorial contraction $X^{(n)} \to Y \to S_1 = T$ or a contraction of fibre type $X^{(n)} = Y \to T \to S_1$. There are two possible ways of starting the construction, and two ways of ending it, leading to Sarkisov links of Type I–IV.

Existence and uniqueness: the 2-ray game is entirely determined by the initial step X_1/S_1. It may happen that the initial step X_1/S_1 does not construct a link – either because an inverse flip demanded by the 2-ray game does not exist or has worse than terminal singularities, or because the final divisorial or fibre type contraction falls out of the Mori category. See paper (5), Section 5.5 and 7.6 for examples.

For surfaces over an algebraically closed field, links are the following familiar transformations: the blowup taking $\mathbb{P}^2$ to $\mathbb{F}_1$, its inverse contraction

$\mathbb{F}_1 \to \mathbb{P}^2$, the well known elementary transformations $\mathbb{F}_k \dashrightarrow \mathbb{F}_{k\pm 1}$ between scrolls over $\mathbb{P}^1$, and the "exchange of factors" of $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ (in other words, the identity on $\mathbb{F}_0$, but viewed as exchanging its two projections). These are exactly the elementary steps in Castelnuovo's proof of Max Noether's theorem (discussed in 4.4 below).

4.3 The Sarkisov program

We explain in conceptual terms how to factor (or "untwist") a birational map $\varphi\colon X \dashrightarrow X'$ between Mori fibre spaces $X \to S$ and $X' \to S'$ as a chain of Sarkisov links. Untwisting is a constructive descending induction: let $\mathcal{H}'$ be a very ample complete linear system on X', chosen at the outset and kept fixed throughout. Following the classical ideas of Cremona, Noether and Hudson, consider the linear system $\mathcal{H} = \varphi_*^{-1}\mathcal{H}'$ on X obtained as the birational transform of $\mathcal{H}'$. Untwisting is the story of how we reduce the singularities and the degree of $\mathcal{H}$ to make φ an isomorphism.

First we prove the *Noether–Fano–Iskovskikh inequalities*, that serve as a sensitive detector to locate the initial step X_1/S_1 of a Sarkisov link if φ is not already an isomorphism: this is either a blowup of a maximal singularity of $\mathcal{H}$, or a way of viewing X over a different base to make $\mathcal{H}$ look simpler. Next, when needed to factor a given map, the Sarkisov link $\psi\colon (X/S) \to (Y/T)$ with given initial step X_1/S_1 always exists.

At the start of the proof, we set up a discrete invariant of φ, its *Sarkisov degree* $\deg\varphi$. We only explain this for a Fano 3-fold X, when $-K_X$ is a $\mathbb{Q}$-basis of $\operatorname{Pic} X$. Then we set $\deg\varphi = n$, where n is the positive rational number for which $\mathcal{H} \subset |-nK_X|$. We prove that the Sarkisov link $\psi\colon X \dashrightarrow Y$ provided by the NFI inequalities decreases the Sarkisov degree, in the sense that the composite map $\varphi\psi^{-1}\colon Y \dashrightarrow X'$ between Y/T and X'/S' has

$$\deg\varphi\psi^{-1} < \deg\varphi.$$

We say that $\varphi\psi^{-1}$ is an *untwisting* of φ by ψ. The factorisation theorem then follows by descending induction on the Sarkisov degree. Of course, we are glossing over many subtle points, including the definition of Sarkisov degree for a strict Mori fibre space $X \to S$, the verification that untwisting by a link decreases $\deg\varphi$, and that a chain of untwistings must terminate. For the details, see paper (6).

4.4 A classical example

We content ourselves with illustrating how these ideas work in the most famous case of all, a birational map from $X = \mathbb{P}^2$ to $X' = \mathbb{P}^2$. Max Noether's inequality states that there are 3 points P_1, P_2, P_3 of $\mathbb{P}^2$ (possibly infinitely near), such that

$$m_1 + m_2 + m_3 \ > \ n = \deg\varphi, \quad \text{where } m_i = \operatorname{mult}_{P_i}\mathcal{H}. \tag{4.4.1}$$

Here $\operatorname{mult}_{P_i}\mathcal{H}$ means the multiplicity of a general element of the linear system $\mathcal{H}$ at P_i. In the general case, when P_1, P_2, P_3 are distinct noncollinear points, we can choose coordinates so that these are the three coordinate points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, and it is easy to check that untwisting by the standard quadratic Cremona involution

$$\psi\colon (x_0 : x_1 : x_2) \dashrightarrow (x_1x_2 : x_0x_2 : x_1x_2)$$

decreases the degree. Indeed, $\psi^{-1}(\text{line})$ is a conic passing through the P_i, so that after untwisting, the degree becomes

$$2n - \sum m_i < n.$$

The gap in this argument is that the points P_i can be infinitely near. This can happen in two or three different ways, and in most of these cases, we can still construct a quadratic transformation centred on a suitably chosen coordinate triangle to untwist our map and decreases its degree. However, there are cases in which no single quadratic Cremona transformation decreases the degree.

This is the starting point of Castelnuovo's proof of Noether's theorem, a direct precursor of the Sarkisov program. Whatever infinitely near points there may be, there always exists one point $P \in \mathbb{P}^2$ with $m_P > \frac{n}{3}$. Of course, this follows from (4.4.1), but it is also easy to prove directly by an easy argument in the spirit of "termination of adjunction" (see paper (6), Theorem 2.4, where the inequality and the argument to prove it are generalised to any Mori fibre space). Blowing up this point by $\mathbb{F}_1 \to \mathbb{P}^1$ is the first link in a Sarkisov chain. It untwists because the Sarkisov degree measures divisors on $\mathbb{P}^2$ in terms of $-K_{\mathbb{P}^2} = \mathcal{O}(3)$, but measures relative divisors on $\mathbb{F}_1$ in terms of the relative $-K_{\mathbb{F}_1/\mathbb{P}^1} = \mathcal{O}(2)$ ([Co], 1.3 contains a more detailed description).

4.5 Birational rigidity

A Fano variety X is *birationally rigid* if for every Mori fibre space $Y \to S$, the existence of a birational equivalence $X \dashrightarrow Y$ implies that $Y \cong X$. Once the Sarkisov program is established (that is, as yet only in dimension ≤ 3), it is equivalent to say that there are either no Sarkisov links out of X, or only self-links $X \dashrightarrow X$.

The notion of birational rigidity for strict Mori fibre spaces is also well studied. However, the definition (paper (6), Definitions 1.2–3) is subtle and somewhat confusing, largely because it treats Mori fibre spaces up to *square birational equivalence.* It covers important results of Sarkisov on conic bundles and Pukhlikov on del Pezzo fibre spaces.

The main result of paper (5) states that a general member of any of the 95 families of Fano 3-fold hypersurfaces is rigid. We show that any birational map $\varphi\colon X \dashrightarrow Y$ factors as a chain of Sarkisov links $X \dashrightarrow X$, followed by an isomorphism $X \cong Y$. The paper is written to be essentially selfcontained. The Sarkisov program is introduced in a context where it is made simpler by various special circumstances. However, at one point, we rely on a technical statement (Theorem 5.3.3) that is proved in Corti's paper (6) (although it goes back essentially to Iskovskikh and Manin, and can be proved by the technique of Pukhlikov's paper (3)). A substantial part of paper (5) is devoted to the classification of links $X \dashrightarrow X$, which we also discuss in Section 5.5 below. It was rather surprising for us to discover that all the links can be described explicitly in terms of just two basic constructions in commutative algebra, which are natural generalisations of the classical Geiser and Bertini involutions of cubic surfaces.

4.6 Beyond rigidity

We now know a handful of varieties having precisely two models as Mori fibre space, either two Fano 3-folds (Corti and Mella [CM]), or two del Pezzo fibrations (Grinenko [Gr]), or one of each (in the case $X_{3,3} \subset \mathbb{P}(1,1,1,1,2,2)$ suggested by Grinenko and Pukhlikov). These varieties are not actually rigid, but nearly so. These and other examples suggest the following idea. Say that a birational map $\varphi\colon X \dashrightarrow X'$ between Mori fibre spaces $X \to S$ and $X' \to S'$ is a *square equivalence* if the following two conditions hold:

(1) There is a birational map $S \dashrightarrow S'$ making the obvious diagram commute. This condition is equivalent to saying that the generic fibre of $X \to S$ is birational to the generic fibre of $X' \to S'$ under φ.

(2) The birational map of (1), from the generic fibre of $X \to S$ to a generic fibre of $X' \to S'$, is in fact biregular.

We define the *pliability* of a Mori fibre space $X \to S$ as the set

$$\mathcal{P}(X/S) = \big\{ \text{Mfs } Y \to T \bigm| X \text{ is birational to } Y \big\} / \text{square equivalence.}$$

Our general philosophy can be described as follows. We would like to describe the pliability $\mathcal{P}(X/S)$ of a Mori fibre space $X \to S$ in terms of its biregular geometry. There are reasons for thinking that $\mathcal{P}(X/S)$ will often have a reasonable description, say as a finite set or a finite union of algebraic varieties. A case division based on the various possibilities for the size of

$\mathcal{P}(X/S)$ can be used as a further birational classification of Mori fibre spaces. Note that our general philosophy can only be stated in the language of Mori theory.

Although the search for counterexamples to the Lüroth theorem played an important part in kick-starting the study of 3-folds around 1970, the last 25 years have seen little progress on criteria for rationality and unirationality, and these questions seem likely to remain intractable in the foreseeable future. The wealth of new examples in Mori theory must in any case cast doubt on the position of these problems as central issues in birational geometry; they date back after all to the golden days of innocence, when the classics (from Cremona through to Iskovskikh) had never really met the typical examples of birational geometry. Rationally connected seems to be the most useful modern replacement for (uni-)rationality, since it is robust on taking surjective image or under deformations, and there are good criteria for it. It is not that we are hostile to the rationality problem; rather, since we are committed to the classification of 3-folds, we need a theoretical framework capable of accomodating all varieties, with all their wealth of individual behaviour. Our suggested notion of pliability is a tentative step in this direction.

5 Some open problems

5.1 Explicit birational geometry

In a sense, we can define explicit birational geometry as the concrete study (including classification) of

(1) divisorial contractions, flips, and Mori fibre spaces;

(2) the way divisorial contractions and flips combine to form the links of the Sarkisov program.

The model is [YPG], Theorem 4.5, which classifies all 3-fold terminal singularities as a reasonably concrete, explicit and finite list of families. It seems reasonable to hope that Mori flips, divisorial contractions, Fano 3-folds and Sarkisov links between 3-fold Mori fibre spaces will eventually succumb to a similar treatment. The overall aim is to make everything else tractable in the same sense as the terminal singularities. This program can be expected to provide useful employment for algebraic geometers over several decades.

5.2 Divisorial contractions

Problem 5.2.1 Fix a 3-fold terminal singularity $P \in Y$ (an analytic germ). Write down all 3-fold divisorial contractions $f\colon (E \subset X) \to (P \in Y)$ by explicit equations.

This seems a rather difficult problem in general. To do something useful with it, it is important to realise that, whereas you are free to pick your favourite singularity $P \in Y$, it is then your responsibility to classify all possible extremal blowups $X \to Y$ of P. The few known cases of this problem are $P \in Y = \frac{1}{r}(1, -1, a)$ treated by Kawamata [Ka], the ordinary node (see paper (6), Chapter 3), the singularity $xy = z^3 + w^3$ of Corti and Mella [CM]. Each of these results has significant applications to the Sarkisov program and 3-fold birational geometry; see paper (6), Section 6.3 and the forthcoming paper [CM]. See also [Ka] for more examples.

Example 5.2.2 Let $P \in Y$ be a nonsingular point and a, b coprime integers; then the weighted blowup $f\colon X \to Y$ with weights $(1, a, b)$ is an extremal divisorial contraction with $f(E) = P$. Corti conjectured in 1993 that this is the complete list.

5.3 What is a flip?

Problem 5.3.1 Classify 3-fold flips $t\colon X \dashrightarrow X^+$

In a sense, this is done in the monumental paper of Kollár and Mori [KM1], but their description is not sufficiently explicit for some applications. The following, taken from [KM1], is the simplest example of flips (see Brown [Br] for many similar families of examples).

Example 5.3.2 Let $f_{m-1}(x_1, x_2)$ be a homogeneous polynomial of degree $m - 1$ in 2 variables, and let $\mathbb{C}^\times$ act on $\mathbb{C}^5$ with weights $1, 1, m, -1, -1$. That is, the action is given by

$$x_1, x_2, x_3, y_1, y_2 \mapsto \lambda x_1, \lambda x_2, \lambda^m x_3, \lambda^{-1} y_1, \lambda^{-1} y_2.$$

Consider the $\mathbb{C}^\times$-invariant affine 4-fold $0 \in A \subset \mathbb{C}^5$ given by

$$x_4 y_1 = f_{m-1}(x_1, x_2).$$

An example of a flipping contraction $X \to Y$ and flip X^+ is obtained by taking the geometric invariant theory quotient (Spec of the ring of invariants) $Y = A/\!/\mathbb{C}^\times$, and setting

$$X = \operatorname{Proj} \bigoplus_{n \leq 0} \mathcal{O}(nK_Y) \quad \text{and} \quad X^+ = \operatorname{Proj} \bigoplus_{n \geq 0} \mathcal{O}(nK_Y)$$

for the two sides of the flip.

Quite generally, all flips arise in this way from an affine Gorenstein 4-fold $0 \in A = \operatorname{Spec} \bigoplus_{n \in \mathbb{Z}} \mathcal{O}(nK_Y)$ with $\mathbb{C}^\times$ action (see [R2]), and the problem is to write manageable equations for A in $\mathbb{C}^\times$-linearised coordinates. These considerations are the starting point of Klaus Altmann's paper (1).

5.4 Mori fibre spaces

Problem 5.4.1 Classify Fano 3-folds (with $B_2 = 1$ and $\mathbb{Q}$-factorial terminal singularities) up to biregular equivalence.

An example, and a beginning of an answer to this problem, is the list of 95 Fano 3-fold (weighted) hypersurfaces of Reid and Iano-Fletcher (see paper (4), 16.6), but also the 86 codimension 2 (weighted) complete intersections (paper (4), 16.7), the 70 codimension 3 Pfaffian cases of Altınok [Al], Table 5.1, p. 69, etc. It is a feature of nonsingular Fano 3-folds that they all are linear sections of standard homogeneous varieties in their Plücker embedding. It is just possible that, when looked at from the correct angle, this beautiful and fundamentally simple structure will extend to singular Fano 3-folds.

5.5 The links of the Sarkisov program

It is in the study of the links of the Sarkisov program, the basis of its applicability, that explicit birational geometry really comes alive. Here we fix a Mori fibre space $X \to S$, and we ask to classify all links $X \dashrightarrow Y$, taking off from X and landing at an arbitrary Mori fibre space $Y \to T$. For X a general member of one of our famous 95 families of Fano 3-fold hypersurfaces, this program is carried out to completion in paper (5). In particular, at the end of the classification, we discover that $Y \cong X$. The links that occur fit into a very small number of known classes. On the other hand, in doing many concrete examples, it is our experience that each new case that we understand involves learning how to do computations (for example, in graded rings or in the geometry of projections), a process that can sometimes be rather tricky.

Example 5.5.1 Consider the Fano variety $X = X_{2,2,2} \subset \mathbb{P}^6$ given as a complete intersection of three sufficiently general quadrics in $\mathbb{P}^6$, and choose a line $L \subset X$. Then there is a link $\tau_L \colon X \dashrightarrow X'$, to a conic bundle $X' \to \mathbb{P}^2$ with discriminant curve of degree 7.

The rational map $X \dashrightarrow \mathbb{P}^2$ is not too difficult to realise. For a point $x \in \mathbb{P}^6$, write Π_x for the 2-plane spanned by L and x. Write $\{Q_\lambda \mid \lambda \in \Lambda\}$ for the net of quadrics vanishing on X. Then the restriction $Q_{\lambda|\Pi_x} = L + \Gamma_\lambda$ is the union of L plus a line Γ_λ, and it is easy to see that $x \notin X$ if and only if $\{\Gamma_\lambda \mid \lambda \in \Lambda\}$ is the whole of $\Pi_x^\vee$. Mapping $x \in X$ to the quadric Q_λ containing all the Π_x (which is unique, in general) gives a rational map $X \dashrightarrow \Lambda \cong \mathbb{P}^2$, whose fibres are conics.

The first point in seeing this process as a Sarkisov link is that we must understand it in explicit biregular terms, and factor the map $X \dashrightarrow \Lambda$ as a chain of flips, flops and divisorial contractions, followed by a Mori fibre space. In fact, if $x_0, \dots, x_6$ are coordinates on $\mathbb{P}^6$ with $L : \{x_0 = x_1 = \cdots = x_4 = 0\}$, the equation of X can be written as

$$M\begin{pmatrix}x_5\\x_6\end{pmatrix}=\mathbf{q}$$

where M is a 3×2 matrix of linear forms in the variables $x_0,\dots,x_4$ and $\mathbf{q}$ is a 3-vector of quadric forms in the same variables. Write $\pi_L\colon X\to\overline{Y}\subset\mathbb{P}^4$ for the projection to $\mathbb{P}^4$. The image $\overline{Y}=\overline{Y}_4$ is the quartic 3-fold given by the equation

$$\det M\mathbf{q}=0.$$

The singular locus of $\overline{Y}_4$ consists of the 44 ordinary nodes

$$\{\operatorname{rank} M\mathbf{q}=1\}.$$

Consequently, letting $Y\to X$ be the blow up of $L\subset X$, the birational morphism $Y\to\overline{Y}$ contracts 44 lines with normal bundle $(-1,-1)$, and denoting the flop $t\colon Y\dashrightarrow Y'$, it is easy to check that the rational map $Y\dashrightarrow\Lambda$ described above becomes a morphism $Y'\to\Lambda$, which is in fact a Mori fibre space and a conic bundle. This explicit construction shows that the map $\tau_L\colon X\dashrightarrow Y'$ is a Sarkisov link of Type II.

Similarly, it is easy to construct self-links $\sigma_C\colon X\dashrightarrow X$ centred on conics and twisted cubics $C\subset X$.

The methods of paper (6), Section 6.6, should prove that the only links $X\dashrightarrow Y$, starting with a general $X=X_{2,2,2}$, are the τ_L and σ_C just described.

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It is our solemn duty, as editors, to apologise to the other authors for our slow pace in preparing this book, which has delayed by several years the appearance of the papers (1)–(4). This is especially reprehensible since the desire to understand their results and assimilate them into our own work has

been a strong motivation for our research (this applies to Klaus Altmann's paper (1), and more especially to Sasha Pukhlikov's paper (3), which provided much of the stimulus for our papers (5) and (6)), and we are deeply conscious of the fact that our own work has improved partly as a result of the delay we have inflicted on theirs. We hope that the quality of the final product can go some way towards compensating for our transgressions.

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