Estimates for the 2D Navier–Stokes equations: the effects of forcing

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Mathematical estimates for the Navier–Stokes equations are traditionally expressed in terms of the Grashof number, which is a dimensionless measure of the magnitude of the forcing and hence a control parameter of the system. However, experimental measurements and statistical theories of turbulence are based on the Reynolds number. Thus, a meaningful comparison between mathematical and physical results requires a conversion of the mathematical estimates to a Reynolds-dependent form. In two dimensions, this was achieved under the assumption that the second derivative of the forcing is square integrable. Nonetheless, numerical simulations have shown that the phenomenology of turbulence is sensitive to the degree of regularity of the forcing. Therefore, we extend the available estimates for the energy and enstrophy dissipation rates as well as the attractor dimension to forcings in the Sobolev space of order s; i.e. forcings whose Fourier coefficients decay with the wavenumber k faster than k^{-s-1}. We consider the range $-1 \leqslant s \leqslant 2$, where $s = 2$ corresponds to the known estimates, and $s = -1$ is the smallest value of s for which weak solutions are known to exist. The main result is the existence of three distinct regimes as a function of the regularity of the forcing.

1. Introduction

The complexity and the limited predictability of turbulent flows justify the use of statistical tools to describe their dynamics. Statistical approaches have indeed been successful in capturing key aspects of turbulence, offering explanations that align well with experimental observations [1–4]. The Navier–Stokes equations, however, are a deterministic (infinite-dimensional) dynamical system. Thus, an important goal of mathematical analysis is to prove the main facts in turbulence theories within a deterministic framework [5–9]. Mathematical results generally take the form of bounds for the time average of Sobolev norms of Navier–Stokes weak solutions; interestingly, statistical predictions often correspond to the saturation of the mathematical bounds.

The early rigorous estimates for the body-forced Navier–Stokes equations were expressed in terms of the generalized Grashof number Gr, a dimensionless measure of the magnitude of the forcing introduced by Foias et al. [10]. This is a natural choice from a mathematical point of view, since the magnitude of the forcing appears as an input parameter for the problem. However, experimental measurements and statistical theories of turbulence are typically expressed in terms of the Reynolds number Re, which is based on the velocity or the energy dissipation rate and therefore emerges as a response of the system. Thus, a meaningful comparison between mathematical and physical results requires a conversion of the mathematical bounds to a Re-dependent form. For periodic boundary conditions and square-integrable forcing, this conversion was carried out by Doering and Foias [11]. They showed that $Gr \leqslant c_1 Re + c_2 Re^2$, where the prefactors depend only on the functional shape of the forcing and are uniform in the other parameters of the system. Moreover, Doering and Foias [11] derived a rigorous upper bound for the time-averaged energy dissipation rate ε that captures the laminar flow behaviour at small Re and is consistent with the dissipative anomaly[1] in three dimensions at large Re. Under the assumption of narrow-band forcing, this estimate was later extended to Sobolev norms of the velocity of any order [13, 14].

As mentioned above, the result in [11] holds for square-integrable forcing. However, in a periodic domain, weak solutions of the Navier–Stokes equations are known to exist under more general conditions, namely the forcing must only be in the Sobolev space $\dot{H}^{-1}$ [15]. Loosely speaking, $\dot{H}^s$ is the space of functions that have zero spatial average and whose Fourier coefficients decay faster than $k^{-s-d/2}$ at large k, where k is the wavenumber and $d = 2, 3$ is the space dimension. Hence, for weak solutions to exist, the Fourier coefficients of the forcing must only decay faster than $k^{1-d/2}$ (as opposed to $k^{-d/2}$ for square-integrable forcing). Although the existence theory ensures that the global energy balance holds in the sense of an energy inequality also for these less regular forcings, it does not immediately

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[1] An estimate of ε for body-forced turbulence in terms of Re was first derived by Foias [12]. That estimate captured the dissipative anomaly in the infinite-Re limit, but it included a spurious volume-dependent prefactor.

clarify how dissipation bounds are modified. In three dimensions, this issue was addressed by Cheskidov, Doering, and Petrov [16]. Their analysis shows a qualitative transition: when the forcing is no longer square-integrable, ε is allowed to grow as a power of Re with an exponent that depends on the degree of regularity of the forcing.

Broad-band forcings have also been considered in the physics literature in order to investigate a potential breakdown of the classical cascade phenomenology. In three dimensions, Kolmogorov's theory assumes that energy injection is concentrated around a characteristic length scale which itself is much larger than the viscous scale. Forcings whose action extends over a wide range of scales, and even reach the viscous range, do not display the necessary scale separation between injection and dissipation. This overlap leads them to generate a different phenomenology. For instance, if the Fourier spectrum of the forcing scales as a power of k – a setting appropriate for turbulent flows generated by fractal grids [17, 18] – the law of finite energy dissipation is violated and ε grows linearly with Re [19]. This is consistent with the result predicted by Cheskidov et al. [16]. Power-law forcings are also at the heart of renormalisation-group (RG) approaches. The canonical setting is white-in-time Gaussian forcing whose covariance in Fourier space scales like k^{4-d-y}, where y is an expansion parameter[2]. Some of the key objectives of the RG approach have been the calculation of the Kolmogorov constant and the dependence of the energy spectrum on the scaling exponent of the forcing [2, 21–23]. The same kind of forcing has been employed beyond the RG framework to investigate the effect of the injection mechanisms on the statistics of the velocity field [24–28]. In particular, the critical threshold $y_c = 4$ was found to separate two regimes: for $y < y_c$ the small scales are forcing-dominated and display nearly dimensional scaling, whereas for $y > y_c$ the small-scale statistics become essentially forcing-independent and exhibit the same anomalous scaling found in flows forced only at large scales [25]. This transition is again consistent with the result of Cheskidov *et al.* [16], since $y_c = 4$ is the value below which the covariance of the forcing ceases to be square-integrable. Thus, these studies demonstrate the strong impact of the forcing regularity on the dissipation of energy in three dimensions.

In two dimensions, the Navier–Stokes equations possess two quadratic invariants in the sense of strong solutions ; namely, energy and enstrophy. This fact strongly modifies the dynamics of the flow. Indeed, the Kraichnan–Leith–Batchelor (KLB) theory states that an inverse transfer of energy to large scales coexists with a direct cascade of enstrophy to small scales (see [29, 30] for a review). This phenomenology suggests that, in a forced state, ε vanishes as viscosity tends to zero, while the time-averaged enstrophy dissipation rate χ remains finite. As in the three-dimensional case, mathematical estimates of ε and χ have typically been expressed in terms of Gr [31–33], but once again the comparison with experimental and numerical results requires a conversion of the results to a Re-dependent form. Rigorous upper bounds in terms of Re were obtained by Alexakis and Doering [34] and Gibbon and Pavliotis [35] for periodic boundary conditions and under the assumption that the forcing is in $\dot{H}^2$; i.e. its second derivative is square integrable. The strategy employed was to apply the methods of Doering and Foias [11] to the vorticity equation instead of the equations for the velocity. The infinite-Re limit of the resulting bounds is consistent with the KLB prediction of a dual cascade. More stringent bounds apply when the forcing is monochromatic or injects energy in a finite set of wavenumbers at a constant rate. In these cases, both ε and χ vanish as Re tends to infinity, and the enstrophy cascade disappears [34, 36, 37]. An analogous result holds for white noise forcing [38]. Incidentally, enstrophy dissipation also vanishes in the inviscid limit when forcing is absent and turbulence decays at long times [39, 40] – see also [41] and references therein.

Returning to the forced case, we have seen that, in two dimensions, the available Re-dependent estimates of ε and χ require that the forcing possesses a relatively high degree of regularity (the forcing must be at least in $\dot{H}^2$). Contrary to the three-dimensional case, to our knowledge there has been no extension of such estimates to less regular forcing. Notwithstanding this, weak solutions are known to exist for forcing in $\dot{H}^{-1}$, and numerical simulations have demonstrated that even in two dimensions the regularity of the forcing has a strong impact on the turbulent flow [42, 43]. In particular, these simulations have clarified certain discrepancies between the RG predictions and the KLB theory and have shown the existence of different dynamical regimes, the KLB scenario being observed only when the forcing is sufficiently regular. Therefore, our aim here is to extend the existing Re-dependent estimates for the time-averaged dissipation rates ε and χ to forcings in $\dot{H}^s$ with $-1 \leqslant s < 2$. To this end, we combine the methods of [34] and [35] with those of [16]. In essence, our strategy is to apply the techniques developed for the three-dimensional Navier-Stokes equations with $\dot{H}^s$-forcing to the vorticity equation in two dimensions.

Another crucial difference between two and three dimensions is that the two-dimensional Navier–Stokes equations on a periodic domain are known to possess a global attractor $\mathcal{A}$ if the forcing is square-integrable. $\mathcal{A}$ is a compact set on which the long-time dynamics is confined [44] (see also [15, 45]). Thus, asymptotically in time, any initial condition outside $\mathcal{A}$ approaches it, while any trajectory starting on $\mathcal{A}$ remains there for all time. The existence of

[2] Using a forcing term with power-law correlation is a strategy for introducing a small parameter into the Navier–Stokes equations. The *functional* RG formalism now allows for non-perturbative approaches. Consequently, the use of a power-law forcing is less common in recent RG studies. See [20] for a review

Forcing	$\chi\ell^3/U^3$	$\varepsilon\ell/U^3$	$\ell\eta_\chi^{-1}$	$\ell\eta_\varepsilon^{-1}$	$d_f(\mathcal{A})$	Refs.
single-k or fixed-ε	Re^{-1}	Re^{-1}	$Re^{1/3}$	$Re^{1/2}$	$Re^{2/3}(1+\ln Re)^{1/3}$	[34, 36, 37]
$s \geqslant 2$	Re^{0}	$Re^{-1/2}$	$Re^{1/2}$	$Re^{5/8}$	$Re(1+\ln Re)^{1/3}$	[34, 35]
$0 \leqslant s < 2$	$Re^{\frac{2-s}{2+s}}$	$Re^{-\frac{s}{s+2}}$	$Re^{\frac{4+s}{3(2+s)}}$	$Re^{\frac{3+s}{2(2+s)}}$	$Re^{\frac{2(4+s)}{3(2+s)}}(1+\ln Re)^{1/3}$	
$-1 \leqslant s < 0$	—	$Re^{-\frac{s}{s+2}}$	—	$Re^{\frac{3+s}{2(2+s)}}$	—	[16]

TABLE I. Large-Re estimates for the two-dimensional Navier–Stokes equations with $\dot{H}^s$-forcing.

such an attractor reflects the effectively finite-dimensional behaviour of an otherwise infinite-dimensional dynamical system. Bounds for the fractal dimension of the attractor, $d_f(\mathcal{A})$, indeed provide a mathematically rigorous notion of the 'degrees of freedom' of the system (see also [46, 47], for an informal introduction). Restricting to periodic boundary conditions – the setting considered here – Constantin, Foias, and Temam [48] were the first to obtain an estimate of $d_f(\mathcal{A})$ in terms of Gr (see also [49] for a simpler proof based on the vorticity rather than the velocity field). Later, Gibbon and Pavliotis [35] converted this estimate into a Re-dependent estimate under the more restrictive assumption of $\dot{H}^2$-forcing. The Re-based estimate of [35] matches the heuristic prediction for the degrees of freedom of two-dimensional turbulence. Here, by using the new bound for χ, we also extend the estimate of $d_f(\mathcal{A})$ in [35] to the case of less regular forcing.

Our results for ε, χ, and $d_f(\mathcal{A})$ are summarized in table 1 in which only the large-Re bounds are shown. While the first line represents the special case of monochromatic or constant-ε forcing, the rest of the table refers to the case of $\dot{H}^s$-forcing. Our analysis reveals the existence of three distinct regimes depending on the value of s. The case $s \geqslant 2$ is that studied in [34, 35]. Our estimates correspond to $0 \leqslant s < 2$. The last line ($-1 \leqslant s < 0$) is a straightforward adaptation of the three-dimensional result of [16] to two dimensions. The rest of this article is organised as follows. In §2 we introduce the functional setting and briefly recall the relevant results from the literature. In §3 we present the derivation of our estimates, and in §4 we offer some concluding remarks.

2. Definitions and previous two-dimensional results

We consider the incompressible Navier–Stokes equations on the periodic domain $\Omega = [0,\ell]^2$:

$$\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \nu\Delta\boldsymbol{u} + \boldsymbol{f}\,, \qquad \nabla\cdot\boldsymbol{u} = 0\,, \tag{1}$$

where $\boldsymbol{u}(\boldsymbol{x},t)$ is the velocity field, $p(\boldsymbol{x},t)$ pressure, ν the kinematic viscosity, and $\boldsymbol{f}(\boldsymbol{x})$ an externally applied, time-independent, divergence-free forcing (in §4, we will comment on the extension of the results to the case of time-dependent forcing). We also assume that both the forcing and the initial velocity have zero mean, so that $\boldsymbol{u}(\boldsymbol{x},t)$ remains zero-mean at all times. To take advantage of the structure of the Navier–Stokes equations in two dimensions, it is convenient to study the evolution of the scalar vorticity

$$\omega = \boldsymbol{e}_3\cdot(\nabla\times\boldsymbol{u})\,. \tag{2}$$

Taking the curl of (1) yields the vorticity equation

$$\partial_t\omega + \boldsymbol{u}\cdot\nabla\omega = \nu\Delta\omega + \phi\,, \tag{3}$$

where $\phi = \boldsymbol{e}_3\cdot(\nabla\times\boldsymbol{f})$.

Given a vector field $\boldsymbol{g}(\boldsymbol{x},t)$, we define its Fourier coefficients $\hat{\boldsymbol{g}}_{\boldsymbol{k}}(t)$ through

$$\boldsymbol{g}(\boldsymbol{x},t) = \sum_{\boldsymbol{k}} \hat{\boldsymbol{g}}_{\boldsymbol{k}}(t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}, \qquad \hat{\boldsymbol{g}}_{\boldsymbol{k}}(t) = \ell^{-2}\int_\Omega \boldsymbol{g}(\boldsymbol{x},t)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\mathrm{d}\boldsymbol{x}\,. \tag{4}$$

Since we only consider zero-mean vector fields, all Fourier series are understood to be over the wave-vectors $\boldsymbol{k} = 2\pi\boldsymbol{n}/\ell$ with $\boldsymbol{n}\in\mathbb{Z}^2$ such that $k = |\boldsymbol{k}| \neq 0$. In particular, if the field $\boldsymbol{g}(\boldsymbol{x},t)$ is divergenceless, then $\boldsymbol{k}\cdot\hat{\boldsymbol{g}}_{\boldsymbol{k}}(t) = 0$ and the following identity holds

$$|(\widehat{\nabla\times\boldsymbol{g}})_{\boldsymbol{k}}|^2 = |\boldsymbol{k}\times\hat{\boldsymbol{g}}_{\boldsymbol{k}}|^2 = k^2|\hat{\boldsymbol{g}}_{\boldsymbol{k}}|^2 - (\boldsymbol{k}\cdot\hat{\boldsymbol{g}}_{\boldsymbol{k}})^2 = k^2|\hat{\boldsymbol{g}}_{\boldsymbol{k}}|^2\,. \tag{5}$$

We aim to study (1) and (3) as the degree of regularity of the forcing varies. To this end, we consider the homogeneous Sobolev space $\dot{H}^s(\Omega)$. A field $\boldsymbol{g} \in \dot{H}^s(\Omega)$ if and only if it has zero mean and its $\dot{H}^s$-norm, defined by

$$\|\boldsymbol{g}\|_{\dot{H}^s}^2 = \sum_{\boldsymbol{k}} (\ell k)^{2s} |\hat{\boldsymbol{g}}_{\boldsymbol{k}}|^2 , \tag{6}$$

is finite. Note that we follow the definition of the $\dot{H}^s$-norm used in [16]; this definition differs from the usual one by a factor of ℓ^{s-1} but has the advantage of simplifying the calculations, since $\|\boldsymbol{g}\|_{\dot{H}^s}$ has the dimension of $|\boldsymbol{g}|$ for all s. The parameter s characterizes the regularity of the field, higher values of s corresponding to smoother fields. In particular, $s = 0$ corresponds to the L^2-norm (here defined with the volume normalization prefactor for consistency with the definition of the $\dot{H}^s$-norm)

$$\|\boldsymbol{g}\|_{L^2}^2 = \ell^{-2} \int_\Omega |\boldsymbol{g}(\boldsymbol{x}, t)|^2 \mathrm{d}\boldsymbol{x} = \sum_{\boldsymbol{k}} |\hat{\boldsymbol{g}}_{\boldsymbol{k}}(t)|^2 = \|\boldsymbol{g}\|_{\dot{H}^0}^2 . \tag{7}$$

The forcing in (1) is thus taken to be in $\dot{H}^s(\Omega)$ and of the form

$$\boldsymbol{f}(\boldsymbol{x}) = F\, \boldsymbol{\Phi}(\ell^{-1}\boldsymbol{x}) , \tag{8}$$

where F is the magnitude and $\boldsymbol{\Phi}$ is a dimensionless field such that

$$\int_\Omega \boldsymbol{\Phi}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = 0 , \qquad \nabla \cdot \boldsymbol{\Phi} = 0 , \qquad \|\boldsymbol{\Phi}\|_{\dot{H}^s} = 1 . \tag{9}$$

We shall refer to $\boldsymbol{\Phi}$ as the ‘shape’ of the forcing. Recall that weak solutions of (1) are known to exist for $s \geqslant -1$ [15].

The time-averaged energy and enstrophy dissipation rates are defined as

$$\varepsilon = \nu \ell^{-2} \left\langle \int_\Omega |\nabla \boldsymbol{u}|^2 \mathrm{d}\boldsymbol{x} \right\rangle , \qquad \chi = \nu \ell^{-2} \left\langle \int_\Omega |\nabla \omega|^2 \mathrm{d}\boldsymbol{x} \right\rangle , \tag{10}$$

respectively, where

$$\langle \cdot \rangle = \limsup_{T \to \infty} \frac{1}{T} \int_0^T \cdot \; \mathrm{d}t . \tag{11}$$

In terms of the $\dot{H}^s$-norms the same quantities are expressed as

$$\varepsilon = \nu \ell^{-2} \left\langle \|\boldsymbol{u}\|_{\dot{H}^1}^2 \right\rangle = \nu \left\langle \|\omega\|_{L^2}^2 \right\rangle , \qquad \chi = \nu \ell^{-2} \left\langle \|\omega\|_{\dot{H}^1}^2 \right\rangle = \nu \ell^{-4} \left\langle \|\boldsymbol{u}\|_{\dot{H}^2}^2 \right\rangle . \tag{12}$$

The length scales η_χ and η_ε corresponding to energy and enstrophy dissipation are

$$\eta_\varepsilon = \left(\frac{\nu^3}{\varepsilon} \right)^{1/4} , \qquad \eta_\chi = \left(\frac{\nu^3}{\chi} \right)^{1/6} . \tag{13}$$

These scales are also known as the Kolmogorov and Kraichnan length scales, respectively. Two relevant dimensionless numbers are the Grashof and Reynolds numbers, together with the the mean-square velocity U

$$Gr = \frac{\ell^3 \, \|\boldsymbol{f}\|_{L^2}}{\nu^2} , \qquad Re = \frac{U\ell}{\nu} , \qquad U^2 = \left\langle \|\boldsymbol{u}\|_{L^2}^2 \right\rangle . \tag{14}$$

The boundedness of U is a consequence of the energy inequality.

We now briefly recall the previous estimates of ε, χ, and the attractor dimension. For $\boldsymbol{f} \in \dot{H}^{-1}$, only a bound for ε is available

$$\varepsilon \leqslant \nu^3 \ell^{-4} Gr_*^2 , \tag{15}$$

where $Gr_* = \nu^{-2} \ell^3 \|\boldsymbol{f}\|_{\dot{H}^{-1}}$ is an alternative Grashof number based on the $\dot{H}^{-1}$-norm [50, 51]. The absence of a bound for χ means that it is not known whether a global attractor exists in this case.

The traditional estimates of ε and χ in terms of Gr hold for $\boldsymbol{f} \in L^2(\Omega)$ and are (see e.g. [5, 6])

$$\varepsilon \leqslant \nu^3 \ell^{-4} Gr^2 , \qquad \chi \leqslant \nu^3 \ell^{-6} Gr^2 . \tag{16}$$

Under the same assumptions on the forcing term, the bound for the attractor dimension is

$$d_f(\mathcal{A}) \leqslant c_0\, Gr^{2/3}(1 + \ln Gr)^{1/3}\,, \tag{17}$$

where c_0 is a positive dimensionless constant [48, 49]. When the forcing is at scales below η_χ the bound in (17) is improved by a bound that is linear in Gr_* [50, 51]. The conversion of (16) and (17) to a Re-dependent form was achieved by assuming more regular forcing. For $\boldsymbol{f} \in \dot{H}^2(\Omega)$, the estimates equivalent to (16) and (17) are [34, 35]

$$\frac{\varepsilon\ell}{U^3} \leqslant Re^{-1/2}\left(c_1 + \frac{c_2}{Re}\right)^{1/2}, \qquad \frac{\chi\ell^3}{U^3} \leqslant c_1 + \frac{c_2}{Re}\,, \tag{18}$$

and

$$d_f(\mathcal{A}) \leqslant c_0 Re(1 + \ln Re)^{1/3}\,, \tag{19}$$

with c_1 and c_2 positive dimensionless constants. The estimates of ε and χ are consistent with a residual dissipation of enstrophy and the absence of a direct cascade of energy in the infinite-Re limit. Regarding the attractor dimension, $d_f(\mathcal{A})$ can be interpreted as a mathematical definition of the number of degrees of freedom of the system [47]. Since η_χ is the smallest length scale in the system, the number of degrees of freedom can in turn be related to the number of boxes of size η_χ that fit into the domain Ω, which leads to the identification $d_f(\mathcal{A}) \sim (\ell/\eta_\chi)^2$. Comparing this interpretation of $d_f(\mathcal{A})$ with (19) yields an estimate of the enstrophy dissipation scale, $\eta_\chi \sim \ell Re^{-1/2}$, consistent with the KLB theory [29]. Finally, in the special case of single-k or constant-ε forcing, the energy and enstrophy dissipation rates are both bounded by Re^{-1}, and a direct cascade of enstrophy must be ruled out in the limit of infinite Re [34, 36, 37].

In the next section, we show how to obtain Re-dependent estimates for less regular forcing, namely for $\boldsymbol{f} \in \dot{H}^s(\Omega)$ with $-1 \leqslant s < 2$.

3. Estimates for $\dot{H}^s$ forcing

We first consider the range $0 \leqslant s \leqslant 2$, for which (1) has strong solutions [15, 45]. The enstrophy balance is obtained by multiplying (3) by ω and integrating over space

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{L^2}^2 + \nu\ell^{-2}\|\omega\|_{\dot{H}^1}^2 = \ell^{-2}\int_\Omega \omega\phi\,\mathrm{d}\boldsymbol{x}\,. \tag{20}$$

By applying the Cauchy–Scharz inequality to the forcing term and using Gronwall's lemma, it can be shown that $\|\omega\|_{L^2}^2$ is bounded for all t and therefore the time average of its time derivative vanishes [15]. It follows that the time-averaged enstrophy dissipation rate satisfies

$$\chi = \ell^{-2}\left\langle \int_\Omega \omega\phi\,\mathrm{d}\boldsymbol{x} \right\rangle\,. \tag{21}$$

To estimate the right-hand side of (21), we use Parseval's theorem, the triangle inequality, and identity (33)

$$\begin{aligned}
\ell^{-2}\left|\int_\Omega \omega\phi\,\mathrm{d}\boldsymbol{x}\right| &= \ell^{-2}\left|\int_\Omega (\nabla\times\boldsymbol{u})\cdot(\nabla\times\boldsymbol{f})\,\mathrm{d}\boldsymbol{x}\right| \\
&\leqslant \sum_{\boldsymbol{k}} |\boldsymbol{k}\times\hat{\boldsymbol{u}}_{\boldsymbol{k}}|\;|\boldsymbol{k}\times\hat{\boldsymbol{f}}_{\boldsymbol{k}}| \\
&= \sum_{\boldsymbol{k}} k^2|\hat{\boldsymbol{u}}_{\boldsymbol{k}}||\hat{\boldsymbol{f}}_{\boldsymbol{k}}|\,.
\end{aligned} \tag{22}$$

We then rearrange the sum and apply the Cauchy–Schwarz inequality

$$\begin{aligned}
\ell^{-2}\left|\int_\Omega \omega\phi\,\mathrm{d}\boldsymbol{x}\right| &\leqslant \ell^{-2}\sum_{\boldsymbol{k}}(\ell k)^2|\hat{\boldsymbol{u}}_{\boldsymbol{k}}||\hat{\boldsymbol{f}}_{\boldsymbol{k}}| \leqslant \ell^{-2}\sum_{\boldsymbol{k}}(\ell k)^{2-s}|\hat{\boldsymbol{u}}_{\boldsymbol{k}}|(\ell k)^s|\hat{\boldsymbol{f}}_{\boldsymbol{k}}| \\
&\leqslant \ell^{-2}\left[\sum_{\boldsymbol{k}}(\ell k)^{2(2-s)}|\hat{\boldsymbol{u}}_{\boldsymbol{k}}|^2\right]^{1/2}\left[\sum_{\boldsymbol{k}}(\ell k)^{2s}|\hat{\boldsymbol{f}}_{\boldsymbol{k}}|^2\right]^{1/2} \\
&= F\ell^{-2}\|\boldsymbol{u}\|_{\dot{H}^{2-s}}\,.
\end{aligned} \tag{23}$$

To relate χ to Re, we need to estimate the time average of $\|\boldsymbol{u}\|_{\dot{H}^{2-s}}$ in terms of χ and U. The following interpolation identity is obtained by rearranging the terms in $\|\boldsymbol{u}\|_{\dot{H}^{2-s}}$ and then using a Hölder inequality

$$\begin{aligned}
\|\boldsymbol{u}\|_{\dot{H}^{2-s}} &= \sum_{\boldsymbol{k}}(\ell k)^{2(2-s)}|\hat{\boldsymbol{u}}_{\boldsymbol{k}}|^2 = \sum_{\boldsymbol{k}}\left[(\ell k)^4|\hat{\boldsymbol{u}}_{\boldsymbol{k}}|^2\right]^{\frac{2-s}{2}}\left(|\hat{\boldsymbol{u}}_{\boldsymbol{k}}|^2\right)^{1-\frac{2-s}{2}} \\
&\leqslant \left(\sum_{\boldsymbol{k}}(\ell k)^4|\hat{\boldsymbol{u}}_{\boldsymbol{k}}|^2\right)^{1-\frac{s}{2}}\left(\sum_{\boldsymbol{k}}|\hat{\boldsymbol{u}}_{\boldsymbol{k}}|^2\right)^{\frac{s}{2}} \\
&= \|\boldsymbol{u}\|_{\dot{H}^2}^{1-\frac{s}{2}}\|\boldsymbol{u}\|_{L^2}^{\frac{s}{2}}\,, \qquad (24)
\end{aligned}$$

where we have used the fact that $0 \leqslant s \leqslant 2$. Taking the time average of both sides and using a Hölder inequality leads to

$$\langle\|\boldsymbol{u}\|_{\dot{H}^{2-s}}\rangle \leqslant \left\langle\|\boldsymbol{u}\|_{\dot{H}^2}^{1-\frac{s}{2}}\|\boldsymbol{u}\|_{L^2}^{\frac{s}{2}}\right\rangle \leqslant \left\langle\|\boldsymbol{u}\|_{\dot{H}^2}^2\right\rangle^{\frac{2-s}{4}}\left\langle\|\boldsymbol{u}\|_{L^2}^{\frac{2s}{2+s}}\right\rangle^{\frac{2+s}{4}}. \qquad (25)$$

An application of Jensen's inequality to the last term (here we use $0 \leqslant s \leqslant 2$ again), together with the definitions of χ and U, gives

$$\begin{aligned}
\langle\|\boldsymbol{u}\|_{\dot{H}^{2-s}}\rangle &\leqslant \left\langle\|\boldsymbol{u}\|_{\dot{H}^2}^2\right\rangle^{\frac{2-s}{4}}\left\langle\|\boldsymbol{u}\|_{L^2}^2\right\rangle^{\frac{s}{2}} \\
&= (\nu\ell^{-4})^{\frac{s-2}{4}}\left\langle\nu\ell^{-4}\|\boldsymbol{u}\|_{\dot{H}^2}^2\right\rangle^{\frac{2-s}{4}}\left\langle\|\boldsymbol{u}\|_{L^2}^2\right\rangle^{\frac{s}{2}} \\
&\leqslant (\nu\ell^{-4})^{\frac{s-2}{4}}\chi^{\frac{2-s}{4}}U^{\frac{s}{2}}\,. \qquad (26)
\end{aligned}$$

Combining (21), (23) and (26) gives

$$\chi \leqslant \ell^{-2}F\left(\nu\ell^{-4}\right)^{\frac{s-2}{4}}\chi^{\frac{2-s}{4}}U^{\frac{s}{2}} \qquad (27)$$

which leads to

$$\chi \leqslant \nu^{-\frac{2-s}{2+s}}\ell^{-\frac{4s}{2+s}}F^{\frac{4}{2+s}}U^{\frac{2s}{2+s}}\,. \qquad (28)$$

It remains to estimate F. We take a smooth time-independent field $\boldsymbol{\psi}$ such that $\nabla\cdot\boldsymbol{\psi} = 0$ and $\int_\Omega \boldsymbol{\Phi}\cdot\boldsymbol{\psi}\,\mathrm{d}\boldsymbol{x} > 0$ ($\boldsymbol{\psi}$ can be, for example, a Galerkin truncation of $\boldsymbol{\Phi}$) and multiply the Navier–Stokes equation (1) by $\boldsymbol{\psi}$. We then integrate by parts, time average, and use the Cauchy–Schwarz inequality

$$\begin{aligned}
\ell^{-2}\left\langle\int_\Omega \boldsymbol{f}\cdot\boldsymbol{\psi}\,\mathrm{d}\boldsymbol{x}\right\rangle &\leqslant \ell^{-2}\left|\left\langle\int_\Omega \boldsymbol{u}\cdot\nabla\boldsymbol{\psi}\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\right\rangle + \nu\left\langle\int_\Omega \boldsymbol{u}\cdot\Delta\boldsymbol{\psi}\,\mathrm{d}\boldsymbol{x}\right\rangle\right| \\
&\leqslant \|\nabla\boldsymbol{\psi}\|_{L^\infty}\left\langle\|\boldsymbol{u}\|_{L^2}^2\right\rangle + \nu\|\Delta\boldsymbol{\psi}\|_{L^2}\left\langle\|\boldsymbol{u}\|_{L^2}\right\rangle \\
&\leqslant \|\nabla\boldsymbol{\psi}\|_{L^\infty}\left\langle\|\boldsymbol{u}\|_{L^2}^2\right\rangle + \nu\|\Delta\boldsymbol{\psi}\|_{L^2}\left\langle\|\boldsymbol{u}\|_{L^2}^2\right\rangle^{1/2} \\
&= \ell^{-1}\|\widetilde{\nabla}\boldsymbol{\psi}\|_{L^\infty}U^2 + \nu\ell^{-2}\|\boldsymbol{\psi}\|_{\dot{H}^2}U\,, \qquad (29)
\end{aligned}$$

where $\widetilde{\nabla} = \ell\nabla$. After rewriting the left-hand side of (29) as

$$\ell^{-2}\int_\Omega \boldsymbol{f}\cdot\boldsymbol{\psi}\,\mathrm{d}\boldsymbol{x} = F\ell^{-2}\int_\Omega \boldsymbol{\Phi}\cdot\boldsymbol{\psi}\,\mathrm{d}\boldsymbol{x}\,, \qquad (30)$$

we find

$$F \leqslant c_1\ell^{-1}U^2 + c_2\nu\ell^{-2}U\,, \qquad (31)$$

together with

$$c_1 = \left(\ell^{-2}\int_\Omega \boldsymbol{\Phi}\cdot\boldsymbol{\psi}\,\mathrm{d}\boldsymbol{x}\right)^{-1}\|\widetilde{\nabla}\boldsymbol{\psi}\|_{L^\infty}, \qquad c_2 = \left(\ell^{-2}\int_\Omega \boldsymbol{\Phi}\cdot\boldsymbol{\psi}\,\mathrm{d}\boldsymbol{x}\right)^{-1}\|\boldsymbol{\psi}\|_{\dot{H}^2}\,, \qquad (32)$$

where c_1 and c_2 are not related to the same coefficients in §2. Finally, combining (28) and (31), we obtain the estimate for the rescaled time-averaged enstrophy dissipation rate

$$\frac{\chi\ell^3}{U^3} \leqslant Re^{\frac{2-s}{2+s}} \left(c_1 + c_2 Re^{-1}\right)^{\frac{4}{2+s}} . \tag{33}$$

Note that the prefactors only depend on the shape of the forcing. For $Re \gg 1$, (33) yields the following estimate for the inverse of the enstrophy dissipation length scale

$$\ell\eta_\chi^{-1} = \ell\left(\frac{\chi}{\nu^3}\right)^{1/6} \leqslant c_1' Re^{\frac{4+s}{3(2+s)}} , \tag{34}$$

with $c_1' = c_1^{2/3(2+s)}$. To estimate the time-averaged energy dissipation rate, we use the interpolation inequality in (24) for $s = 1$

$$\|\boldsymbol{u}\|_{\dot{H}^1} \leqslant \|\boldsymbol{u}\|_{\dot{H}^2}^{1/2} \|\boldsymbol{u}\|_{L^2}^{1/2} . \tag{35}$$

We now square both members, time average, and use the Cauchy–Schwarz inequality to obtain

$$\left\langle \|\boldsymbol{u}\|_{\dot{H}^1}^2 \right\rangle \leqslant \langle \|\boldsymbol{u}\|_{\dot{H}^2} \|\boldsymbol{u}\|_{L^2} \rangle \leqslant \left\langle \|\boldsymbol{u}\|_{\dot{H}^2}^2 \right\rangle^{1/2} \left\langle \|\boldsymbol{u}\|_{L^2}^2 \right\rangle^{1/2} . \tag{36}$$

After multiplying by $\nu\ell^{-2}$, we find

$$\varepsilon \leqslant \nu^{1/2}\chi^{1/2} U . \tag{37}$$

The estimate for χ in (33) can be used to bound the right-hand side

$$\frac{\varepsilon\ell}{U^3} \leqslant Re^{-\frac{s}{2+s}} \left(c_1 + c_2 Re^{-1}\right)^{\frac{2}{2+s}} . \tag{38}$$

The inverse of the energy dissipation length scale can now be estimated using (38) in the definition of η_ε

$$\ell\eta_\varepsilon^{-1} = \ell\left(\frac{\varepsilon}{\nu^3}\right)^{1/4} \leqslant c_1'' Re^{\frac{3+s}{2(2+s)}} , \qquad c_1'' = c_1^{\frac{1}{2(2+s)}} . \tag{39}$$

We conclude this study of the case $0 \leqslant s \leqslant 2$ with the estimate of the attractor dimension $d_f(\mathcal{A})$. We recall that $d_f(\mathcal{A})$ is related to η_χ via the following bound [48]

$$d_f(\mathcal{A}) \leqslant c_3 \left(\ell\eta_\chi^{-1}\right)^2 \left[1 + \ln\left(\ell\eta_\chi^{-1}\right)\right]^{1/3} . \tag{40}$$

Plugging (34) into (40) gives

$$d_f(\mathcal{A}) \leqslant c_3' Re^{\frac{2(4+s)}{3(2+s)}} \left(1 + \ln Re^{\frac{4+s}{3(2+s)}}\right)^{1/3} \leqslant c_3' Re^{\frac{2(4+s)}{3(2+s)}} \left(1 + \ln Re\right)^{1/3} , \tag{41}$$

where c_3' is a constant related to c_1' and c_3. The scaling exponent varies between unity for $s = 2$ and $4/3$ for $s = 0$.

If $-1 \leqslant s < 0$, the above derivation does not work as a bound for χ is not available. However, we can obtain a bound for ε by working with the velocity equation instead of the vorticity equation and following the same procedure used in [16] for three dimensions. The final result is a bound of the same form as in (38).

4. Concluding remarks

Our results are summarised in table 1 in which only the large-Re bounds are shown. Leaving aside the special case of single-k or constant-ε forcing, three distinct regimes can be identified as the degree of regularity of the forcing is varied. For $s \geqslant 2$, the estimates obtained in [34, 35] are consistent with the dual-cascade scenario. In the second regime ($0 < s < 2$), a direct cascade of energy must still be ruled out in the $Re \to \infty$ limit. However, the amount of energy injected at small scales increases as s decreases, so that ε decreases monotonically as a function of Re, but increasingly slowly. In this regime, χ is allowed to grow with Re. As energy injection at small scales increases, the bound on the number of degrees of freedom of the system (quantified by the attractor dimension) also grows from Re

for $s = 2$ to $Re^{4/3}$ for $s = 0$. Finally, for $-1 < s < 0$, the time-averaged energy dissipation rate can increase with Re, similar to what is found in three dimensions for non-square-integrable forcing [16, 19].

It is noteworthy that three dynamical regimes are also observed in the two-dimensional Navier–Stokes equations with power-law *stochastic* forcing [42, 43]. The covariance of the forcing in Fourier space is taken proportional to k^{2-y} with $y \geqslant 0$. If $y > 6$, the enstrophy flux is constant, and a direct cascade of enstrophy is observed. In the range $4 < y < 6$, there is scale-by-scale balance of the energy and enstrophy fluxes. In addition, the enstrophy flux increases with k, whereas the energy flux decays. For $0 \leqslant y < 4$, the system displays an inverse energy cascade. While some similarities with our results may exist for the two regimes with more regular forcing, a systematic comparison of the two systems may not be appropriate. Indeed, our forcing is deterministic whereas in [42, 43] the forcing is stochastic and the analysis explicitly uses its Gaussian and white-in-time properties. Moreover, the range $0 \leqslant y < 2$ is not covered by our study : for weak solutions to exist, the Fourier coefficients of the forcing must decay with k.

To conclude, we note that our estimates for $0 \leqslant s < 2$ can be generalised to the case of deterministic but time-dependent forcing. The only steps that change in the derivation of the estimates are (29) and (30). If the forcing is time dependent, the field $\boldsymbol{\psi}$ should also be time dependent ($\boldsymbol{\psi}$ can again be a suitable Galerkin truncation of $\boldsymbol{\Phi}$, in both space and time). Moreover, F should be replaced with the root-mean-square of the time-dependent magnitude of the forcing. The methods outlined in section 3 of [34] can then be used to obtain a bound for F similar to that in (31). Only the prefactors c_1 and c_2 change, but not the scaling of F with Re. Consequently, the final bounds for χ, ε, and $d_f(\mathcal{A})$ retain the same form as in the case when the forcing is time independent. For $s \geqslant 2$, the contribution to c_1 due to the time dependence of $\boldsymbol{\psi}$ could be interpreted as the ratio of the eddy turnover time of the flow to the characteristic time scale of the forcing [34]. Such an interpretation is not possible when the forcing is less regular. Finally, even the estimate of ε for $-1 \leqslant s < 0$ does not change form when the forcing is time dependent, since once again only the prefactors in the bound for F are modified, while the rest of the proof remains the same.

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