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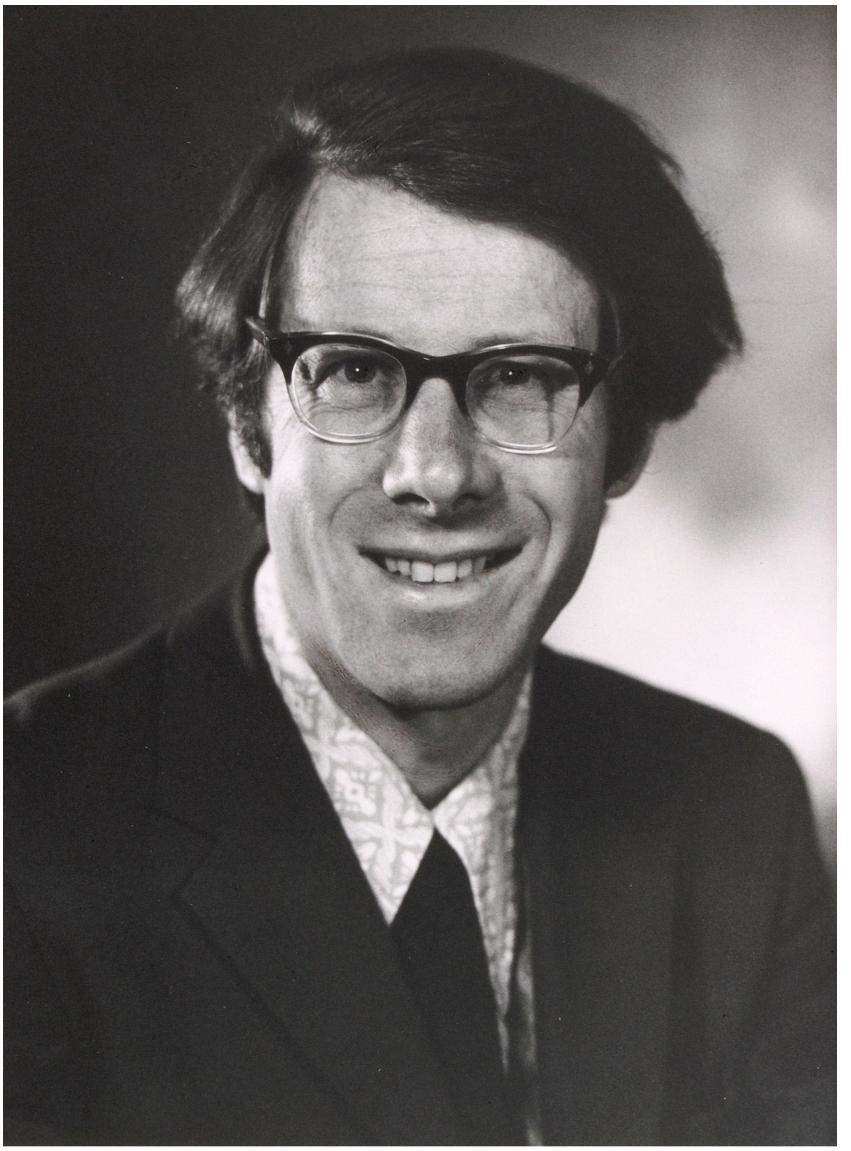

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Elected FRS 1974

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In 1947 the award of a Royal Scholarship at Imperial College's Department of Mathematics launched Trevor Stuart from a working-class family in Leicester into a career in the Aerodynamics Division of the National Physical Laboratory (NPL) just at a time when post-war government scientific institutions were growing rapidly. The outstanding nature of his early work as a pioneer of the theory of hydrodynamic stability was formally recognized in 1961 when, at the age of 32, he became the youngest Senior Principal Scientific Officer with a Special Merit Award in the Scientific Civil Service. The reorganization and transfer of NPL's Aerodynamics Division to RAE Farnborough ultimately led to Trevor moving to a chair in his old department at Imperial in 1966. Trevor's pioneering work on fluid flow instabilities in systems where initially small disturbances rapidly grow to finite values was a key development in the theoretical understanding of the first stage (weakly nonlinear) of transition to full-scale turbulence. Plane parallel flow and circular Couette flow were two examples to which he devoted much time and energy. Although reserved in manner, Trevor was a natural leader and a good judge of scientific potential in younger colleagues. In the latter part of his career he served on the Council of the Royal Society, as a head of department (twice), as a Dean of the Royal College of Science within Imperial, as a chair of the (then) Science and Engineering Research Council Mathematics Committee and as President of the London Mathematical Society.

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https://doi.org/10.1098/rsbm.2025.0022

Scientific career at the National Physical Laboratory

Now that the archives have been opened, and once secret histories can see the light of day, there is universal recognition of the immense contribution that scientists made to the Allied victory in the Second World War (WW2). In the first decade after the war the situation in which UK government scientific institutions found themselves has been deftly summarized by Emmeline Ledgerwood (Ledgerwood 2022) in her study of the history of the Scientific Civil Service:

> The demands of the Second World War for technical expertise and innovation had cemented the position of government scientists and engineers as vital contributors to national technological advancement and the provision of expert advice. After the war ended, joining the Scientific Civil Service was recognized as a respectable way to pursue a technical or scientific career in the UK, and in the immediate post-war period had the added advantage of being a substitute for National Service.

John Trevor Stuart (universally known as Trevor) was one of that immediate post-WW2 generation whose scientific careers began in this manner. His parents were Horace and Phyllis Stuart (née Potter), who resided in the city of Leicester. Horace had been drafted into the British Army at the age of 18 in 1917 and had subsequently suffered severe leg wounds that required extensive surgery and a long recuperation. After the war he became a foreman in a factory that made hosiery machinery. Phyllis was a professional dressmaker and was also a gifted amateur pianist. Trevor, their second son, attended Leicester's Gateway School. For financial reasons, it would normally have been out of the question for a son of such a family to have attended university, but the fact that Trevor won a Royal Scholarship at Imperial College London meant that he was able to begin his undergraduate studies in the Department of Mathematics in 1947. His first choice had been physics, but, when he discovered these places had been taken, he gladly accepted the offer of a scholarship in mathematics. The wartime regulations that had temporarily reduced degrees to two years were still in operation, so it was only four years later, in 1951, that he emerged with a PhD. As a graduate student, he had been supported by a University of London postgraduate studentship. His first paper, published in 1951, was co-authored with his supervisor David Meksyn (1).

From Imperial, Trevor immediately joined the Aerodynamics Division of the National Physical Laboratory (NPL), which is located between Teddington and Hampton Hill in London. He was assigned to the theoretical group, led by W. P. Jones, which also included Les Woods,* a former Rhodes Scholar at Oxford, on secondment for two years from the scientific branch of the Royal New Zealand Air Force. Trevor's early work marked him out as a major player in what was then called the hydrodynamic stability of fluid flows. Initially he was influenced by H. B. Squire (FRS 1957) to work on transition to turbulence on swept-back wings. Then, at the suggestion of G. K. Batchelor (FRS 1957) he was urged to consider the flow between two circular cylinders, the inner of which rotated with the outer one stationary. Circular Couette flow was to become a recurring theme in his career (see below). Keith Moffatt (FRS 1986) has said : 'I first encountered Trevor when he lectured at the International Congress of Mathematics in Edinburgh 1958, when I was employed as a

* Leslie Woods eventually became a professor of mathematics at Oxford and a Fellow of Balliol College.

blackboard cleaner. He lectured on nonlinear stability of circular Couette flow, which was pioneering work at that time (personal observation in 2024).'

Trevor's commitment to both NPL and Christine—his future wife (and NPL employee)—and the significant impact of his early work can be illustrated by the fact that over a short space of a time in the mid to late 1950s he was offered an assistant lectureship in the Department of Applied Mathematics and Theoretical Physics at Cambridge, a lectureship in the Cambridge Department of Engineering, together with faculty positions at Stanford University and the Massachusetts Institute of Technology (MIT). He turned them all down to remain at NPL and marry Christine in 1957. Indeed, in 1956 they had deferred their marriage by a year to allow him to take up a visiting lectureship at MIT for the year 1956–1957 at the invitation of C. C. Lin. There he formed life-long friendships with David Benney, Willem Malkus and Lou Howard.

It is inevitable that we are influenced by institutional attitudes at the beginning of our careers. The post-war years of the late 1940s and early 1950s in the government Scientific Civil Service must have been a comparatively heady time with its wealth of new recruits. Trevor himself said that at that time NPL was a wonderful place for a younger scientist to develop. Even so, the management style employed and the work environment was a world away from the twenty-first century. The universal lack of building materials meant that there was a severe shortage of working space. Stories abound of laboratories in which young recruits worked in myriad rows of desks in hangar-like buildings and Nissen huts. It sounds like a rudimentary version of the modern-day open-plan office, except the Civil Service rules were crafted for a different era. If a person achieved a promotion, the rules said they could have a hat/umbrella stand. Overnight, the men in brown coats would place the stand next to a particular desk in the hangar. If that person received a further promotion, in the morning the desk and hat/umbrella stand would be seen to be standing on a tiny square of carpet with all the other desks adjusted to avoid it. At NPL, all incoming letters addressed to individuals, unless marked 'personal', first went to the Director's office, where they were read by an official and were only forwarded if deemed appropriate, and even then to the individual's group leader. Before journal submission, all papers had to be vetted by a higher authority, and occasionally disappeared for years into a black hole. Despite the distinguished reputations of many senior staff, this rigidly hierarchical environment was a reflection of the out-dated Edwardian managerial conventions still functioning in government institutions in the immediate post-war period. They seem strangely amusing to the modern mind, but this was the world in which Trevor was raised and in which he thrived. Mercifully, they are a distant memory compared with the modern, streamlined and world-leading metrology institute that NPL is today. Reference is made further down to the gradual transfer of NPL's Aerodynamics Divison to RAE Farnborough in the late 1960s (completed in 1971). The National Maritime Institute was spun off NPL in 1976 and eventually became British Maritime Technology.

Evidence of the outstanding impact of Trevor's early work was the invitation to present one of the four general lectures at the (quadrennial) International Congress of Mechanics held in Stresa (1960) (4). To be invited at the age of just 31 to give such a lecture was a signal honour and a recognition of his seminal achievements during the 1950s. Institutional recognition followed in 1961, when, at the age of 32, he became the youngest Senior Principal Scientific Officer with a Special Merit Award in the Scientific Civil Service. The

pinnacle of this work was his classic 1960 paper (3), which developed the ideas laid out in an earlier paper (2) and which introduced the idea of what is called the weakly nonlinear regime. This opened up a new world of how to approach flow instabilities. In fact, it was a genuine milestone that permanently changed the subject of fluid mechanics and beyond. A wider exposition is set out below. It should be noted that adjacent to the 1960 paper was the paper of his NPL collaborator John Watson (Watson 1960).

In the early 1960s at NPL, on the first floor of Cromer House,† Trevor and John Watson began to develop these ideas in collaboration with Tony Davey, who undertook an early computational initiative to solve the Orr–Sommerfeld equation. The Stuart–Watson papers were the first to open up a crack in the edifice of how to deal with transition to turbulence in nonlinear systems by working directly with the Navier–Stokes equations. Indeed, the next 30 years turned out to be a golden time, as similar asymptotic methods were developed for the theory of boundary layers. It was during the period 1957–1965 that Michael Gaster (FRS 1985), first as a PhD student at Queen Mary College and then on the staff at Cranfield Aeronautics Department, began to discuss with Trevor how to adapt these ideas for flow over a wing where disturbances grow downstream. Initially, Trevor was reluctant to accept the link that Mike had found between the temporal and spatial evolution of disturbances by using Fourier methods. Later, however, he accepted their validity and encouraged the methods used to obtain them (Gaster 1962).

All the major government laboratories of those postwar decades housed various groups who acted as an informal interface with the UK university research system and were led by senior scientists who had earned distinction in their fields. Many of these went on to be elected as Fellows of the Royal Society, as Trevor was in 1974. This equilibrium was up-ended in the mid 1980s when the government of the day decided that their laboratories should move towards ‘near market research’, and instituted severe budget cuts. Groups that could not justify their existence through industrial connections were closed down. NPL went through an early version of this change in policy in 1966, when the Ministry of Aviation became concerned over the state of the many small British aviation companies that were feeling the stiff breeze of US and French competition. Their response was to encourage mergers while giving direct technical help. NPL’s Aerodynamics Division was reorganized in order to place a greater emphasis on shorter-term research, and much of it was transferred to RAE Farnborough. Moreover, the discontinuation of funding for laminar flow research in the USA induced the UK authorities to follow suit in 1966 by bringing to a close some 15 years of outstanding work by Trevor’s Laminar Flow group.‡ His association with NPL thus came to a formal end, and the next chapter of his career began when he moved to a chair back in his old Department of Mathematics at Imperial College. Figure 4.

† It was in Cromer House, now demolished, where Sylvia Skan performed her pioneering numerical work on boundary layers that led to the Falkner–Skan equation (Stewartson 1953).

‡ ‡A history of the NPL Aerodynamics Division has been written by R. C. Pankhurst (Pankhurst 1972), who was its Superintendent during the period 1964–1970.

Figure 1. Trevor on holiday in Wales From the Stuart family collection.

Personal and anecdotal recollections

As a personality, Trevor was a very private man who was not socially gregarious, although the few who knew him well were aware of his natural kindness. Only on rare occasions could he be persuaded to come over to the Queen's Arms, the local Imperial watering-hole, just off Queen's Gate Terrace. His natural reserve contrasted greatly with his wife Christine (married in 1957; see figure 2), who had a lovably extrovert and occasionally quirky personality. They were a perfect match of opposites. Indeed he only seemed comfortable in public when he was with her. She was born Christine Mary Tracy (1930–2017), and her family home was located in Staines Road in Twickenham. Christine attended Gumley House School in Old Isleworth and then joined the Shock Tube group at NPL, where she performed experimental work under the guidance of Herbert Pearcey. During that time she undertook a part-time extra-mural London external degree in mathematics at Kingston Technical Institute (now Kingston University), which earned her a promotion. She and Trevor met through NPL social events—their first date was a prom at the Royal Albert Hall. In 1968 the Stuart family moved from Walton-on-Thames to Wimbledon to a brand new house that lay at the end of a small cul-de-sac close to the All England Tennis Club. The Stuart family, consisting of Trevor, Christine, Andrew (born 1962), David (born 1964) and Katherine (born 1968), seems to be unusually gifted mathematically: the 'mathematical gene' has been passed first to Andrew, who is currently a professor at California Institute of Technology in the Department of Computing and Mathematical Sciences, and also to David,

who is a professor of applied mathematics in Cambridge. Trevor was elected FRS in 1974 and Andrew in 2020. The number of living parent/offspring Fellows of the Royal Society has been very small given the total size of the Fellowship in the last century.[¶]

Not only was Trevor a very private man, but he also had an intensely strong sense of public propriety. Not for him the colourful language and behaviour of more flamboyant colleagues. He was not the type to form easy casual friendships with colleagues, but he was very close to our much-loved colleague Derek Moore (FRS 1990), who had a wicked Geordie sense of humour. Despite, or perhaps because of, his reserved personality, he could occasionally spring a surprise. One of us (JG) records that over lunch he and Trevor were once discussing the college's collection of portraits of past Rectors displayed in their formal residence at 170 Queen's Gate. He (JG) made a mildly negative remark about the official portrait of Brian, Lord Flowers FRS (Rector of Imperial 1973–1985) by Kyffin Williams RA. In reply, he received a gentle rebuke from Trevor followed by a 15 minute homily on the work of Williams in particular and of modern art in general. The point was reinforced a few weeks later on a visit to the Stuart household in Wimbledon, when he noticed a well thumbed copy of Ernst Gombrich's (Gombrich 1950) *The story of art* on a table-top. Likewise, Trevor was also knowledgeable about various aspects of classical piano music, which, according to his son Andrew, bonded him to both his mother and his wife Christine. Some people wear their enthusiasms like lapel badges, to be talked about incessantly and eye-glazingly in public, whereas Trevor was quite the opposite.

Later career at Imperial College London

The dramatic expansion of the entire UK university system after the Robbins Report (1963) was reflected in the growth of the Imperial Department of Mathematics under the headship of Harry Jones FRS. Trevor's arrival in 1966 coincided with that of the analytic number theorist Klaus Roth FRS—the UK's first Fields Medallist (1958)—and the statistician Sir David Cox (FRS 1973). However, the department also had to adjust to the loss of Abdus Salam FRS's (1979 Nobel Prize in Physics with Steven Weinberg and Sheldon Glashow) distinguished Mathematical Physics Group (including Tom Kibble (FRS 1980), among others), who had moved to the Blackett Laboratory in the early part of that decade; the group has continued to flourish to this day under the title of the Theory Group. This significant change meant that Harry Jones and Dick Scorer, together with Trevor on his arrival, oversaw the growth in applied mathematics more towards the areas of aero-acoustics and hydrodynamic stability rather than mathematical physics. Among the names associated with this time are: Sir James Lighthill FRS,[#] John 'Shôn' Ffowcs-Williams, David Crighton (FRS 1993), Frank Leppington, Derek Moore and Mike Howe; and, later, Colin Atkinson (FRS 1998), Frank Berkshire, Philip Hall, Frank Smith (FRS 1984) and Ian Walton. It was

[¶] Examples include: Sir George Darwin (FRS 1879) was the son of Charles Darwin (FRS 1839); Sir Lawrence Bragg (FRS 1921) was the son of Sir William Bragg (FRS 1907); Sir George Thomson (FRS 1930) was the son of Sir Joseph (J. J.) Thomson (FRS 1884); Sir John Kingman (Honorary FRS 2021) is the son of Sir John Kingman (FRS 1971); Daniel Wolpert (FRS 2012) is the son of Lewis Wolpert (FRS 1980); and Ben Simons (FRS 2021) is the son of John Simons (FRS 1989).

[#] Sir James was a Royal Society Research Professor at Imperial College London between 1964 and 1969. He had been Director of RAE Farnborough in the years 1959–1964.

Figure 2. Trevor and Christine on the day of their marriage in 1957. From the Stuart family collection.

in this latter period that Trevor began to hold his Friday afternoon seminar sessions in his office, where ideas had to be expounded from first principles on the blackboard. On occasion these were attended by others outside Imperial, such as Keith Stewartson FRS and Mike Gaster.

Trevor served as head of department twice. The first time was for five years (1974–1979), when Lord Flowers was Rector, and the second was a three-year stint between 1983 and 1986 under Eric (later Sir Eric) Ash FRS. The first period, although longer, was less onerous. Without its enormous medical school, the college was smaller and was not divided into faculties. This meant he reported directly to Lord Flowers, with whom he got on well, so decisions could be easily reached. However, in the early/mid 1980s, the heavy university funding cuts imposed by Margaret Thatcher (Honorary FRS 1983)'s government created a fraught environment in every direction. By 1986 he was glad to pass over the headship of the department to Frank Leppington and recuperate. Moreover, he had been hit hard by the relatively early deaths of Keith Stewartson (in 1983) and Dick DiPrima (in 1984). Dick in particular had not only been a valued collaborator but a very close friend, so this loss affected Trevor in a deeply personal way. Moreover, his work style relied on collaborative ventures, so the loss of both men affected his output.

During the 1980s he served a term on the Council of the Royal Society as well as a term as chair of the Science and Engineering Research Council (now EPSRC) Mathematics Committee. In the period 1990–1993, he then served a three-year term as Dean of the Royal College of Science within Imperial, a position that had no management duties but, as an independently elected senior academic, he was involved in the supervision of the academic promotions process, and served on appointments, appeals and prize committees. Trevor was well suited to this role and excelled in it. He had always had an administrative gift, which he had dutifully exercised earlier in his two spells as head of department, and would again exercise in the year 2000 when he was asked to be President of the London

Mathematical Society after the death of David Crighton. Nevertheless, he was uninterested in higher administrative university positions, and resisted attempts to lure him into applying for vice-chancellorship appointments. Despite this, his well developed sense of fairness, decisiveness and care with the paperwork meant that he was often approached to sit on external committees or act as an external assessor for UK university chair appointments.

To those of us who knew him well, his reserved, gentleman-like personality did not prevent him, particularly in his younger days, from expressing robust views on a variety of subjects. Nevertheless, one of his early collaborators at NPL, Mike Gaster, says that 'Trevor was always tough but rigorously fair'. As is often the case, he mellowed with age. In fact, for all his inbred Civil Service instincts and the Edwardian sense of hierarchy that it imbued, and which he never entirely left behind, he was a natural academic at heart. Many of his generation of senior applied mathematicians had a disdain for (or fear of) pure mathematics and reacted accordingly if anyone dared to use abstract ideas. Trevor was the opposite. In his own quiet way he actively encouraged new ideas, regardless of their source, and showed evident joy when an abstract piece of work could be put to good use. He also had an uncanny flair for picking winners. He would regularly attend the seminars given by job candidates across most disciplines in the department. He would then write notes for the committee and had the uncanny knack of picking out the winner in about 90% of cases, a gift that would have made him a fortune on the racecourse.

In 1984 he was awarded the Senior Whitehead Prize by the London Mathematical Society. He was also awarded honorary Doctor of Science degrees from Brown University and the University of East Anglia. Together with Derek Moore, he took formal retirement in 1996, but continued as a Senior Research Investigator. He was elected a Fellow of Imperial College (2012) and served a four-year term as editor of *Biographical Memoirs of Fellows of the Royal Society* (2012–2016). In his later years he still appeared at seminars, posing penetrating questions for the speakers. He went missing only when he began to care tirelessly and lovingly for Christine during a progressive and lengthy illness in her last few years. After her death (17 November 2017), he returned to attend seminars and enjoy dinners at the Royal Society Dining Club. Moreover, while religion had played little role in the Stuart family, both Trevor and Christine in their later years, and after Christine's death Trevor alone, found the sense of community at the neighbouring St Mary's Wimbledon to be a welcome and supportive component of their lives. As he weakened towards the end, he received strong support from Katherine Thomasset, his daughter, and from Christine's sister Irene, from his carer Glenda and from this church.

The development of weakly nonlinear theory

In general terms, the subject of hydrodynamic stability investigates how relatively simple flows can drastically lose their simplicity to exhibit more complex spatial–temporal characteristics, ultimately entering seemingly chaotic states that are loosely referred to as 'turbulent'. Such flows appear naturally in a wide range of areas, such as the breakup of liquid jets, in layers of fluid heated from below and flow between rotating cylinders, as well as flows through channels/pipes or over wings/fuselages. Sudden changes take place in two

forms: either as control parameters are varied, or as the fluid migrates from one part of the flow field to another.

Linear stability theory, which considers infinitesimal disturbances as a first step of inquiry, had been developed by some of the great figures of their times, including William Thomson (FRS 1851; Lord Kelvin 1992), John Strutt (FRS 1873; Lord Rayleigh 1873), Hermann von Helmholtz (ForMemsRS 1860), Arnold Sommerfeld (FRS 1926), Werner Heisenberg (FRS 1955), Geoffrey 'G. I.' Taylor (FRS 1919) and Ludwig Prandtl (FRS 1928). To the above list of luminaries one can add Trevor's name. In the late 1950s and early 1960s he opened a new chapter with his seminal work on what is now called weakly nonlinear theory, which has become a standard method taught in graduate school courses.

To fully appreciate its origin, essence and impact, let us look at the state of the subject in the 1950s. Through the effort and ingenuity of Heisenberg, Tollmien, Schlichting and Lin in the period 1924–1945, the solution of the linear stability problems for channel flow and boundary layers had been resolved by developing asymptotic techniques applied to the Orr–Sommerfeld equation. They had shown convincingly that viscous effects induce instability, despite the fact that they dissipate the disturbance kinetic energy into heat. Moreover, the value of a critical Reynolds number Re_c had been predicted with reasonable accuracy for the onset of instability in each flow. Using digital computers, numerical solutions began to emerge. However, although the eigenvalue problems solved were far more complex than their counterparts in Rayleigh–Bernard convection, in Taylor–Couette flow or in quantum mechanics, the solutions turned out to be far less successful. In fact, severe problems remained. In channel flow, the onset of turbulence occurs at $\mathrm{Re}_c \approx 2000$, well below the predicted value of about 5800, whereas in the Blasius boundary layer, small-amplitude disturbances start to amplify at the predicted location, but transition commences much farther downstream. For plane Couette and pipe flows, the evidence led to the conclusion that they are both linearly stable at all Reynolds numbers, and yet, in reality, transition to turbulence takes place at the moderate Reynolds numbers of 300 and 3000, respectively. Nonlinear effects were suggested as holding the key to explaining the discrepancies (Landau 1944). It transpired that it was too naive an idea to equate linear instabilities with the onset of turbulence. The latter is a nonlinear world, subtler by far than the linear kingdom, and had been little explored.

In his 1958 paper (2), Trevor considered non-equilibrium solutions, with an analysis based on the energy equation for the perturbation. In order to represent disturbances undergoing nonlinear evolution, the notion of an 'amplitude equation' was introduced for the first time with a solution expressed in terms of a separation of variables. This is a product of the amplitude function, a 'shape function' of the transverse coordinate, and a carrier-wave factor involving the streamwise coordinate and the time variable. By taking the shape function to be the eigenfunction given by linear stability theory, and by using it to evaluate the production–dissipation as well as the kinetic energy, Trevor arrived at a nonlinear amplitude equation of the form postulated by (Landau 1944) on a phenomenological basis. In 1944 the Soviet physicist Lev Davydovitch Landau (FRS 1960) had proposed a phenomenological argument to explain the transition to turbulence of an oscillating system near a Hopf bifurcation and wrote down the Stuart–Watson equation by simply appealing to general scaling arguments (Landau 1944). This paper was referenced by Trevor (3). However, Landau made no connection with the Navier–Stokes equations,

and it was Trevor who developed the idea in this context. Once again, only the mean-flow distortion contributed to the nonlinear term. This approach has a clear physical picture and requires only simple algebra to evaluate the coefficients in the amplitude equation. Interestingly, as the derivation does not formally require the disturbance to be nearly neutral, it may be extended to amplifying and damped modes, in which case the time derivative term in the mean-flow distortion equation would be retained, leading to an amplitude equation of integro-differential form. Owing to these appealing attributes, this approach became popular and has been applied (with modification) to study nonlinear evolution of disturbances in free shear layers (Liu 1989).

In this early phase of weakly nonlinear stability theory, the disturbance amplitude and growth rate were taken to be small but finite. The approximation of truncating the expansion at the lowest order needs to be checked. This becomes unnecessary if the linear growth rate is asymptotically small because a systematically asymptotic approach, whose validity is *a priori* guaranteed, may be developed. This key observation, made first by Trevor, formed the foundation of weakly nonlinear stability theory. The classical weakly nonlinear stability theory expounded in his most famous paper (3) was formulated primarily for exactly parallel flows and on the basis of a finite Reynolds number Re controlling the stability. To fix the idea, consider the channel flow. Let $(\alpha, 0, \omega)$ be a planar mode that is neutral at critical Reynolds number Re_c. At a value of Re close to Re_c, its linear growth rate is $O|Re - Re_c| \ll 1$, and so a slow time variable, $\tau = (Re - Re_c)t$, is introduced. Under the influence of nonlinearity, the evolution is no longer exponential but can instead be described by an amplitude function $A(\tau)$. The disturbance with a small magnitude $\epsilon \ll 1$ can be expressed as an *asymptotic series*. The nonlinear self-interaction generates a mean-flow distortion and second harmonic of $O(\epsilon^2)$. Each of them interacts with the $O(\epsilon)$ fundamental mode to produce an $O(\epsilon^3)$ forcing on the fundamental component (and the third harmonic as well). The evolution is influenced by nonlinearity when this forcing is comparable to the $O(\epsilon(Re - Re_c))$ modulation, i.e. when $Re - Re_c = O(\epsilon^2)$. At $O(\epsilon^3)$, there arises an inhomogeneous system, acceptable solutions that exist only when a solvability condition is satisfied, since the homogeneous system admits a non-trivial solution (i.e. the eigenmode). Imposition of this condition through the use of the adjoint eigenfunction led to the evolution equation for the amplitude function $(\kappa_1, \kappa_2 > 0)$,

$$A'(\tau) = \kappa_1 A - \kappa_2 |A|^2 A, \tag{1}$$

which (Landau 1944) had postulated. Equation (1) has come to be known as the Stuart–Watson–Landau or Stuart–Landau equation since its derivation. The idea and the methods employed formed the basis for what has become known as 'weakly nonlinear theory'. Later it was recognized that equation (0.1) stands as the normal form for bifurcation of co-dimension 1, and is consistent with the Centre Manifold and Hopf-bifurcation theorems, which were rigorously proven (but only) for nonlinear dynamical systems of finite dimension (Guckenheimer & Holmes 1983). In a related manner, Stewartson & Stuart (9) considered two-dimensional streamwise *localized disturbances* in a shear flow with Reynolds number Re close to the critical value Re_c. Such a disturbance propagates downstream to form a

wavepacket with its envelope translating at a group velocity c_g. The amplitude function now depends on $\xi = \epsilon^{1/2}(x - c_g t)$ as well as on τ. Stewartson & Stuart (9) showed that $A(\tau, \xi)$ satisfies a modulation equation of nonlinear Schrödinger type. Generalization to three-dimensional disturbances localized in both the streamwise and spanwise directions was made by (Davey *et al*. 1974).

Trevor's vision and ideas sowed the seed for the development of high-Reynolds-number weakly nonlinear theory, which was further encouraged by his ability to nurture and attract younger researchers of outstanding talent. The venture into this area was begun at Imperial College in the late 1970s by Frank Smith and Philip Hall, and continued by Stephen J. Cowley well into the 1990s. They investigated a wide range of nonlinear scenarios, of which only a few can be mentioned below, with a full picture being given in a review (Cowley & Wu 1994).

(Smith 1979) and (Hall & Smith 1984) considered nonlinear development and mutual interactions of Tollmien–Schlicting (TS) waves near the lower-branch neutral curve. These waves concentrate in the wall layer (lower deck) and so interact to drive a mean-flow distortion. The latter spreads out to a thicker outer layer, a feature akin to acoustic streaming (4). For a plane wave, the distortion in the outer layer does not contribute to the leading-order effect, and consequently an amplitude equation of Stuart–Landau type was derived. However, for a spanwise modulated wavetrain (Smith & Walton 1989), or a pair of oblique TS-waves (Hall & Smith 1989), the interaction drives a spanwise-dependent mean flow, which becomes unbounded towards the outer edge of the lower deck and thus acquires a much larger amplitude in the outer or diffusion layer, so-called because the distortion is governed by a diffusion equation. Moreover, it interacts with the fundamental, contributing the leading-order nonlinear term, which is non-local. (Hall 1983) considered nonlinear evolution of Görtler vortices at high Görtler numbers. Without rapid streamwise variation, these vortices are distinct from TS-waves. Nevertheless, the essence of nonlinear effects remains the same: the fundamental interacts with itself to generate a mean-flow distortion, which in turn influences the fundamental. (Haynes & Cowley 1986) considered nonlinear development of Rossby waves in a temporally or spatially developing shear flow, showing that the motion of the critical level itself could become important.

One of us (XW) was fortunate enough to be Trevor's last PhD student, joining the venture of developing high-Reynolds-number nonlinear theory in the late 1980s when it was still in full swing. Attention was beginning to shift towards the upper-branch instability of boundary layers and free shear flows. The salient feature is that instability modes have a critical layer, where several otherwise negligible physical effects become important simultaneously. In particular, dominant nonlinear interactions take place there locally but are capable of affecting the overall development of the modes, provided the modal amplitude reaches a threshold. The fundamental processes—generation of mean-flow distortions and harmonics as well as their impact on the mode through cubic interaction—remain similar to those in the classical weakly nonlinear theory, and can be treated by a similar procedure. However, there are significant differences. The first is that the slow growth of the mode itself, referred to as a non-equilibrium effect, may appear in the leading-order governing equations in the critical layer. The second is that the length/time scale and the threshold amplitude for the nonlinear evolution depend on the singular nature of the inviscid solutions at the critical level as well as on the composition of the modes when modal interactions are

considered (Cowley & Wu 1994). Systematic studies have been performed for instability modes in a variety of important shear flows (see the review paper by (Wu 2019)).

Taylor–Couette–Mallock flow

Students of fluid flows usually find that their first encounter with sub/supercritical bifurcations is the theory surrounding the Stuart–Watson–Landau (Equation 1). Trevor's seminal contributions in those directions make it easy to overlook another branch (excuse the pun) of research close to Trevor's heart; namely, nonlinear bifurcation phenomena in centrifugal flows. These produce branches of solutions that, in turn, pave the way for a more orderly transition from one state to another but ultimately lead to complex phenomena that are generally called 'turbulent'. In retrospect, his passion for nonlinear bifurcation theory had always been present at Imperial and was showcased fully in the Imperial Fluid Dynamics Seminar that was organized by one of us (DP), who took it over in 2009 after a move from the US to the UK. Trevor was a regular participant up to 2020, when Covid hit. Afterwards, his decreased mobility made his attendance more sporadic. There were several memorable comedic moments when young high-flying researchers were questioned by Trevor in his courteous but penetrating way about their results. Afterwards, when they inquired about the identity of the older gentleman, their jaws dropped when they were informed that he was Trevor Stuart of 'Stuart–Landau' fame!

Taylor–Couette–Mallock flow describes the motion of a fluid confined between two concentric cylinders with either or both cylinders rotating about their common axis. For an extended account of the early history of the phenomenon, see the article by Russell Donnelly (Donnelly 1991) and also the paper by (Golubitsky *et al*. 2000). It is named, first after Geoffrey 'G. I.' Taylor (1886–1975), who provided the first theoretical explanation of experiments in 1923 (Taylor 1923), second after Maurice Couette (1858–1943), and third after Henry Mallock (1851–1933; FRS 1903). In his 1986 *SIAM Review* paper (13), Trevor referred to G. I. Taylor's seminal paper (Taylor 1923) as 'epoch-making'. Taylor formulated the linear stability problem and solved the eigenvalue problem by using eigenfunction expansions and truncations to show that above a critical Taylor number (a dimensionless measure of the speed of the inner cylinder, say) the flow becomes unstable to axially periodic disturbances. Excellent agreement was found with experiments on values of the onset Taylor number. However, to follow the bifurcation(s) into the nonlinear regime, to consider secondary instabilities and to look for a route to transition to chaos, it was found that increased rotational speed was required.

This bifurcation route to complexity was elucidated beautifully in Trevor's earlier review paper (7) written in 1971. He identified four bifurcation problems: (a) determination of a critical Taylor number where Couette flow first becomes unstable; (b) calculation of the bifurcated vortex flow and wavenumber selection; (c) the stability of the vortex flow to wavy disturbances to determine a second bifurcation point; and (d) calculation of the new wavy-vortex type state that emerges as a stable solution. Without modern computational facilities, analytical pathways had to be found. Problem (a) was addressed by G. I. Taylor, while Trevor himself solved problem (b), where he used the methods he had developed for parallel shear flows to derive a weakly nonlinear amplitude equation (2) that predicted the supercritical saturation of the linear instabilities first described by Taylor. Perhaps more

importantly, the resulting finite-amplitude flow induces a Reynolds stress that, in turn, modifies the mean circumferential flow and the torque needed on the inner cylinder to keep the motion. Trevor calculated this and made the first direct comparison of his nonlinear theory with Taylor's experiments with excellent agreement (2)—see his fig. 5. The stability of the Taylor vortices to disturbances both in the axial and azimuthal directions (with a travelling component in the latter) was addressed by Trevor with Davey and DiPrima (6), thus explaining some experimental observations: e.g. the last image in figure 3. Problem (d) is analytically highly challenging. Trevor produced an ansatz that accounts for the problem's symmetries (always motivated by the experimental observations) to derive and analyse a consistent set of four coupled nonlinear Stuart–Watson type equations (4, 7).

In further pioneering work with his longstanding collaborator Dick DiPrima, Trevor began looking at the Taylor vortex problem in situations where the inner cylinder is placed off-centre. They used modified bipolar coordinates together with a conformal transformation to calculate the steady two-dimensional flow in appropriate asymptotic regimes (10). In a companion paper by the same authors (11), partial differential equation stability equations emerged that required non-local methods. Trevor and DiPrima used ingenious asymptotic canonical limits for small clearance ratio and small eccentricity to solve the non-local stability problem. It appears that this was one of the first studies of non-local stability phenomena that broke the recourse to local stability or parallel flow approximations. They considered the supercritical nonlinear flow at the first bifurcation of eccentric rotating cylinder flows (11), and the result was improved by going to higher order in work by Trevor with Eagles and DiPrima (12). Given the computational power now available, it has become standard practice to solve such problems numerically, but the fact that Trevor and Dick DiPrima solved such a problem more than 50 years ago is a testament to their mathematical prowess and inventiveness.

This section would not be complete without a brief account of the influence that Trevor's work on the Taylor–Couette–Mallock problem has had in other closely related areas. The common thread is the presence of centrifugal instabilities due to flow over curved boundaries. Trevor established Imperial as an international leader in the study of such flows, and trained many PhD students, some of whom have continued this work—e.g. Philip Hall and Xuesong Wu. Seminal work on the stability of boundary layers to Görtler vortices was carried out by Philip Hall (Hall 1983), and studies on centrifugal instabilities of Stokes layers and vortex formation in flow between two concentric rotating spheres by Ian Walton (Walton 1978) and oscillatory flows in curved pipes by one of us (DP) (Papageorgiou 1987).

It is quite clear that Trevor's contributions to nonlinear hydrodynamics drove the field forward at a time when computational power was still a nascent dream. In the particular case of Taylor–Couette–Mallock flows, the analytical developments paved the way for large-scale modern computational explorations, which are now being carried out at Taylor numbers six orders of magnitude larger than critical values, making the problem a test-bed in turbulence discovery science—see the review by (Grossmann *et al*. 2016). In addition, direct numerical simulations have shown that features such as Feigenbaum cascades and behaviour pertinent to low-order dynamical systems and one-dimensional partial differential equations (PDEs) govern the subcritical regime of Taylor–Couette flow. It is pleasing to note that one of the corresponding authors in Wang *et al*. (2025a, b;), Kengo Deguchi, is

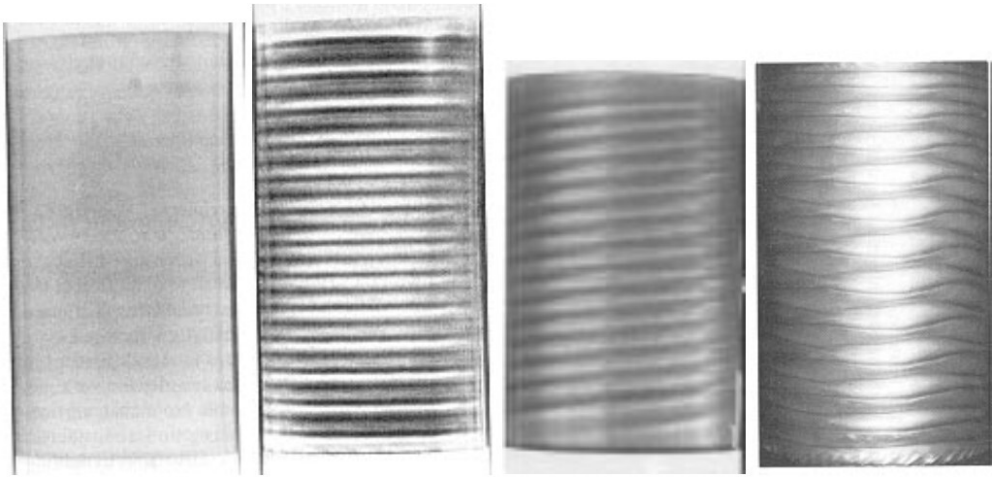

Figure 3. The Taylor–Couette–Mallock experiment. From left to right the rotation increases: laminar Couette flow, Taylor vortex flow, spiral vortex flow and wavy vortex flow. Reproduced from (Golubitsky *et al*. 2000).

Trevor's 'academic grandchild', given that Kengo spent several years with us at Imperial working with Philip Hall in the EPSRC Centre for Laminar Flow Control.

Kelvin–Stuart vortices

During his recent sabbatical to Imperial College London from MIT's Department of Mechanical Engineering, Triantaphyllos Akylas shared reminiscences of Trevor's generosity to him in his early career. Triantphyllos had once been a PhD student of David Benney, whom Trevor visited at MIT in 1981. Trevor took the time to visit the graduate student offices to speak with Triantphyllos while he was working on wind-driven surface waves as a shear-flow instability. He fondly recalls Trevor's genuine interest in this work and his support, which continued in the following years. Over dinner, when asked by two of us (DC, DP) what he viewed as Trevor's most famous result, Triantaphyllos did not hesitate : 'The cat's eyes!' (figure 4). He was referring to what has become known in the fluid dynamics community as 'Stuart vortices', originating in Trevor's 1967 paper (5). That paper mostly concerned the stability properties of a tanh-function shear-flow profile. It was motivated by a theoretical contribution made by Schade (1964) on the nonlinear stability of inviscid shear layers, and by more recently reported experimental results. The paper in the American Mathematical Society volume edited by W. H. Reid (8) contains further remarks on this topic.

Trevor divided his interesting and unusual paper (5) into two parts, having recognized with characteristic insight that the considerations of the second part were of much broader theoretical significance than the more specific hydrodynamic stability concerns of part 1. Nevertheless, he pointed out connections between the two. The Stuart vortex solution, presented in part 2, is a one-parameter family of steady exact solutions of the nonlinear

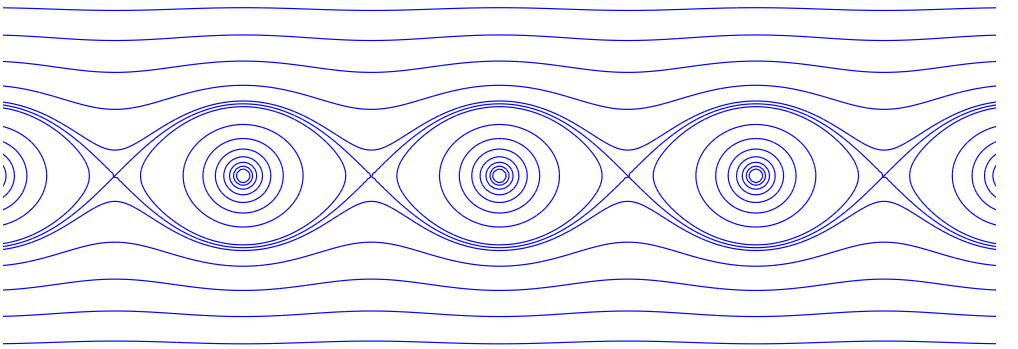

Figure 4. Kelvin–Stuart cat's eyes.

Euler equations. Trevor called the parameter C ($1 \leq C < \infty$), furnishing a solution. In one limit, when $C = 1$, the solution coincides with the tanh-function shear profile whose stability properties motivated the article in the first place; in the opposite limit, as $C \to \infty$, the solution class merges continuously into a singly periodic point-vortex row sitting in otherwise irrotational flow (see figure 4), which is a classical equilibrium known to be linearly unstable to subharmonic perturbations of twice the period (Lamb 1932; Saffman 1992). Philip Saffman (FRS 1998), an authority on vortex dynamics, called this a 'pairing mode instability' because it explains the observation of pairing in the phenomenon of vortex merging. The streamline patterns of a periodic point-vortex row are characterized by lens-shaped separatrices and closed recirculating cores. They were once studied by Lord Kelvin, which has earned them the designation 'Kelvin cat's eyes'. They can also be found in Lamb's treatise on hydrodynamics (Lamb 1932). In view of the subsequent impact of Trevor's work, the attribution is often modified to 'Kelvin–Stuart cat's eyes' (Liao *et al.* 2023). Saffman's monograph includes a dedicated section on Stuart vortices, reflecting the place those solutions now hold in the fluid dynamics canon (Saffman 1992). History has proven Trevor right when he underscored the broader theoretical significance of the material in his part two (5). In closing his paper, he offered the following remark: 'The solutions presented here, because they are an exact consequence of certain equations, are of theoretical and illustrative value. Especially it is helpful to see a solution of Laplace's equation, namely that for the flow due to a set of point vortices equi-spaced along a line, related to a class of rotational flows. This result, in itself, gives added meaning to the concept of point-vortex solutions of the irrotational-flow equations.'

This insightful observation, made in 1967, can be seen as the seed of a rich array of further developments made over the intervening years. Trevor noticed that the stream-function–vorticity relation for a unidirectional tanh-function shear profile is tantamount to that streamfunction which satisfies the elliptic Liouville equation, which is a canonical quasi-linear partial differential equation with applications in space plasma physics, high-energy physics, differential geometry and even theories of quantum gravity; his insight added fluid dynamics to this list. His class of solutions can be viewed as a one-parameter branch of solutions of Liouville's equation connecting the unidirectional tanh-function shear profile to the periodic point-vortex row. In a fascinating and quite recent development,

it is now known that there is a zoo of other solution branches, inter-related in mathematically interesting ways, all satisfying Liouville's equation and constituting equilibria of the nonlinear Euler equations. In common with Trevor's vortex solution, many share the feature of terminating in point-vortex equilibria in otherwise irrotational flow, and often with surprisingly exotic configurations (Krishnamurthy *et al*. 2021). These branches are all mathematical cousins of the Stuart vortex solution branch.

In a different direction, this class of solutions has also been generalized to flows on several other surfaces besides the plane, the first such being the identification of Stuart vortices on the surface of a sphere (Crowdy 2004). Despite topological differences from the plane, it turns out that there is a natural generalization of Trevor's solutions to a spherical surface involving a suitably modified Liouville equation that also admits analytical solutions characterized by cat's-eye streamline patterns and sharing the feature of terminating, as the relevant parameters change, in pure point-vortex equilibria in otherwise irrotational flows (Crowdy 2004). As a pioneer of hydrodynamic stability theory, Trevor made the following conjecture: since the $C = 1$ tanh-function shear layer and the $C = \infty$ point-vortex row were known to be unstable to the pairing mode instability, then it is likely that so too are some of the solutions on his connecting branch. Many partial results addressing this notion have appeared along the way, but a paper providing a complete answer to Trevor's conjecture has appeared only very recently (Liao *et al*. 2023). Evidently, after nearly 60 years, Trevor's paper continues to inspire.

Acknowledgements

We have had considerable help from the Stuart family, Andrew, David and Katherine, for which we are grateful. We thank Mike Gaster for his memories of his early collaborations with Trevor. Moreover, we also thank John Norbury (Oxford) and Frank Berkshire (Imperial) for their critical readings of the manuscript.

The frontispiece portrait photograph was taken in 1974 by Godfrey Argent and is © The Royal Society.

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