Pisa Examples

Alessio Corti

London, 30th May 2008

Abstract

This is a list of exercises to go with my course[1] "New methods in orbifold Gromov-Witten theory" in Pisa, June 15–22 2008. The aim is to develop common-sense and feeling for Gromov-Witten theory of stacks through simple examples rather than general formalism.[2]

Contents

1 Orbi-curves

This is a series on the basics of orbi-curves. An orbi-curve is a nodal twisted curve $\big(\mathfrak{C}, x_i(r_i)\big)$ where all points with non-trivial stabiliser are marked with an isomorphism $G_{x_i} = \mu_{r_i}$ (sometimes I omit the marked points from the notation).

[1]I apologise to those involved but this text contains no references, but I do want to thank my collaborators Tom Coates, Hiroshi Iritani and Hsian-Hua Tseng who taught me almost all I know about Gromov-Witten theory

[2]I expect that this text contains several misprints. Sorry.

(1) (i) Persuade yourselves that the orbifold fundamental group of a smooth orbi-curve $(\mathfrak{C}, x_i(r_i))$ is

$$\pi_1^{\mathrm{orb}}\mathfrak{C} = \pi_1\big(\mathfrak{C} \smallsetminus \{x_i\}\big)/\langle\gamma_i^{r_i}\rangle$$

where γ_i are small loops around the punctures.

(ii) Let $\mathfrak{C}$ be a smooth orbi-curve and G a finite group. Show that to give a representable morphism $\mathfrak{C} \to BG$ is equivalent to give a group homomorphism $\pi_1^{\mathrm{orb}}\mathfrak{C} \to G$ which sends each γ_i to an element of order r_i. The data is also equivalent to give a principal G-bundle on $\mathfrak{C}$, that is a space $\pi\colon G \curvearrowright E \to C$ (where C is the coarse moduli space of $\mathfrak{C}$) which is a principal G-bundle over $C \smallsetminus \{x_i\}$ and has inertia group μ_{r_i} above x_i.

(2) Show Riemann-Roch and Serre duality for an orbi-curve $\mathfrak{C}$. For example, if L is a line bundle, then we get representations of μ_{r_i} on the fibre L_{x_i} of L at x_i and a Riemann-Roch formula

$$\chi(\mathfrak{C}, L) = \deg L + 1 - g - \sum \frac{k_i}{r_i}$$

(3) If $(\mathfrak{C}; x_i(r_i))$ is a n-pointed orbi-curve and $f\colon (\mathfrak{C}; x_i(r_i)) \to \mathfrak{X}$ is a stable representable morphism, then $f^*T_{\mathfrak{X}}$ makes sense is an orbi-bundle and $\mu_{r_i} = G_{x_i}$ acts through the representation into $G_{f(x)}$; we label the representation at x_i by its weights $0 \leq w_{i,j} < r_i$; persuade yourself that the expected dimension of the moduli space is

$$\begin{aligned}\dim \mathfrak{X}_{0,n,\beta} &= \chi\big(\mathfrak{C}, f^*T_{\mathfrak{X}}\big) + n - 3 = \\ &= -K_{\mathfrak{X}} \cdot \beta + \dim \mathfrak{X} + n - 3 - \sum_{i=1}^{n} \sum_{j=1}^{\dim \mathcal{X}} \frac{w_{i,j}}{r_i}\end{aligned}$$

[Hint. Denote by

$$\mathbb{L}_f^\bullet = f^*\Omega^1_{\mathfrak{X}} \to \Omega^1_{\mathfrak{C}}$$

the cotangent complex of the morphism f. The deformation theory of f is controlled by the hyperext algebra $\mathrm{Ext}^\bullet\big(\mathbb{L}_f, \mathcal{O}_{\mathfrak{C}}\big)$.]

(4) Let $f\colon (\mathfrak{C}, x_i(r_i)) \to \mathfrak{X}$ be a stable morphism. Let us assume that f is a embedding locally at every point of $\mathfrak{C}$. In this case, there is a locally free sheaf N_f, the *normal bundle* of f, defined by the sequence

$$0 \to N_f^\vee \to \Omega^1_{\mathfrak{X}} \to \Omega^1_{\mathfrak{C}} \to 0$$

This needs to be taken with a pinch of salt: show that, when $\mathfrak{C}$ has nodes, the natural sheaf homomorphism $T_{\mathfrak{X}} \to N_f$ is not surjective.

(5) Find concrete models for the moduli stacks of stable morphisms of degree ≤ 2 from orbi-curves to $\mathbb{P}(1,1,2)$. Determine which components have the correct dimension, which are smooth as Deligne-Mumford stacks, and the nature of all the singular points.

[Degree two is very tough, but try to do at least the case of morphisms of degree $3/2$.]

(6) Give a sensible definition of a "stacky" *topological Euler number* of a smooth stack curve. State some properties of the topological Euler number. Let C be a smooth proper curve and G a finite group acting on C: calculate the stacky topological Euler number of the stack $[C/G]$ in terms of vertices, edges and faces of a G-invariant cellular decomposition of C.

(7) (i) Given a Deligne-Mumford stack $\mathfrak{X}$, build a model for the simplicial stack made of moduli stacks $\mathfrak{X}_{0,\bullet,0}$ of genus 0 $\bullet$-pointed stable morphisms of degree 0 in terms of "higher inertia" stacks $\mathfrak{X}_{\bullet}$. Carefully identify all degeneneracy and face maps.

(ii) Build a model for the moduli stack $\mathfrak{X}_{1,1,0}$ of genus 1 1-pointed stable morphisms of degree 0. Be careful: this is rather tricky. For instance if $\mathfrak{X} = BG$, then $\mathfrak{X}_{1,1,0}$ is a moduli stack of G-twisted covers.

2 Gromov-Witten invariants of stacks: basic examples

In each of the following questions, you are asked to compute the small quantum orbifold cohomology of a simple explicit stack $\mathfrak{X}$ naïvely from the definition. This is hard work but it does give a "body" to a very abstract formalism.

If $\mathbf{w} = (w_0, \ldots, w_n)$ is an integer vector and $\mathfrak{X} = \mathbb{P}^{\mathbf{w}}$ the corresponding weighted projective space, than the components of the inertia stack are in 1-to-1 correspondence with the set

$$F = \left\{ \frac{k_i}{w_i} \mid i = 0, \ldots, n; \quad 0 \leq k_i < w_i \right\}$$

We denote by $\mathfrak{X}_{0,n,d}(f_1,\dots,f_n)$ the connected component of $\mathfrak{X}_{0,n,d}$ of stable morphisms which "evaluate" in the components of inertia corresponding to $f_1,\dots,f_n \in F$.

I often confuse degree in cohomology with degree in the Chow ring—please sort out the factors of 2 on your own.

(8) $\mathfrak{X} = \mathbb{P}(1,1,3)$ (i) Show that $H^\bullet_{\mathrm{orb}}\mathfrak{X}$ is generated as a vector space by classes $\mathbf{1}, \eta_{\frac{1}{3}}, A = \mathcal{O}(1), \eta_{\frac{2}{3}}, A^2$ in cohomology degrees $0, 2/3, 1, 4/3, 2$.

(ii) Show directly from the definition that

$$\eta_{\frac{1}{3}} \cup \eta_{\frac{1}{3}} = \eta_{\frac{2}{3}}, \quad \text{and} \quad \eta_{\frac{1}{3}} \cup \eta_{\frac{2}{3}} = A^2 = \frac{1}{3}\mathrm{pt}.$$

[Hint. For the first one look for constant representable morphisms in $\mathfrak{X}_{0,3,0}(\frac{1}{3}^3)$. Note that the last point evaluates with inversion in $\mathbb{P}(3)_{\frac{2}{3}}$. The moduli space has virtual dimension $0+2-2/3-2/3-2/3=0$; etc.

For the second look for constant representable morphisms in $\mathfrak{X}_{0,3,0}(1/3,2/3,0)$; the expected dimension of the moduli space is $0+2-2/3-4/3=0$; the relevant component of the moduli space is isomorphic to $\mathbb{P}(3)$; by definition

$$\eta_{\frac{1}{3}} \cup \eta_{\frac{2}{3}} = e_{3\,*}\mathbf{1} = \frac{1}{3}\mathrm{pt}$$

—whatever it is, it has degree $\int_{\mathbb{P}(3)} \mathbf{1} = 1/3$.]

Finally, it is clear that $\eta_{\frac{2}{3}} \cup \eta_{\frac{2}{3}} = 0$; indeed, $\mathfrak{X}_{0,3,0}(\frac{2}{3}^3)$ has virtual dimension $0+2-4/3-4/3-4/3<0$.

(iii) First note that $\operatorname{codim} q = \frac{1+1+3}{1\times 1\times 3} = 5/3$ (why?). In the basis above, write down the matrix of quantum multiplication by A:

$$M = \begin{pmatrix} 0 & aq^{\frac{1}{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & bq^{\frac{1}{3}} & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & cq^{\frac{1}{3}} \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

and show that $a=b=c=1/3$ by interpreting the unknown entries a,b,c in terms of stacky Gromov-Witten invariants.

[Hints: First, from $A * \eta_{\frac{1}{3}} = \mathbf{1}aq$ and integrating against A^2:

$$1/3\, aq = \deg A^2\, aq = \langle \mathbf{1}, A^2\rangle\, aq = \langle A * \eta_{\frac{1}{3}}, A^2\rangle = \langle A, \eta_{\frac{1}{3}}, A^2\rangle_{1/3}\, q$$

or

$$a = 3\int_{\mathfrak{X}_{0,3,1/3}} e_1^*(A)\cup e_2^*(\eta_{\frac{1}{3}})\cup e_3^*(A^2)\cap e(E).$$

Here $\mathfrak{X}_{0,3,1/3}$ parameterises representable morphisms with image a curve of degree $1/3$ on $\mathfrak{X}$; knowing what these curves are, we must be looking at $\mathfrak{X}_{0,3,1/3}(0,1/3,0)$; the virtual dimension is

$$\dim \mathfrak{X}_{0,3,1/3}(0,1/3,0) = 5/3+2-2/3 = 3.$$

The virtual dimension is the actual dimension and the problem is unobstructed; you can integrate:

$$a = 3\int_{\mathfrak{X}_{0,3,1/3}} e_1^*(A)\cup e_2^*(\eta_{\frac{1}{3}})\cup e_3^*(A^2) = \\ = \int_{\mathfrak{X}_{0,2,1/3}} e_1^*(A)\cup e_2^*(\eta_{\frac{1}{3}}) = \int_{\mathbb{P}(1,3)} A = \frac{1}{3}$$

Sanity check: $a = 3\langle A, \eta_{\frac{1}{3}}, A^2\rangle_{1/3} = \langle A, \eta_{\frac{1}{3}}, 3A^2\rangle_{1/3} =$ (by the divisor axiom) $= 1/3\langle \eta_{\frac{1}{3}}, \mathrm{pt}\rangle_{1/3} = 1/3$: there is just one orbi-line of degree $1/3$ that passes through the singular point and one additional general point. The corresponding stable morphism has no automorphisms, hence this line contributes with "multiplicity 1" to $\langle \eta_{\frac{1}{3}}, \mathrm{pt}\rangle_{1/3}$.

Second, from $A * \eta_{\frac{2}{3}} = bq\,\eta_{\frac{1}{3}}$, we derive

$$\frac{1}{3}b = b\,\langle \eta_{\frac{1}{3}}, \eta_{\frac{2}{3}}\rangle = \langle A, \eta_{\frac{2}{3}}, \eta_{\frac{2}{3}}\rangle_{1/3}$$

The relevant moduli space is $\mathfrak{X}_{0,3,1/3}(1,2/3,2/3)$; it has expected dimension

$$5/3+2-4/3-4/3 = 1.$$

The only way to achieve this is by gluing a morphism in $\mathfrak{X}_{0,3,0}(\frac{2}{3}^3)$ with one in $\mathfrak{X}_{0,2,1/3}(1/3,0)$; the two orbi-curves glue as a nontrivial twisted curve and the map is constant on the first component. This space is two dimensional; we have to deal with a one-dimensional obstruction bundle. Note that the first component is in $\mathfrak{X}_{0,3,0}(\frac{2}{3}^3)$ which has negative virtual dimension $0+2-3\times(4/3) = -2$ but it is still there. Fortunately, we can calculate b using the associativity relations:

$$A^2 * \eta_{\frac{2}{3}} = A*(A*\eta_{\frac{2}{3}}) = bqA*\eta_{\frac{1}{3}} = abq^2\mathbf{1}, \quad \text{hence} \\ abq^2 = \langle A^2 * \eta_{\frac{2}{3}}, \mathrm{pt}\rangle = \langle A^2, \eta_{\frac{2}{3}}, \mathrm{pt}\rangle_{\frac{2}{3}}\, q^2.$$

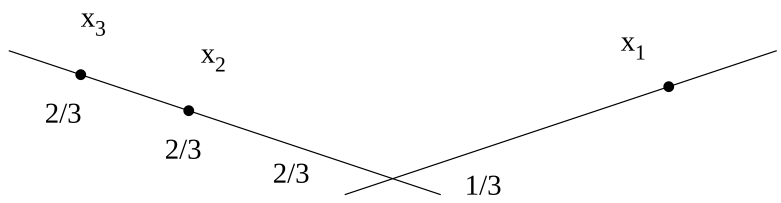

Figure 1: $\mathfrak{X}_{0,3,\frac{1}{3}}\left(0, \frac{2}{3}, \frac{2}{3}\right)$

We calculate an integral over $\mathfrak{X}_{0,3,2/3}(0, 2/3, 0)$; the generic point of this moduli space is a stable morphism from a reducible curve with three components: The key thing to keep in mind is that the corre-

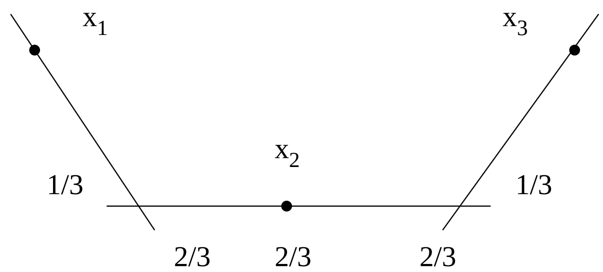

Figure 2: $\mathfrak{X}_{0,3,\frac{2}{3}}\left(0, \frac{2}{3}, 0\right)$

sponding morphism always has a μ_3 of automorphisms over $\mathbb{P}(1,1,3)$, coming from the central component $\mathfrak{C}$ on which the morphism is constant. The central component maps to $B\mu_3$ and it carries an induced μ_3-bundle; this bundle has a μ_3 of automorphisms which survive as nontrivial automorphisms of $\mathfrak{C}$ over $B\mu_3$. Having said this, we can now calculate b:

$$ab = \frac{1}{3}b = \frac{1}{3}\int_{\mathfrak{X}_{0,3,2/3}(0,2/3,0)} e_1^*(\mathrm{pt}) \cup e_2^*(\eta_{\frac{2}{3}}) \cup e_3^*(\mathrm{pt}) = \frac{1}{3}\int_{\mathbb{P}(3)} \mathbf{1} = \frac{1}{9}$$

that is, $b = 1/3$.

Third, show that $c = a$. Indeed, from $A * A^2 = cq\,\eta_{\frac{2}{3}}$, we get

$$1/3\, cq = cq \deg\langle \eta_{\frac{2}{3}} \cup \eta_{\frac{1}{3}} \rangle = cq \langle \eta_{\frac{2}{3}}, \eta_{\frac{1}{3}} \rangle = \langle A, A^2, \eta_{\frac{1}{3}} \rangle_{1/3}\, q$$

and $c = 3\langle A, A^2, \eta_{\frac{1}{3}}\rangle_{1/3} = \langle A, \mathrm{pt}, \eta_{\frac{1}{3}}\rangle_{1/3} = 1/3\langle \mathrm{pt}, \eta_{\frac{1}{3}}\rangle_{1/3} = 1$ as before.]

(iv) Let $D = q\frac{d}{dq}$ and consider the *quantum differential equation*

$$D\Psi = \Psi M \quad \text{for} \quad \Psi\colon \mathbb{C}^\times \to \operatorname{End} H^\bullet_{\mathrm{orb}}(\mathfrak{X}, \mathbb{C}).$$

In the given basis, write $\Psi = (\psi_0, \dots, \psi_n)$ where ψ_i are column vectors; find the ordinary differential equation satisfied by ψ_0.

[Hint.

$$\begin{aligned}3^3(D-2/3)(D-1/3)D^3\psi_0 = 3^3(D-2/3)(D-1/3)D^2\psi_2 = \\ 3^3(D-2/3)(D-1/3)D\psi_4 = 3^2(D-2/3)(D-1/3)q^{1/3}\psi_3 = \\ 3^2q^{1/3}(D-1/3)D\psi_3 = 3q^{1/3}(D-1/3)q^{1/3}\psi_1 = \\ 3q^{2/3}D\psi_1 = q\psi_0]\end{aligned}$$

(9) $\mathfrak{X} = \mathbb{P}(2,2,2)$ (i) Persuade yourself that $\mathfrak{X} = \mathbb{P}(2,2,2)$ is the moduli stack of square roots of $\mathcal{O}(1)$ on $\mathbb{P}^2$. In other words, $\mathbb{P}(2,2,2)$ is characterised by the following universal property: there is an étale morphism $\pi\colon \mathfrak{X} \to \mathbb{P}^2$ of degree $1/2$ and, for any stack $\mathfrak{Y}$, to give a morphism $f\colon \mathfrak{Y} \to \mathfrak{X}$ is equivalent to give a morphism $g\colon \mathfrak{Y} \to \mathbb{P}^2$, a line bundle L on $\mathfrak{Y}$, and an isomorphism

$$L^{\otimes 2} \to g^*\mathcal{O}(1).$$

(This should help you maintain sanity as you work on this question.)

(ii) Show that the orbifold cohomology of $\mathfrak{X}$ is generated by classes $A = c_1\mathcal{O}(1)$ of codimension 2 and $\int_{\mathfrak{X}} A^2 = 1/8$, and w of codimension 0, with the relations

$$w^2 = \mathbf{1}, \quad A^3 = 0.$$

(iii) Show that, in the basis

$$\mathbf{1}, A, A^2, w, wA, wA^2$$

of $H^\bullet_{\mathrm{orb}}\mathfrak{X}$, quantum multiplication by A is given by the matrix:

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{8}q^{\frac{1}{2}} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{8}q^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

[Hint. For instance:

$$\begin{aligned}\langle A, A^2, wA^2\rangle_{\frac{1}{2}} &= \frac{1}{2}\langle A^2, wA^2\rangle_{\frac{1}{2}} = \\ = \frac{1}{2}\int_{\mathfrak{X}_{0,2,\frac{1}{2}}(1/2,0)} e_1^*(A^2)e_2^*(wA^2) &= \frac{1}{2}\int_{\mathfrak{X}\times\mathbb{P}^2} \mathrm{pr}_1^*(A^2)(s\circ \mathrm{pr}_2)^*(A^2) = \\ &= \frac{1}{2}\Big(\int_{\mathfrak{X}} A^2\Big)\Big(\int_{\mathbb{P}^2} s^*A^2\Big) = \frac{1}{2}\times\frac{1}{8}\times\frac{1}{4}\end{aligned}$$

where $s\colon \mathbb{P}^2 \to \mathfrak{X}$ is the non-existing "section" of degree 2.]

(iv) Conclude that the small quantum orbifold cohomology of $\mathfrak{X}$ is generated by w, A with relations:

$$w^2 = 1, \quad A^3 = \frac{1}{8}q^{\frac{1}{2}}w$$

(v) Show that ψ_0—cf. Q 8(4)—satisfies the "expected" hypergeometric differential operator

$$8D^3(2D-1)^3 - q$$

where $D = q\frac{d}{dq}$.

(10) Cubic surface (i) Let $X = X_3^2 \subset \mathbb{P}^3$ be a nonsingular cubic surface. Let A be the class of a hyperplane section and consider the subspace of $H^\bullet X$ with basis $\mathbf{1}, A, A^2 = 3\mathrm{pt}$. Show that quantum multiplication by A preserves this subspace and it is given by the matrix

$$M = \begin{pmatrix} 0 & 108q^2 & 756q^3 \\ 1 & 9q & 108q^2 \\ 0 & 1 & 0 \end{pmatrix}$$

[Hint. The relevant enumerative information is: $\langle A, A, A\rangle_1 = 27$, $\langle A, A, A^2\rangle_2 = 12\langle \mathrm{pt}\rangle_2 = 12\times 27$, $\langle A, A^2, A^2\rangle_3 = 27\times\langle \mathrm{pt},\mathrm{pt}\rangle_3 = 27\times 84$.]

(ii) This is one of the simples examples of a mirror theorem: $e^{6q}\psi_0$—cf. Q 8(4)—satisfies the hypergeometric operator

$$D^3 - 3q(3D+1)(3D+2).$$

(11) Degree two del Pezzo $X = X_4^2 \subset \mathbb{P}(1^3,2)$ Make a similar discussion for the del Pezzo surface of degree 2, $X = X_4^2 \subset \mathbb{P}(1^3,2)$:

(i) In the obvious basis

$$M = \begin{pmatrix} 0 & 552q^2 & 7,488q^3 \\ 1 & 28q & 552q^2 \\ 0 & 1 & 0 \end{pmatrix}$$

(For instance, $7,488 = 6 \times 1248$, where 1248 is the number of cubics through two general points of X.)

(ii) Make contact with the appropriate hypergeometric differential operator $D^3 - 4q(4D+1)(4D+3)$.

(12) $\mathfrak{X} = X_3^2 \subset \mathbb{P}(1^3,2)$ (i) Note that $\mathfrak{X}$ can be written as

$$\big(yx_0 + a_1(x_2,x_3) = 0\big) \subset \mathbb{P}(1^3,2)$$

Build a mental picture of $\mathfrak{X}$ by studying the obvious birational map $\mathfrak{X} \dashrightarrow \mathbb{P}^2$: $\mathfrak{X}$ is obtained by blowing up three collinear points and contracting the proper transform of the line $(x_0 = 0)$. Let $A = \mathcal{O}_{\mathfrak{X}}(1)$. In particular, on $\mathfrak{X}$, there are:

- Three 'lines' of A-degree $1/2$;
- Three fibrations by 'conics' of A-degree 1;
- One map to $\mathbb{P}^2$.

(ii) Convince yourself that the orbifold cohomology of $\mathfrak{X}$ has basis $\mathbf{1}, A, \eta, A^2$ in degrees $0,1,1,2$; $\deg A^2 = 3/2$ and $\deg \eta^2 = 1/2$.

(iii) Use the information in (i) to show that quantum multiplication by A is given in this basis by the following matrix where $\operatorname{codim} q = 2$:

$$M = \begin{pmatrix} 0 & q\langle A,A,\mathrm{pt}\rangle_1 & 0 & 0 \\ 1 & 0 & \frac{2}{3}q^{\frac{1}{2}}\langle A,\eta,A\rangle_{\frac{1}{2}} & q\langle A,\mathrm{pt},A\rangle_1 \\ 0 & 2q\langle A,A,\eta\rangle_{\frac{1}{2}} & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 3q & 0 & 0 \\ 1 & 0 & \frac{1}{2}q^{\frac{1}{2}} & 3q \\ 0 & \frac{3}{2}q^{\frac{1}{2}} & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

(iv) Verify directly that the cyclic vector ψ_0—cf. Q 8(4)—satisfies the 'expected' hypergeometric operator

$$2D^3(2D-1) - 3q(3D+1)(3D+2).$$

[Hint: First rewrite $D\psi = \psi M$ in the new basis

$$\begin{aligned}
\phi_0 &= \psi_0 \\
\phi_1 &= D\psi_0 = \psi_1 \\
\phi_2 &= \psi_3 + \frac{3}{2} q^{\frac{1}{2}} \psi_2 \\
\phi_3 &= \psi_2
\end{aligned}$$

A small calculation shows that the equation in the new basis is:

$$D\Phi = \Phi \begin{pmatrix} 0 & 3q & 0 & 0 \\ 1 & 0 & \frac{15}{4}q & \frac{1}{2}q^{\frac{1}{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{3}{4}q^{\frac{1}{2}} & 0 \end{pmatrix}$$

In this form it is easy to calculate the equation satisfied by $\phi_0 = \psi_0$.]

(v) Show that the quantum products calculated in (iii), together with associativity, determine the whole small quantum cohomology ring. In particular, show that this determines the curious Gromov-Witten number:

$$\langle \eta, \eta, \eta \rangle_{\frac{1}{2}} = -\frac{3}{4}.$$

(vi) The direct calculation of the number in (v) leads to a beautiful case study in excess intersection theory: the expected dimension of the

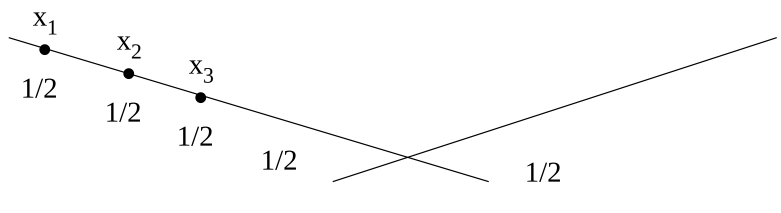

Figure 3: $\mathfrak{X}_{0,3,1/2}\left(\frac{1}{2}^3\right)$

moduli space is $1+2-3\times 1 = 0$; however, the picture shows an actual moduli space of dimension 1 (the four points on the component on which the morphism to X is constant). If you feel brave enough, calculate $\langle \eta, \eta, \eta \rangle_{\frac{1}{2}} = -\frac{3}{4}$ by a study of the virtual class.

3 Toric stacks practice

(13) Show that $\operatorname{Pic}\mathbb{P}_{r_1,r_2} = \mathbb{Z} \oplus \mathbb{Z}/_{\mathrm{hcf}(r_1,r_2)\mathbb{Z}}$ directly from the definition.

(14) Consider the action of $G = \mathbb{C}^{\times\,2}$ on $\mathbb{C}^4$ by the weights

$$\begin{pmatrix} 1 & 1 & 0 & -n \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

(where $n > 0$ is a positive integer).

(i) Persuade yourself that the group of G-linearised line bundles is $\mathbb{L}^\vee = \mathbb{Z}^2$.

(ii) Show that there are two stability conditions for linearised line bundles, carefully write down the unstable loci for each, and identify the two geometric quotients as $\mathbb{P}(1,1,n)$ and $\mathbb{F}_n$.

(iii) Choose coordinate charts around the two cusps of the Kähler moduli space and write down the GKZ operators in each chart.

(15) Consider the n-dimensional weighted projective space $\mathbb{P}^{\mathbf{w}} = \mathbb{P}(w_0, \dots, w_n)$ with the obvious diagonal actions by $\mathbb{C}^{\times\,n+1}$ (ineffective) and by the quotient n-dimensional torus $\mathbb{T}^n = \mathbb{C}^{\times\,n+1}/\mathbb{C}^\times$ (effective).

(i) Persuade yourself that a degree d "$\mathbb{T}$-fixed" (whatever this means!) morphism $f\colon \mathbb{P}^1 \to \mathbb{P}^{\mathbf{w}}$ can be written in the form:

$$\begin{cases} x_i = z_0^{dw_i} \\ x_j = z_1^{dw_j} \\ x_k = 0 \quad \text{if} \quad k \neq i,j \end{cases}$$

(ii) Prove that, as a representation:

$$\chi_{\mathbb{T}}(f^* T_{\mathbb{P}^{\mathbf{w}}}) = \Big(\bigoplus_k \bigoplus_{\substack{a+b=dw_k \\ a,b\geq 0}} \mathbb{C}\big(\frac{a}{dw_i}\chi_i + \frac{b}{dw_j}\chi_j - \chi_k\big)\Big)\Big/\mathbb{C}$$

$$\in H^\bullet_{\mathbb{T}}\{\mathrm{pt}\}$$

(The lattice M of characters of $\mathbb{T}$ fits in an exact sequence:

$$0 \to M \to \mathbb{Z}^{n+1} \overset{(\mathbf{w})}{\to} \mathbb{Z}.)$$

[Hint. Start with the $\mathbb{T}$-equivariant Euler sequence on $\mathbb{P}^{\mathbf{w}}$:

$$0 \to \mathcal{O} \to \oplus_k \mathcal{O}(w_k P - \chi_k) \to T_{\mathbb{P}^{\mathbf{w}}} \to 0.]$$

(iii) Use the general theory developed in class to generalise this calculation to arbitrary stable representable morphism $\mathbb{P}_{r_1,r_2} \to \mathbb{P}^{\mathbf{w}}$.

4 Quantum Cohomology systematics

(16) Using the recipe given in class, write down presentations for the extended quantum equivariant orbifold cohomology of $\mathbb{P}(1,2)$ and $\mathbb{P}(1,1,3)$. Study the classical and non-equivariant limits and persuade yourself that everything fits together nicely.

(17) The purpose of this question is to demonstrate, by looking at the simple case $\mathfrak{X} = \mathbb{P}(1,1,2)$, that knowledge of the small J-function is sufficient (in fact, the first few coefficients of the asymptotic expansion in $1/z$ are sufficient) to determine the small quantum cohomology.

(i) Following the recipe given in class, write the small J-function of $\mathbb{P}(1,1,2)$ as:

$$J(t;z) = ze^{\frac{Pt}{z}}\Big(\mathbf{1}+\sum_{d\geq 1} \frac{Q^d e^{dt}}{(P+z)^2\cdots(P+dz)^2(2P+z)\cdots(2P+2dz)}+$$
$$+\mathbf{1}_{\frac{1}{2}}\sum_{d\geq 0} \frac{Q^{d+\frac{1}{2}}e^{d+\frac{1}{2}}}{\big(P+\frac{1}{2}z\big)^2\cdots\big(P+(d+\frac{1}{2})z\big)^2\big(2P+z\big)\cdots\big(2P+(2d+1)z\big)}\Big)$$

(ii) Show that the small J-function satisfies the differential equation:

$$2z^4\frac{d^3}{dt^3}\Big(2\frac{d}{dt}-1\Big) - Qe^t = 0$$

(iii) Show that:

$$J(t;z) = z\mathbf{1} + P^2\frac{t^2}{2z} + \mathbf{1}_{\frac{1}{2}}\frac{4Q^{\frac{1}{2}}e^{\frac{t}{2}}}{z^2} + \mathbf{1}\frac{Qe^t}{2z^3} + O\Big(\frac{1}{z^4}\Big)$$

(iv) Consider the small J-function as a column vector in the basis $\mathbf{1}, P, P^2, \mathbf{1}_{\frac{1}{2}}$. Define the S-*matrix* as follows:

$$S := \Big(J, z\frac{d}{dt}J, z^2\frac{d^2}{dt^2}J, 2Q^{-\frac{1}{2}}e^{-\frac{t}{2}}z^3\frac{d^3}{dt^3}J\Big)$$

Use the expansion above to shows that $S(0) = \text{identity}$.

(v) Prove that the S-matrix, as defined above, satisfies the differential system

$$z\frac{d}{dt}S = SM, \quad \text{where} \quad M = \begin{pmatrix} 0 & 0 & 0 & \frac{Q^{\frac{1}{2}}e^{\frac{t}{2}}}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{Q^{\frac{1}{2}}e^{\frac{t}{2}}}{2} & 0 \end{pmatrix}$$

(vi) Convince yourself that M is the matrix of quantum multiplication by P in the basis $\mathbf{1}, P, P^2, \mathbf{1}_{\frac{1}{2}}$.

[Hint. You must use the general theory; in particular, the properties of the big J-function.]

(18) Consider the 2-dimensional toric stacks $\mathfrak{X}$ with fan and divisor diagrams:

$$\rho = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -3 & -2 \end{pmatrix} : \mathbb{Z}^2 \to N = \mathbb{Z}^2, \quad \text{and}$$

$$D = \begin{pmatrix} 1 & 3 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix} : \mathbb{Z}^{*\,2} \to \mathbb{L}^\vee = \mathbb{Z}^2.$$

(i) Find an explicit identification of the coarse moduli space X with the ruled surface $\mathbb{F}_3$. Denote by A and B the natural divisors on $\mathbb{F}_3$–the fibre and the negative section. Show that $\mathfrak{X}$ can be interpreted as the moduli space of square roots of B—cf. Q 9(i), and make sure that $\mathfrak{X}$ contains a substack $\{x_4 = 0\}$ supported on B and isomorphic to $\mathbb{P}(2,2)$. Identify the integral Chow ring $\mathrm{CH}^\bullet(\mathfrak{X}, \mathbb{Z})$ with the subring of $\mathrm{CH}^\bullet(X, \mathbb{Q})$ multiplicatively generated by A and $B/2$; the cycle class of $\mathbb{P}(2,2) \subset \mathfrak{X}$ is $B/2$.

(ii) Show that $\mathrm{NE}\,\mathfrak{X} \subset \mathbb{L}_\mathbb{R} = \mathbb{R}^2$ is the cone of vectors

$$\begin{pmatrix} l_1 \\ l_2 \end{pmatrix} \in \mathbb{R}^2 \quad \text{such that} \quad \begin{cases} l_1 \geq 0, & \text{and} \\ 3l_1 + 2l_2 \geq 0. \end{cases}$$

(iii) Show that $\Lambda\mathrm{E}\,\mathfrak{X}$ is the set of vectors

$$\begin{pmatrix} l_1 \\ l_2 \end{pmatrix} \in \mathrm{NE}\,\mathfrak{X} \quad \text{such that} \quad \begin{cases} l_1 \in \mathbb{Z} & \text{and} \\ l_2 \in \frac{1}{2}\mathbb{Z}, \end{cases}$$

and the reduction function is $v\left(\begin{pmatrix} l_1 \\ l_2 \end{pmatrix}\right) = \langle \frac{l_2}{2} \rangle$.

(iv) Write down the small non-equivariant I-function of $\mathfrak{X}$ in the natural integral basis $P_1 = D_1$, $P_2 = D_4$ of $\operatorname{Pic}\mathfrak{X}$—and dual basis for $N_1(\mathfrak{X}, \mathbb{Z})$. Stare at it. Then show that:

$$I^{\mathrm{sm}}(Q, s; z) \;=\; z\mathbf{1} + s_1 P_1 + s_2 P_2 + Q_1 Q_2^{-\frac{3}{2}} e^{s_1 - \frac{3}{2} s_2} \mathbf{1}_{\frac{1}{2}} + O\Big(\frac{1}{z}\Big) \quad (1)$$

This shows that condition $\sharp$ does not hold. More relevant, it is impossible to calculate the small quantum orbifold cohomology of $\mathfrak{X}$ without using the extended I-function.

(v) For $S = \left\{\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right\}$, write down the S-extended I-function. Calculate the mirror map.

(vi) Calculate the first few terms of the asymptotic expansion in $1/z$ of the S-extended J-function $J^S_{\mathfrak{X}}$ and hence write a presentation of the small quantum cohomology of $\mathfrak{X}$. (Good luck. I can't tell you what you are supposed to get since I didn't do this: in fact, I appreciate if you show me your calculation.)