Vorticity and symplecticity in Lagrangian fluid dynamics

By Thomas J. Bridges[1], Peter E. Hydon[1] & Sebastian Reich[2]

[1]*Department of Mathematics and Statistics, University of Surrey, Guildford, Surrey GU2 7RY, UK*

[2]*Department of Mathematics, Imperial College, London SW7 2BZ, UK*

The relationship between potential vorticity (PV) and the symplectic form for the Lagrangian form of the shallow water equations is established. Starting with the symplectic form, the PV is found by the pull-back operation to the reference space. At first sight, the encoding of PV in the symplectic form appears to be independent of the particle relabelling symmetry. We carry the analysis a step further in two ways. Using the conservation of symplecticity as a starting point, the fluxes of symplecticity are differential forms, and a complete pull-back of the flux forms leads to a geometric description of PV conservation. Secondly we use symmetry methods to give a rigorous connection between particle relabelling, symplecticity and PV conservation. Generalisations of this discovery to semi-geostrophic flow and three-dimensional Lagrangian fluid flows, and the connection with Ertel's theorem are also discussed.

Keywords: geophysical fluid dynamics, potential vorticity, multisymplectic, particle relabelling, symmetry

1. Introduction

In Lagrangian fluid dynamics, the 1-form

$$\boldsymbol{\alpha} = \langle \mathbf{u}, \mathrm{d}\mathbf{x} \rangle = \mathbf{u} \cdot \mathrm{d}\mathbf{x} \tag{1.1}$$

plays the dual role of a circulation density and the generator for a canonical symplectic structure. To illustrate this point, consider the equations governing incompressible Lagrangian fluid dynamics in two space dimensions with unit density,

$$\begin{aligned} \frac{\partial x_1}{\partial t} &= u_1\,, \qquad \frac{\partial u_1}{\partial t} = -\frac{\partial x_2}{\partial m_2}\frac{\partial p}{\partial m_1} + \frac{\partial x_2}{\partial m_1}\frac{\partial p}{\partial m_2}\,, \\ \frac{\partial x_2}{\partial t} &= u_2\,, \qquad \frac{\partial u_2}{\partial t} = \frac{\partial x_1}{\partial m_2}\frac{\partial p}{\partial m_1} - \frac{\partial x_1}{\partial m_1}\frac{\partial p}{\partial m_2}\,. \end{aligned}$$

Here $x_1 = m_1$ and $x_2 = m_2$ at $t = 0$, where $\mathbf{m} = (m_1, m_2)$ are Lagrangian mass coordinates, p is the pressure, and the incompressibility constraint is

$$\det(\mathbf{x_m}) = 1, \qquad \text{where} \quad \mathbf{x_m} = \begin{bmatrix} \frac{\partial x_1}{\partial m_1} & \frac{\partial x_1}{\partial m_2} \\ \frac{\partial x_2}{\partial m_1} & \frac{\partial x_2}{\partial m_2} \end{bmatrix}\,.$$

This system is a constrained Hamiltonian system; the Hamiltonian is

$$H(\mathbf{x},\mathbf{u},p) = \int S\,\mathrm{d}\mathbf{m}\,, \qquad \text{where} \quad S = \tfrac{1}{2}\langle \mathbf{u},\mathbf{u}\rangle + (1-\det(\mathbf{x_m}))\,p\,, \tag{1.2}$$

and the canonical representation is

$$\begin{bmatrix} 0 & -\mathbf{I} & 0 \\ \mathbf{I} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{u} \\ p \end{pmatrix}_{,t} = \begin{pmatrix} \delta H/\delta \mathbf{x} \\ \delta H/\delta \mathbf{u} \\ \delta H/\delta p \end{pmatrix}. \tag{1.3}$$

The symplectic form for this system is the integral of the exterior derivative of the circulation 1-form

$$\overline{\boldsymbol{\omega}} = \int \mathrm{d}\boldsymbol{\alpha}\,\mathrm{d}\mathbf{m} = \int \mathrm{d}\mathbf{u}\wedge\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{m} = \int (\,\mathrm{d}u_1\wedge\mathrm{d}x_1 + \mathrm{d}u_2\wedge\mathrm{d}x_2\,)\,\mathrm{d}\mathbf{m}\,, \tag{1.4}$$

where the line over $\boldsymbol{\omega}$ signifies that it is averaged over $\mathbf{m}$-space.

On the other hand, in the classical description, the circulation is obtained by integrating the 1-form $\boldsymbol{\alpha}$ along a loop in the $(\mathbf{x},\mathbf{u})-$space, and by Stokes Theorem, the circulation can be related to a surface integral of the vorticity (cf. Batchelor (1967, p. 93). Comparing this classical observation with (1.4) suggests that it is natural to expect some connection between vorticity and symplecticity.

When stripped of the integral, the *density of the symplectic form*, $\boldsymbol{\omega} = \mathrm{d}\boldsymbol{\alpha}$, has the following precise relation with vorticity,

$$\psi^*\,\boldsymbol{\omega} = \Omega\, dm_1\wedge dm_2\,, \tag{1.5}$$

where Ω is the vorticity in Lagrangian coordinates,

$$\Omega \;=\; \frac{\partial x_1}{\partial m_2}\frac{\partial u_1}{\partial m_1} - \frac{\partial x_1}{\partial m_1}\frac{\partial u_1}{\partial m_2} + \frac{\partial x_2}{\partial m_2}\frac{\partial u_2}{\partial m_1} - \frac{\partial x_2}{\partial m_1}\frac{\partial u_2}{\partial m_2}\,,$$

and ψ^* is the *pull-back* mapping which is obtained by restricting the phase space coordinates $\mathbf{x},\mathbf{u}$ to the label space $\mathbf{m}$, i.e., $\mathbf{x}=\mathbf{x}(m_1,m_2)$, $\mathbf{u}=\mathbf{u}(m_1,m_2)$. Hence the pull-back is defined by

$$\psi^*(\mathrm{d}\mathbf{x}) = \frac{\partial \mathbf{x}}{\partial m_1}\mathrm{d}m_1 + \frac{\partial \mathbf{x}}{\partial m_2}\mathrm{d}m_2\,, \qquad \psi^*(\mathrm{d}\mathbf{u}) = \frac{\partial \mathbf{u}}{\partial m_1}\mathrm{d}m_1 + \frac{\partial \mathbf{u}}{\partial m_2}\mathrm{d}m_2\,,$$

with obvious generalization to higher dimension, and to other functions (cf. Cartan (1983, p. 28). The proof of (1.5) is therefore

$$\begin{aligned}
\psi^*\,\boldsymbol{\omega} &= \psi^*(\mathrm{d}\mathbf{u}\wedge\mathrm{d}\mathbf{x}) = \psi^*(\mathrm{d}\mathbf{u})\wedge\psi^*(\mathrm{d}\mathbf{x}) \\
&= \left(\frac{\partial \mathbf{u}}{\partial m_1}\mathrm{d}m_1 + \frac{\partial \mathbf{u}}{\partial m_2}\mathrm{d}m_2\right)\wedge\left(\frac{\partial \mathbf{x}}{\partial m_1}\mathrm{d}m_1 + \frac{\partial \mathbf{x}}{\partial m_2}\mathrm{d}m_2\right) \\
&= \sum_{i=1}^{2}\left\{\left(\frac{\partial u_i}{\partial m_1}\mathrm{d}m_1 + \frac{\partial u_i}{\partial m_2}\mathrm{d}m_2\right)\wedge\left(\frac{\partial x_i}{\partial m_1}\mathrm{d}m_1 + \frac{\partial x_i}{\partial m_2}\mathrm{d}m_2\right)\right\} \\
&= \Omega\, dm_1\wedge dm_2\,.
\end{aligned}$$

A similar argument applied to the shallow water equations in geophysical fluid dynamics shows that the pull-back of the symplectic form density there results in the potential vorticity. The details of this construction are given in §2.

While the above examples show a precise connection between vorticity or potential vorticity and symplecticity, there are several unanswered questions. The phase space variables $\mathbf{x}, \mathbf{u}$ depend on (m_1, m_2, t). Why pull back to the label space $\mathbf{m} = (m_1, m_2)$? Why not pull back to the full reference space (m_1, m_2, t)? Indeed, this more general pull-back leads to both vorticity conservation and energy conservation. Generalisations to $(\mathbf{m}, t) \in \mathbb{R}^4$ are also considered and lead to a symplectic derivation of Ertel's Theorem.

Further geometry is obtained by stripping away the integral in the 2-form which defines the symplectic structure, leading to a conservation law which can be generated on the reference space, a *conservation law for symplecticity*. This conservation law arises in a natural way when the equations governing Lagrangian fluid dynamics and geophysical fluid dynamics are formulated in a multisymplectic setting. In the multisymplectic setting, the fluxes of symplecticity arise as 2-forms. The symplectic 2-form then satisfies a conservation law of the form

$$\frac{\partial}{\partial t}\boldsymbol{\omega} + \frac{\partial}{\partial m_1}\boldsymbol{\kappa}_1 + \frac{\partial}{\partial m_2}\boldsymbol{\kappa}_2 = 0\,, \tag{1.6}$$

where $\boldsymbol{\kappa}_1$ and $\boldsymbol{\kappa}_2$ are a pair of 2-forms that represent the flux of symplecticity. Pulling back this equation to the label space, $\mathbf{m}$, leads to

$$\frac{\partial}{\partial t}\psi^*\boldsymbol{\omega} + \frac{\partial}{\partial m_1}\psi^*\boldsymbol{\kappa}_1 + \frac{\partial}{\partial m_2}\psi^*\boldsymbol{\kappa}_2 = 0\,, \tag{1.7}$$

where $\psi^*\boldsymbol{\kappa}_1$ and $\psi^*\boldsymbol{\kappa}_2$ are 2-forms on the label space that represent the flux of potential vorticity.

It is in the investigation of the fluxes in the symplectic conservation law that the importance of particle relabelling and the homentropic nature of the fluid arises. For example, this formulation leads to the result that potential vorticity (or vorticity) is conserved on particle paths if and only if the divergence of the pull-back of the flux 2-forms to the label space vanish identically, that is if

$$\mathrm{div}(\psi^*\boldsymbol{\kappa}_l, \psi^*\boldsymbol{\kappa}_2) = \boldsymbol{\nabla}_{\mathbf{m}} \cdot \psi^*\boldsymbol{\kappa} = 0\,, \qquad \boldsymbol{\kappa} = (\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2)^T\,. \tag{1.8}$$

We will show that a *sufficient* condition for the condition (1.8) to be satisfied is the the Hamiltonian function be invariant under particle relabelling or homentropic. Note that this derivation of PV conservation is fundamentally different from the ones typically given in fluid dynamics (see, for example, Salmon (1998, p. 302).

In §2, the Lagrangian form of the shallow water equations is treated in detail, and the following features are treated in turn: the encoding of PV in symplecticity, the multisymplectic structure that leads to a geometric form of PV conservation, the pull-back as a generator of PV and energy conservation, and the use of symmetry methods to establish rigorously the connection between particle relabelling and PV conservation. The ideas generalize in a straightforward way to other systems in Lagrangian fluid dynamics. We briefly consider the semi-geostrophic equations and three-dimensional Lagrangian incompressible fluid dynamics.

2. Multisymplectic GFD: shallow water equations

In this section, the shallow water equations will be considered in the Lagrangian particle-path representation. The connection between potential vorticity and symplecticity will be established, as will conditions for conservation of potential vorticity along particle paths, by constructing the flux 2-forms associated with conservation of symplecticity.

As in the introduction, let $\mathbf{x} \in \mathbb{R}^2$, $\mathbf{u} \in \mathbb{R}^2$ represent the position and velocity of the fluid particles, parametrized by Lagrangian mass coordinates $\mathbf{m} = (m_1, m_2)$, and let $h \in \mathbb{R}$ represent the fluid depth. The system is rotating at constant angular velocity $f/2$.

Henceforth, we use Latin indices to denote components of $\mathbf{x}$ and $\mathbf{u}$, and Greek indices to denote components of $\mathbf{m}$; the usual summation convention is adopted. Total derivatives with respect to t or m_α, for which all variables are regarded as functions of (m_1, m_2, t), are denoted by the subscript t or α after a comma. For example,

$$x_{i,\alpha} = \frac{\partial x_i(\mathbf{m}, t)}{\partial m_\alpha}, \qquad x_{i,\alpha t} = \frac{\partial^2 x_i(\mathbf{m}, t)}{\partial m_\alpha \partial t}, \qquad \text{etc.}$$

Partial derivatives with respect to any other variable are written in full.

Define the Jacobian matrix

$$\mathbf{x_m} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \qquad \text{where} \qquad x_{i\alpha} \equiv x_{i,\alpha}.$$

Conservation of mass reduces to $h\tau = 1$ where $\tau = \det(\mathbf{x_m}) = x_{11}x_{22} - x_{12}x_{21}$. The governing equations for $\mathbf{u}$ and $\mathbf{x}$ are

$$u_{1,t} = -(x_{22} \cdot p)_{,1} + (x_{21} \cdot p)_{,2} + f u_2, \tag{2.1}$$

$$u_{2,t} = (x_{12} \cdot p)_{,1} - (x_{11} \cdot p)_{,2} - f u_1, \tag{2.2}$$

$$x_{1,t} = u_1, \tag{2.3}$$

$$x_{2,t} = u_2. \tag{2.4}$$

The pressure is defined by $p = -e'(\tau)$ where $e(\tau)$ is the *specific internal energy*, in homentropic form. The classical shallow-water equations are recovered by taking $e(\tau) = g/(2\tau)$.

In terms of $e(\tau)$, the equations of motion amount to

$$u_{1,t} - f x_{2,t} = \left(\frac{\partial e(\tau)}{\partial x_{11}}\right)_{,1} + \left(\frac{\partial e(\tau)}{\partial x_{12}}\right)_{,2}, \tag{2.5}$$

$$u_{2,t} + f x_{1,t} = \left(\frac{\partial e(\tau)}{\partial x_{21}}\right)_{,1} + \left(\frac{\partial e(\tau)}{\partial x_{22}}\right)_{,2}, \tag{2.6}$$

$$x_{1,t} = u_1, \tag{2.7}$$

$$x_{2,t} = u_2. \tag{2.8}$$

This system has a Hamiltonian representation with canonical coordinates $(\mathbf{x}, \mathbf{u})$:

$$\begin{bmatrix} -f\mathbf{J} & -\mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{u} \end{pmatrix}_t = \begin{pmatrix} \delta H/\delta \mathbf{x} \\ \delta H/\delta \mathbf{u} \end{pmatrix}, \qquad \mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where

$$H(\mathbf{x},\mathbf{u}) = \int S\,\mathrm{d}\mathbf{m}\,, \qquad S = \tfrac{1}{2}\langle \mathbf{u},\mathbf{u}\rangle + e(\tau)\,. \tag{2.9}$$

The symplectic form for this system is

$$\overline{\boldsymbol{\omega}} = \int \boldsymbol{\omega}\,\mathrm{d}\mathbf{m}\,, \qquad \boldsymbol{\omega} = \mathrm{d}u_1\wedge\mathrm{d}x_1 + \mathrm{d}u_2\wedge\mathrm{d}x_2 + f\mathrm{d}x_1\wedge\mathrm{d}x_2\,.$$

We will prove that $\psi^*\boldsymbol{\omega} = q\,\mathrm{d}m_1\wedge\mathrm{d}m_2$, where q is the potential vorticity. However, we first recast the system into a multisymplectic formulation in order to deduce further information.

(*a*) *A multisymplectic formulation*

Introduce the additional variables

$$w_{i\alpha} \equiv -\frac{\partial S}{\partial x_{i\alpha}} = -\frac{\partial e(\tau)}{\partial x_{i\alpha}} = -e'(\tau)\frac{\partial \tau}{\partial x_{i\alpha}} = p(\tau)\frac{\partial \tau}{\partial x_{i\alpha}}\,,$$

and the extended set of dependent variables

$$\boldsymbol{z} = (x_1, x_2, u_1, u_2, w_{11}, w_{12}, w_{21}, w_{22}, x_{11}, x_{12}, x_{21}, x_{22})^T, \tag{2.10}$$

so that $\boldsymbol{z}\in\mathbb{R}^{12}$. Then (2.5)-(2.8) can be written as the following system of first-order PDEs:

$$\begin{aligned}
-u_{1,t} + fx_{2,t} - w_{11,1} - w_{12,2} &= 0,\\
-u_{2,t} - fx_{1,t} - w_{21,1} - w_{22,2} &= 0,\\
x_{i,t} &= u_i\,, && i=1,2,\\
x_{i,\alpha} &= x_{i\alpha}\,, && i=1,2, \quad \alpha=1,2,\\
0 &= \frac{\partial e(\tau)}{\partial x_{i\alpha}} + w_{i\alpha}\,, && i=1,2, \quad \alpha=1,2.
\end{aligned}$$

This system is of the form

$$\mathbf{K}\boldsymbol{z}_{,t} + \mathbf{L}\boldsymbol{z}_{,1} + \mathbf{M}\boldsymbol{z}_{,2} = \boldsymbol{\nabla}_{\boldsymbol{z}} S_M(\boldsymbol{z}), \tag{2.11}$$

where

$$S_M(\boldsymbol{z}) = S + x_{i\alpha}w_{i\alpha} = \tfrac{1}{2}\langle \mathbf{u},\mathbf{u}\rangle + e(\tau) + x_{i\alpha}w_{i\alpha}\,. \tag{2.12}$$

Here $\mathbf{K},\mathbf{L},\mathbf{M}\in\mathbb{R}^{12\times 12}$ are skew-symmetric matrices with nonzero entries

$$\begin{gathered}
K_{1,2}=f,\ \ K_{1,3}=-1,\ \ K_{2,1}=-f,\ \ K_{2,4}=-1,\ \ K_{3,1}=1,\ \ K_{4,2}=1,\\
L_{1,5}=-1,\ \ L_{2,7}=-1,\ \ L_{5,1}=1,\ \ L_{7,2}=1,\\
M_{1,6}=-1,\ \ M_{2,8}=-1,\ \ M_{6,1}=1,\ \ M_{8,2}=1.
\end{gathered}$$

A PDE of the form (2.11) for which $\mathbf{K}$, $\mathbf{L}$, $\mathbf{M}$ are skew-symmetric matrices is called multisymplectic, because there is a distinct symplectic operator associated with time and each label coordinate. See Bridges (1997a,1997b) for general properties of such systems.

Note that $S_M(\boldsymbol{z})$ in (2.12) is defined locally (for it involves no integrals), and it differs from the classical Hamiltonian function density S by the third term. The following identities hold:

$$x_{i\alpha}\frac{\partial \tau}{\partial x_{i\beta}} = \tau\delta_{\alpha\beta}\,, \qquad x_{i\alpha}\frac{\partial \tau}{\partial x_{j\alpha}} = \tau\delta_{ij}\,,$$

where δ is the Kronecker delta. Therefore

$$x_{i\alpha}w_{i\beta} = -\tau e'(\tau)\delta_{\alpha\beta}\,, \qquad x_{i\alpha}w_{j\alpha} = -\tau e'(\tau)\delta_{ij}\,, \tag{2.13}$$

and so the multisymplectic density S_M can be reduced to the form

$$S_M(\boldsymbol{z}) = \tfrac{1}{2}\langle \mathbf{u}, \mathbf{u}\rangle + e(\tau) - 2\tau e'(\tau). \tag{2.14}$$

However, in order for (2.11) to yield the multisymplectic shallow water equations, $S_M(\boldsymbol{z})$ must be expressed in terms of the multisymplectic coordinates, i. e. in the form (2.12).

The abstract formulation (2.11) gives rise to a conservation law of symplecticity (Bridges, 1997b)

$$(\mathrm{d}\boldsymbol{z}\wedge\mathbf{K}\mathrm{d}\boldsymbol{z})_{,t} + (\mathrm{d}\boldsymbol{z}\wedge\mathbf{L}\mathrm{d}\boldsymbol{z})_{,1} + (\mathrm{d}\boldsymbol{z}\wedge\mathbf{M}\mathrm{d}\boldsymbol{z})_{,2} = 0, \tag{2.15}$$

which, for the present system, amounts to

$$\boldsymbol{\omega}_{,t} + \boldsymbol{\kappa}_{\alpha,\alpha} = 0\,, \tag{2.16}$$

where

$$\boldsymbol{\omega} = \mathrm{d}u_i\wedge\mathrm{d}x_i + f\,\mathrm{d}x_1\wedge\mathrm{d}x_2 \quad \text{and} \quad \boldsymbol{\kappa}_\alpha = \mathrm{d}w_{i\alpha}\wedge\mathrm{d}x_i\,, \qquad \alpha = 1,2.$$

(*b*) *Pullback and the geometry of conservation laws*

In this section, we explore in detail the connection between conservation of symplecticity, the pull-back operation, and conservation laws, in the setting of the shallow water equations.

The potential vorticity for the shallow water system is

$$q = h^{-1}\left(f + \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) = f\tau + x_{i2}u_{i,1} - x_{i1}u_{i,2}\,,$$

where the second expression is in Lagrangian mass coordinates satisfying $\tau h = 1$. Let Ψ^* represent the pull-back operation from the phase space to the full reference space, $(\mathbf{m}, t)$. Application to the symplectic 2-form results in

$$\begin{aligned}
\Psi^*\boldsymbol{\omega} &= (u_{i,t}\,\mathrm{d}t + u_{i,\beta}\,\mathrm{d}m_\beta)\wedge(x_{i,t}\,\mathrm{d}t + x_{i\gamma}\,\mathrm{d}m_\gamma)\\
&+ f\,(x_{1,t}\,\mathrm{d}t + x_{1\beta}\,\mathrm{d}m_\beta)\wedge(x_{2,t}\,\mathrm{d}t + x_{2\gamma}\,\mathrm{d}m_\gamma)\\
&= \{x_{i,t}u_{i,\beta} - x_{i\beta}u_{i,t} + f(x_{1\beta}x_{2,t} - x_{1,t}x_{2\beta})\}\,\mathrm{d}m_\beta\wedge\mathrm{d}t\\
&\quad + \{x_{i2}u_{i,1} - x_{i1}u_{i,2} + f(x_{11}x_{22} - x_{12}x_{21})\}\,\mathrm{d}m_1\wedge\mathrm{d}m_2\\
&= \{x_{i,t}u_{i,\beta} - x_{i\beta}u_{i,t} + f(x_{1\beta}x_{2,t} - x_{1,t}x_{2\beta})\}\,\mathrm{d}m_\beta\wedge\mathrm{d}t + q\,\mathrm{d}m_1\wedge\mathrm{d}m_2\,,
\end{aligned}$$

and

$$\begin{aligned}\Psi^*\boldsymbol{\kappa}_\alpha &= (w_{i\alpha,t}\,\mathrm{d}t + w_{i\alpha,\beta}\,\mathrm{d}m_\beta)\wedge(x_{i,t}\,\mathrm{d}t + x_{i\gamma}\,\mathrm{d}m_\gamma)\\ &= \{x_{i,t}w_{i\alpha,\beta} - x_{i\beta}w_{i\alpha,t}\}\,\mathrm{d}m_\beta\wedge\mathrm{d}t + \{x_{i2}w_{i\alpha,1} - x_{i1}w_{i\alpha,2}\}\,\mathrm{d}m_1\wedge\mathrm{d}m_2\,.\end{aligned}$$

Evaluating $(\Psi^*\boldsymbol{\omega}, \Psi^*\boldsymbol{\kappa}_\alpha)$ on the set of solutions of the shallow water equations, and taking (2.13) into account, we finally obtain

$$\begin{aligned}\Psi^*\boldsymbol{\omega} &= \{u_iu_{i,\beta} + x_{i\beta}w_{i\gamma,\gamma}\}\,\mathrm{d}m_\beta\wedge\mathrm{d}t + q\,\mathrm{d}m_1\wedge\mathrm{d}m_2\\ &= \{u_iu_{i,\beta} + (x_{i\beta}w_{i\gamma})_{,\gamma} - x_{i,\beta\gamma}w_{i\gamma}\}\,\mathrm{d}m_\beta\wedge\mathrm{d}t + q\,\mathrm{d}m_1\wedge\mathrm{d}m_2\\ &= \left\{\tfrac{1}{2}\langle\mathbf{u},\mathbf{u}\rangle - \tau e'(\tau) + e(\tau)\right\}_{,\beta}\mathrm{d}m_\beta\wedge\mathrm{d}t + q\,\mathrm{d}m_1\wedge\mathrm{d}m_2\,,\end{aligned}$$

and

$$\begin{aligned}\Psi^*\boldsymbol{\kappa}_\alpha &= \{(u_iw_{i\alpha})_{,\beta} - (x_{i\beta}w_{i\alpha})_{,t}\}\,\mathrm{d}m_\beta\wedge\mathrm{d}t\\ &\qquad + \{(x_{i2}w_{i\alpha})_{,1} - (x_{i1}w_{i\alpha})_{,2}\}\,\mathrm{d}m_1\wedge\mathrm{d}m_2\\ &= \Big\{(u_iw_{i\alpha})_{,\beta} + \delta_{\alpha\beta}\left(\tau e'(\tau)\right)_{,t}\Big\}\,\mathrm{d}m_\beta\wedge\mathrm{d}t\\ &\qquad + \Big\{\delta_{\alpha1}\left(\tau e'(\tau)\right)_{,2} - \delta_{\alpha2}\left(\tau e'(\tau)\right)_{,1}\Big\}\,\mathrm{d}m_1\wedge\mathrm{d}m_2\,.\end{aligned}$$

The pull-back operator Ψ^* commutes with total differentiation with respect to t, m_1, or m_2. Consequently

$$\Psi^*\boldsymbol{\kappa}_{\alpha,\alpha} = \Big\{(u_iw_{i\alpha})_{,\alpha\beta} + \left(\tau e'(\tau)\right)_{,t\beta}\Big\}\,\mathrm{d}m_\beta\wedge\mathrm{d}t\,;$$

note that the $\mathrm{d}m_1\wedge\mathrm{d}m_2$ component is zero. Therefore the multisymplectic conservation law (2.16) is pulled back to

$$\begin{aligned}0 = \Psi^*(\boldsymbol{\omega}_{,t} + \boldsymbol{\kappa}_{\alpha,\alpha}) &= q_{,t}\,\mathrm{d}m_1\wedge\mathrm{d}m_2\\ &+ \Big\{\left(\tfrac{1}{2}\langle\mathbf{u},\mathbf{u}\rangle + e(\tau)\right)_{,t} + (u_iw_{i\alpha})_{,\alpha}\Big\}_{,\beta}\mathrm{d}m_\beta\wedge\mathrm{d}t\,.\end{aligned}$$

Separating this conservation law into its three components, we obtain

$$q_{,t} = 0, \tag{2.17}$$

$$\Big\{\left(\tfrac{1}{2}\langle\mathbf{u},\mathbf{u}\rangle + e(\tau)\right)_{,t} + (u_iw_{i\alpha})_{,\alpha}\Big\}_{,\beta} = 0, \qquad \beta = 1,2. \tag{2.18}$$

The $\mathrm{d}m_1\wedge\mathrm{d}m_2$ component (2.17) encapsulates the conservation of potential vorticity on particle paths. Note that the label-space divergence of the flux of potential vorticity is zero. In the next subsection, the geometry of this material conservation of PV will be examined in more detail.

The other conservation laws generated automatically by the pull-back in (2.18) are label-space derivatives of the local energy conservation law:

$$\left(\tfrac{1}{2}\langle\mathbf{u},\mathbf{u}\rangle + e(\tau)\right)_{,t} + (u_iw_{i\alpha})_{,\alpha} = 0\,. \tag{2.19}$$

This conservation law can be derived from (2.18), as we show in the next subsection. It implies that the total energy $\mathcal{E}$ is conserved, i.e.,

$$\partial_t\mathcal{E} = \partial_t\int \tfrac{1}{2}\langle\mathbf{u},\mathbf{u}\rangle + e(\tau)\,\mathrm{d}\mathbf{m} = 0,$$

provided that appropriate boundary conditions are given.

(c) *Symmetries and multisymplectic conservation laws*

If a system of equations stems from a variational principle, Noether's theorem links conservation laws with symmetries of the Lagrangian form (which are called variational symmetries). Noether's theorem has been extended to Hamiltonian systems (see OLVER (1993), and is applied to particle relabelling symmetries in PADHYE & MORRISON (1996), PADHYE (1998) and ALBERT (1997a,1997b). A restricted form of Noether's theorem can also be derived for multisymplectic systems (cf. BRIDGES, 1997a, App. C).

The shallow water equations (2.5)-(2.8) possess various point symmetries, which can be found systematically using the approach pioneered by Lie (see HYDON, 2000, for a simple introduction). For example, the shallow water equations are invariant under translations $\mathbf{x} \mapsto \mathbf{x} + \varepsilon$ for any $\varepsilon \in \mathbb{R}^2$; these symmetries lead to the unremarkable conservation laws (2.5) and (2.6).

The symmetries that lead to conservation of potential vorticity and total energy are particle relabelling and time-translation respectively. It can be shown that these are the only symmetries that have no effect upon (and are independent of) the variables $\mathbf{x}$ and $\mathbf{u}$. In this section, we demonstrate that these symmetries are encoded in the multisymplectic conservation law, explaining why the pull-back of this law produces conservation of total energy and potential vorticity.

A particle relabelling transformation is a diffeomorphism of the label space:

$$\Gamma : (m_1, m_2) \mapsto (\hat{m}_1(m_1, m_2),\, \hat{m}_2(m_1, m_2))\,.$$

Neither $\mathbf{x}$ nor $\mathbf{u}$ are affected by any such transformation. However, the Hamiltonian density S depends on an arbitrary function $e(\tau)$, so the transformation is a symmetry for every $e(\tau)$ only if τ is invariant. (It is easy to verify that this necessary condition is also sufficient.) By definition,

$$\tau = \det(\mathbf{x}_{\mathbf{m}}) = \det(\mathbf{x}_{\hat{\mathbf{m}}}) \det(\hat{\mathbf{m}}_{\mathbf{m}}) \quad \text{where} \quad \det(\hat{\mathbf{m}}_{\mathbf{m}}) = \hat{m}_{1,1}\hat{m}_{2,2} - \hat{m}_{1,2}\hat{m}_{2,1}\,.$$

Therefore τ is invariant if and only if

$$\det(\hat{\mathbf{m}}_{\mathbf{m}}) = 1, \tag{2.20}$$

so that Γ is an area-preserving transformation. The set of all particle relabelling symmetries is an infinite-dimensional Lie (pseudo-) group. Every one-parameter Lie subgroup of this group has a generator of the form

$$X = \phi_{,2}\frac{\partial}{\partial m_1} - \phi_{,1}\frac{\partial}{\partial m_2}\,,$$

for some smooth function $\phi(\mathbf{m})$. A function $F(\mathbf{m}, t, \mathbf{x}, \mathbf{u})$ is invariant under the subgroup generated by X if and only if $XF = 0$. If F also depends upon derivatives of $\mathbf{x}$ and $\mathbf{u}$, the same condition for invariance applies once X has been prolonged to take account of the group action on derivatives. For example, τ is a function of the first derivatives $x_{i\alpha}$. The prolongation of the generator to these derivatives is

$$X^{(1)} = \phi_{,2}\frac{\partial}{\partial m_1} - \phi_{,1}\frac{\partial}{\partial m_2} + \{x_{i2}\phi_{,1\alpha} - x_{i1}\phi_{,2\alpha}\}\frac{\partial}{\partial x_{i\alpha}}\,, \tag{2.21}$$

and therefore

$$\begin{aligned} X^{(1)}\tau &= x_{22}(x_{12}\phi_{,11} - x_{11}\phi_{,21}) - x_{21}(x_{12}\phi_{,12} - x_{11}\phi_{,22}) \\ &\quad -x_{12}(x_{22}\phi_{,11} - x_{21}\phi_{,21}) + x_{11}(x_{22}\phi_{,12} - x_{21}\phi_{,22}) = 0\,, \end{aligned}$$

which is to be expected, because we obtained the relabelling symmetries by requiring that τ is invariant.

Before we examine the link between the particle relabelling symmetries and conservation of potential vorticity within the multisymplectic framework, it is instructive to seek generalizations of the Hamiltonian density S that retain the particle relabelling symmetries. We restrict attention to densities of the form

$$\tilde{S} = \tilde{S}(t, \mathbf{x}, \mathbf{u}, \mathbf{x_m}),$$

so that the resulting Hamiltonian system can be recast as a multisymplectic system, using the matrices $\mathbf{K}, \mathbf{L}, \mathbf{M}$ that were defined in §2(a). Consequently, the conservation law (2.16) holds for the same 2-forms $\boldsymbol{\omega}$ and $\boldsymbol{\kappa}_\alpha$ that were used earlier, bearing in mind that the auxiliary variables $w_{i\alpha}$ are now defined as follows:

$$w_{i\alpha} = -\frac{\partial \tilde{S}}{\partial x_{i\alpha}}\,.$$

If $\tilde{S}$ has no explicit dependence on t, it is invariant under translations in time. The density $\tilde{S}$ is invariant under the particle relabelling symmetries generated by X if and only if

$$X^{(1)}\tilde{S} \equiv \left(X^{(1)}t\right)\frac{\partial \tilde{S}}{\partial t} + \left(X^{(1)}x_i\right)\frac{\partial \tilde{S}}{\partial x_i} + \left(X^{(1)}u_i\right)\frac{\partial \tilde{S}}{\partial u_i} + \left(X^{(1)}x_{i\alpha}\right)\frac{\partial \tilde{S}}{\partial x_{i\alpha}} = 0.$$

But

$$X^{(1)}t = X^{(1)}x_i = X^{(1)}u_i = 0,$$

so $\tilde{S}$ can depend arbitrarily on t, $\mathbf{x}$, and $\mathbf{u}$. Given any smooth function $F(\mathbf{x_m})$,

$$\begin{aligned} X^{(1)}F(\mathbf{x_m}) &= \{x_{i2}\phi_{,1\alpha} - x_{i1}\phi_{,2\alpha}\}\frac{\partial F(\mathbf{x_m})}{\partial x_{i\alpha}} \\ &= \phi_{,11}x_{i2}\frac{\partial F(\mathbf{x_m})}{\partial x_{i1}} + \phi_{,12}\left(x_{i2}\frac{\partial F(\mathbf{x_m})}{\partial x_{i2}} - x_{i1}\frac{\partial F(\mathbf{x_m})}{\partial x_{i1}}\right) \\ &\quad -\phi_{,22}x_{i1}\frac{\partial F(\mathbf{x_m})}{\partial x_{i2}}\,, \end{aligned}$$

so for $F(\mathbf{x_m})$ to be invariant under arbitrary particle relabelling transformations, the following conditions must hold:

$$x_{i2}\frac{\partial F(\mathbf{x_m})}{\partial x_{i1}} = 0, \qquad x_{i2}\frac{\partial F(\mathbf{x_m})}{\partial x_{i2}} - x_{i1}\frac{\partial F(\mathbf{x_m})}{\partial x_{i1}} = 0, \qquad x_{i1}\frac{\partial F(\mathbf{x_m})}{\partial x_{i2}} = 0.$$

By using the method of characteristics, it is easy to show that F is invariant if and only if it is a function of τ alone. Therefore $\tilde{S}$ admits the group of particle relabelling symmetries provided that it is a function of t, $\mathbf{x}, \mathbf{u}$, and τ only. Note that this result is based solely on mathematical considerations. There may be good

physical reasons for restricting the class of allowable functions still further. For example, the governing equations include

$$x_{i,t} = \frac{\partial \tilde{S}}{\partial u_i}\,,$$

so if $\mathbf{u}$ is to represent the particle velocity, the most general form that $\tilde{S}$ can take is

$$\tilde{S} = \tfrac{1}{2}\langle \mathbf{u}, \mathbf{u}\rangle + \sigma(t, \mathbf{x}, \tau)\,,$$

for some function σ.

Our main observation is that *whatever functional form* $\tilde{S}(t, \mathbf{x}, \mathbf{u}, \tau)$ *takes, potential vorticity is conserved on the solutions of the system* (2.11), where now

$$S_M = \tilde{S} + x_{i\alpha} w_{i\alpha}\,.$$

This PV conservation can be seen by substituting (2.11) into

$$\begin{aligned}
\Psi^*\boldsymbol{\omega} &= \{x_{i,t}u_{i,\beta} - x_{i\beta}u_{i,t} + f(x_{1\beta}x_{2,t} - x_{1,t}x_{2\beta})\}\ \mathrm{d}m_\beta \wedge \mathrm{d}t + q\,\mathrm{d}m_1 \wedge \mathrm{d}m_2\,,\\
\Psi^*\boldsymbol{\kappa}_\alpha &= \{x_{i,t}w_{i\alpha,\beta} - x_{i\beta}w_{i\alpha,t}\}\ \mathrm{d}m_\beta \wedge \mathrm{d}t + \{x_{i2}w_{i\alpha,1} - x_{i1}w_{i\alpha,2}\}\ \mathrm{d}m_1 \wedge \mathrm{d}m_2\,,
\end{aligned}$$

and rearranging terms in the same way as in the previous section. After a slightly messy calculation, we obtain

$$\Psi^*\boldsymbol{\omega} = \left\{\tilde{S} - \tau\frac{\partial \tilde{S}}{\partial \tau}\right\}_{,\beta} \mathrm{d}m_\beta \wedge \mathrm{d}t + q\,\mathrm{d}m_1 \wedge \mathrm{d}m_2\,,$$

$$\begin{aligned}
\Psi^*\boldsymbol{\kappa}_\alpha &= \left\{(u_i w_{i\alpha})_{,\beta} + \delta_{\alpha\beta}\left(\tau\frac{\partial \tilde{S}}{\partial \tau}\right)_{,t}\right\}\mathrm{d}m_\beta \wedge \mathrm{d}t\\
&\quad + \left\{\delta_{\alpha 1}\left(\tau\frac{\partial \tilde{S}}{\partial \tau}\right)_{,2} - \delta_{\alpha 2}\left(\tau\frac{\partial \tilde{S}}{\partial \tau}\right)_{,1}\right\}\mathrm{d}m_1 \wedge \mathrm{d}m_2\,.
\end{aligned}$$

Therefore the pull-back of the multisymplectic conservation law is

$$\Psi^*(\boldsymbol{\omega}_{,t} + \boldsymbol{\kappa}_{\alpha,\alpha}) = \left\{\tilde{S}_{,t} + (u_i w_{i\alpha})_{,\alpha}\right\}_{,\beta} \mathrm{d}m_\beta \wedge \mathrm{d}t + q_{,t}\,\mathrm{d}m_1 \wedge \mathrm{d}m_2 = 0.$$

Hence if the Hamiltonian density is invariant under arbitrary area-preserving particle relabelling transformations, potential vorticity is conserved. Furthermore, the remaining components of the conservation law lead to the result

$$\Psi^*\left(\tilde{S}_{,t} + (u_i w_{i\alpha})_{,\alpha}\right) = f(t),$$

for some function f. From the equations of motion, we obtain

$$f(t) = \Psi^*\left(\frac{\partial \tilde{S}}{\partial t}\right),$$

so $f(t) = 0$ if $\tilde{S}$ is invariant under translations in time. For the shallow water equations, this leads to the local energy conservation law (2.19).

How is the conservation of potential vorticity encoded in the multisymplectic 2-forms? To answer this question, it is sufficient to restrict attention to the $\mathrm{d}m_1 \wedge \mathrm{d}m_2$ component. First note that

$$\boldsymbol{\omega} = \mathrm{d}u_i \wedge \mathrm{d}x_i + f\,\mathrm{d}x_1 \wedge \mathrm{d}x_2$$

is invariant under particle relabelling symmetries, because x_i and u_i are invariant and the exterior derivative d is essentially coordinate-invariant. Neither of the flux 2-forms $\boldsymbol{\kappa}_\alpha$ is invariant; however, we have shown that if the Hamiltonian density is invariant under particle relabelling then the $\mathrm{d}m_1 \wedge \mathrm{d}m_2$ component of $\Psi^* \boldsymbol{\kappa}_{\alpha,\alpha}$ vanishes. Finally, the pull-back of $\boldsymbol{\omega}$ to label space is the product of the potential vorticity and the 2-form $\mathrm{d}m_1 \wedge \mathrm{d}m_2$ (which is itself invariant under the relabelling symmetries). Consequently $q_{,t} = 0$.

3. Symplecticity and the geometry of vorticity for other models

In this section, we briefly describe the implications of the constructions in §2 based on conservation of symplecticity for other models in Lagrangian fluid dynamics. For three-dimensional Lagrangian fluid dynamics the main result is that the pull-back to the reference space of the symplectic form leads to Ertel's Theorem.

(*a*) *Semi-geostrophic shallow water equations*

In the semi-geostrophic approximation (cf. Salmon (1988), Roulstone & Norbury (1994), Roulstone & Sewell (1997)), the shallow water equations (2.5)-(2.8) are replaced by

$$u_{1,t} - f x_{2,t} = \left(\frac{\partial e(\tau)}{\partial x_{11}}\right)_{,1} + \left(\frac{\partial e(\tau)}{\partial x_{12}}\right)_{,2}, \tag{3.1}$$

$$u_{2,t} + f x_{1,t} = \left(\frac{\partial e(\tau)}{\partial x_{21}}\right)_{,1} + \left(\frac{\partial e(\tau)}{\partial x_{22}}\right)_{,2}, \tag{3.2}$$

$$0 = f u_1 - \left(\frac{\partial e(\tau)}{\partial x_{21}}\right)_{,1} - \left(\frac{\partial e(\tau)}{\partial x_{22}}\right)_{,2}, \tag{3.3}$$

$$0 = f u_2 + \left(\frac{\partial e(\tau)}{\partial x_{11}}\right)_{,1} + \left(\frac{\partial e(\tau)}{\partial x_{12}}\right)_{,2}, \tag{3.4}$$

where

$$e(\tau) = \frac{g}{2\tau}. \tag{3.5}$$

Here $\mathbf{u}$ is the geostrophic velocity, not the particle velocity (which is $\mathbf{x}_{,t}$). This system of equations is equivalent to the following Hamiltonian system:

$$\begin{bmatrix} -f\mathbf{J} & -\mathbf{I} \\ \mathbf{I} & -f^{-1}\mathbf{J} \end{bmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{u} \end{pmatrix}_t = \begin{pmatrix} \delta H/\delta \mathbf{x} \\ \delta H/\delta \mathbf{u} \end{pmatrix}, \qquad \mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{3.6}$$

where

$$H(\mathbf{x},\mathbf{u}) = \int S\,\mathrm{d}\mathbf{m}\,, \qquad S = \tfrac{1}{2}\langle \mathbf{u},\mathbf{u}\rangle + e(\tau)\,. \tag{3.7}$$

The skew-symmetric operator on the left-hand side of (3.6) is not invertible. This degeneracy reflects the constraints in the semi-geostrophic formulation. In other words, the gradient of H is required to be in the range of the skew-symmetric operator; this solvability condition imposes the constraints. A closed but degenerate two-form is called a *pre-symplectic form*, and for this system the pre-symplectic form is

$$\overline{\boldsymbol{\omega}} = \int \boldsymbol{\omega}\,\mathrm{d}\mathbf{m}\,, \quad \boldsymbol{\omega} = \mathrm{d}u_1 \wedge \mathrm{d}x_1 + \mathrm{d}u_2 \wedge \mathrm{d}x_2 + f\mathrm{d}x_1 \wedge \mathrm{d}x_2 + f^{-1}\mathrm{d}u_1 \wedge \mathrm{d}u_2\,.$$

This two-form differs from the symplectic form for the standard shallow-water equations by the last term only. In the multisymplectic formulation, the analysis of §2 goes through with the following minor changes.

1. The skew-symmetric matrix $\boldsymbol{K}$ has two extra nonzero components:

$$K_{3,4} = f^{-1}, \qquad K_{4,3} = -f^{-1}.$$

2. The potential vorticity is

$$q = f^{-1}\left\{ (u_1 - fx_2)_{,1}\,(u_2 + fx_1)_{,2} \;-\; (u_1 - fx_2)_{,2}\,(u_2 + fx_1)_{,1} \right\}.$$

   (As before, the potential vorticity is the $\mathrm{d}m_1 \wedge \mathrm{d}m_2$ component of $\Psi^*\boldsymbol{\omega}$; the corresponding component of $\Psi^*\boldsymbol{\kappa}_{\alpha,\alpha}$ is zero on solutions, so q is conserved on particle paths.)

3. The energy conservation law, which can be deduced from the $\mathrm{d}m_\alpha \wedge \mathrm{d}t$ components of the pull-back of the symplectic conservation law, is

$$S_{,t} + \left(x_{i,t} w_{i\alpha}\right)_{,\alpha} = 0.$$

(*b*) *Three-dimensional ideal fluid dynamics*

The Euler equations for incompressible fluid dynamics in three dimensions, written in terms of the Lagrangian mass coordinates $\mathbf{m} = (m_1, m_2, m_3)$, are

$$x_{i,t} = u_i\,, \qquad x_{i,\alpha} u_{i,t} = -p_{,\alpha}\,, \qquad \det(\mathbf{x}_\mathbf{m}) = 1.$$

Here $\mathbf{x}_\mathbf{m}$ is the Jacobian matrix, so the incompressibility condition amounts to

$$\tau \equiv \varepsilon_{\alpha\beta\gamma}\, x_{1\alpha}\, x_{2\beta}\, x_{3\gamma} = 1,$$

where $\varepsilon_{\alpha\beta\gamma}$ is the alternating tensor and $x_{i\alpha} = x_{i,\alpha}$.

The Euler equations have the same Hamiltonian formulation (1.3) in two and three dimensions. From (1.2), the Hamiltonian density is

$$S = \tfrac{1}{2}\langle \mathbf{u},\mathbf{u}\rangle + (1 - \det(\mathbf{x}_\mathbf{m}))\,p\,.$$

As usual, we construct the multisymplectic formulation by defining the auxiliary variables

$$w_{i\alpha} = -\frac{\partial S}{\partial x_{i\alpha}}\,.$$

Then the Euler equations can be rewritten as

$$\begin{aligned} -u_{i,t} - w_{i\alpha,\alpha} &= 0, \\ x_{i,t} &= u_i\,, \\ x_{i,\alpha} &= x_{i\alpha}\,, \\ 0 &= \frac{\partial S}{\partial x_{i\alpha}} + w_{i\alpha}\,, \\ 0 &= 1-\tau. \end{aligned}$$

Let $\mathbf{z} = (x_i, u_i, w_{i\alpha}, x_{i\alpha}, p)^{\mathrm{T}}$, where the variables are ordered by the value of i (and, where there are two subscripts, by the value of α for each i). Then the Euler equations amount to

$$\mathbf{K}\boldsymbol{z}_{,t} + \mathbf{L}^{\alpha}\boldsymbol{z}_{,\alpha} = \boldsymbol{\nabla}_{\boldsymbol{z}} S_M(\boldsymbol{z}),$$

where $\mathbf{K}, \mathbf{L}^1, \mathbf{L}^2, \mathbf{L}^3$ are skew-symmetric matrices whose only nonzero components are

$$\begin{aligned} K_{1,4} &= K_{2,5} = K_{3,6} = -1, & K_{4,1} &= K_{5,2} = K_{6,3} = 1, \\ L^1_{1,7} &= L^1_{2,10} = L^1_{3,13} = -1, & L^1_{7,1} &= L^1_{10,2} = L^1_{13,3} = 1, \\ L^2_{1,8} &= L^2_{2,11} = L^2_{3,14} = -1, & L^2_{8,1} &= L^2_{11,2} = L^2_{14,3} = 1, \\ L^3_{1,9} &= L^3_{2,12} = L^3_{3,15} = -1, & L^3_{9,1} &= L^3_{12,2} = L^3_{15,3} = 1, \end{aligned}$$

and where $S_M = S + x_{i\alpha}w_{i\alpha}$. The conservation law of symplecticity is

$$\boldsymbol{\omega}_{,t} + \boldsymbol{\kappa}_{\alpha,\alpha} = 0,$$

where

$$\boldsymbol{\omega} = \mathrm{d}\boldsymbol{z} \wedge \mathbf{K}\mathrm{d}\boldsymbol{z} = \mathrm{d}u_i \wedge \mathrm{d}x_i \quad \text{and} \quad \boldsymbol{\kappa}_{\alpha} = \mathrm{d}\boldsymbol{z} \wedge \mathbf{L}^{\alpha}\mathrm{d}\boldsymbol{z} = \mathrm{d}w_{i\alpha} \wedge \mathrm{d}x_i\,.$$

The pull-back of the multisymplectic conservation law to $(\mathbf{m}, t)$ space is

$$\begin{aligned} 0 = \Psi^*\left(\boldsymbol{\omega}_{,t} + \boldsymbol{\kappa}_{\alpha,\alpha}\right) &= \varepsilon_{\alpha\beta\gamma}\,\Omega_{\gamma,t}\,\mathrm{d}m_\alpha \wedge \mathrm{d}m_\beta \\ &+ \left\{\left(\tfrac{1}{2}\langle\mathbf{u},\mathbf{u}\rangle\right)_{,t} + (u_i w_{i\beta})_{,\beta}\right\}_{,\alpha} \mathrm{d}m_\alpha \wedge \mathrm{d}t\,, \end{aligned}$$

where

$$\Omega_\gamma = \varepsilon_{\lambda\mu\gamma}\,x_{i\mu}u_{i,\lambda} = \varepsilon_{\lambda\mu\gamma}\,\left(x_{i\mu}u_i\right)_{,\lambda}\,. \tag{3.8}$$

(This result derived in essentially the same way as the conservation laws in §2; details are omitted for the sake of brevity.) Thus the pull-back leads to conservation of total energy,

$$\left(\tfrac{1}{2}\langle\mathbf{u},\mathbf{u}\rangle\right)_{,t} + (u_i w_{i\beta})_{,\beta} = 0,$$

and conservation of $\boldsymbol{\Omega} = (\Omega_1, \Omega_2, \Omega_3)^{\mathrm{T}}$:

$$\boldsymbol{\Omega}_{,t} = 0\,.$$

Now, from (3.8),

$$\Omega_\gamma = \varepsilon_{\lambda\mu\gamma}\, x_{i\mu} x_{j\lambda} \frac{\partial u_i}{\partial x_j},$$

which can be written in the form

$$\mathbf{\Omega} = -\mathbf{x}_\mathbf{m}^{-1}(\boldsymbol{\nabla}_\mathbf{x} \times \mathbf{u}). \tag{3.9}$$

In contrast to corresponding result for the two-dimensional Euler equations (cf. §1), $\mathbf{\Omega}$ is not the vorticity. Ertel's theorem (cf. MÜLLER 1995, SALMON 1998) is an immediate consequence of (3.9). Let θ be any materially conserved quantity, so that $\theta_{,t} = 0$. Then the vector product

$$Q = \Omega_\gamma \theta_{,\gamma}$$

is also materially conserved: $Q_{,t} = 0$. Unlike $\mathbf{\Omega}$, the quantity Q has a representation in terms of the Eulerian variables $\mathbf{x}$ and $\mathbf{u}$ only:

$$Q = -\langle \boldsymbol{\nabla}_\mathbf{x}\theta, \boldsymbol{\nabla}_\mathbf{x} \times \mathbf{u}\rangle.$$

For compressible homentropic fluids, the only significant modification is to replace (3.9) by $\mathbf{\Omega} = -\tau \mathbf{x}_\mathbf{m}^{-1}(\boldsymbol{\nabla}_\mathbf{x} \times \mathbf{u})$; then

$$Q = \frac{-\langle \boldsymbol{\nabla}_\mathbf{x}\theta, (\boldsymbol{\nabla}_\mathbf{x} \times \mathbf{u})\rangle}{\rho},$$

where $\rho = 1/\tau$ the density of the fluid.

References

Albert, H.-F. 1997a *On new hydrodynamic conservation laws related to the particle relabeling symmetry*, Z. angew. Math. Mech. **77** 945–954.

Albert, H.-F. 1997b *Point symmetries for Lagrangian fluid dynamics*, Z. angew. Math. Mech. **77** 955–958.

Batchelor, G.K. 1967 *An Introduction to Fluid Dynamics*, Cambridge University Press.

Bridges, T.J. 1997a *Multi-symplectic structures and wave propagation*, Math. Proc. Camb. Phil. Soc. **121** 147–190.

Bridges, T.J. 1997b *A geometric formulation of the conservation of wave action and its implications for signature and the classification of instabilities*, Proc. Roy. Soc. Lond. A **453** 1365–1395.

Cartan, H. 1983 *Differential Forms*, Hermann Publishers, Paris.

Hydon, P.E. 2000 *Symmetry Methods for Differential Equations*, Cambridge University Press, Cambridge.

Müller, P. 1995 *Ertel's potential vorticity theorem in physical oceanography*, Rev. Geophysics **33** 67–97.

Olver, P.J. 1993 *Applications of Lie Groups to Differential Equations*, 2nd edition, Springer-Verlag, New York.

Padhye, N.S. 1998 *Topics in Lagrangian and Hamiltonian fluid dynamics: relabeling symmetry and ion-acoustic wave stability*, PhD Dissertation, University of Texas.

Padhye, N.S. & Morrison, P.J. 1996 *Fluid element relabeling symmetry* Physics Letters A **219** 287–292.

Roulstone, I. & Norbury, J. 1994 *A Hamiltonian structure with contact geometry for the semi-geostrophic equations*, J. Fluid Mech. **272** 211-233.

Roulstone, I. & Sewell, M.J. 1997 *The mathematical structure of theories of semi-geostrophic type*, Phil. Trans. R. Soc. Lond. A **355** 2489–2517.

Salmon, R. 1988 *Semi-geostrophic theory as a Dirac bracket projection*, J. Fluid Mech. **196** 345–358.

Salmon, R. 1998 *Lectures on Geophysical Fluid Dynamics*, Oxford University Press, Oxford.