Transmission line equations as Hamiltonian partial differential equations: theory and implications for circuit simulation

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Transmission line equations as Hamiltonian partial differential equations: theory and implications for circuit simulation

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Zusammenfassung. Eine Vielzahl physikalischer Vorgänge kann durch Hamiltonsche Bewegungsgleichungen beschrieben werden. Ein wesentliches Merkmal solcher Prozesse ist die Energieerhaltung. Es zeigt sich, daß die numerische Simulation konservativer Systeme speziell angepaßte Techniken erfordert. In dieser Arbeit wird dies am Beispiel der Leitungsgleichung demonstriert. Dabei wird auch auf die Eigenschaften verschiedener Leitungsersatzschaltungen eingegangen.

Abstract. A variety of physical processes can be modeled by Hamiltonian dynamical systems. Such processes are, among others, characterized by conservation of energy. It can be shown that this requires special techniques for the numerical simulation. In this note, we demonstrate this for the transmission line equations. Our discussion will also shed some new light on various lumped transmission line models as used in circuit simulations.

1 Introduction

Deterministic dynamical processes can be divided into those that conserve an energy function (conservative systems) and those that do not (dissipative systems). An example of the later category is provided by electronic circuits containing resistors, while Newton's equations describing the planetary motion belong to the realm of conservative systems. It turns out that most conservative systems fit into the framework of Hamiltonian mechanics, i.e., are of the following structure: A Hamiltonian dynamical system consists of a triple $(M, \{\cdot,\cdot\}, H)$ where the phase space M is a smooth manifold (linear or nonlinear, finite or infinite dimensional), $H : M \to R$ is the Hamiltonian function (or energy function), and $\{\cdot,\cdot\}$ is a Poisson bracket, a bilinear, skew-adjoint operator satisfying the Jacobi identity and the Leibnitz rule. (Olver (1986) or Marsden & Ratiu (1994) is a good introduction to the structure of Hamiltonian ODEs' and PDE's.) In finite dimensions, the bracket can be written as

$$\{F,G\}(x) = (\nabla F(x))^T J(x)(\nabla G(x)) \tag{1}$$

where $F,G : M \to R$, $x \in M$, and J is called the Poisson tensor. Using this terminology, the equations of motion are now given by

$$\frac{d}{dt}x = \{\mathrm{Id}, H\}(x) = J(x)\nabla H(x)\,, \tag{2}$$

Id the identity operator, i.e., $\mathrm{Id}(x) = x$.

Example. Consider the motion of a planet in a central field (Kepler problem). In polar coordinates, the time evolution of the radius r is given by the second order differential equation

$$\frac{d^2}{dt^2}r = \frac{M^2}{r^3} - \frac{m}{r^2}\,, \tag{3}$$

m the mass of the planet and M the magnitude of the perpendicular momentum. Upon introducing the conjugate momentum $p = mdr/dt$, this equation can be recasted as a Hamiltonian dynamical system

$$\frac{d}{dt}r = p/m \tag{4}$$

$$\frac{d}{dt}p = \frac{M^2}{mr^3} - \frac{1}{r^2} \tag{5}$$

with energy function

$$E(r,p) = \frac{p^2}{2m} + \frac{M^2}{2mr^2} - \frac{1}{r} \tag{6}$$

and structure matrix

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{7}$$

$\diamond$

Besides conserving energy, Hamiltonian systems possess a rich geometric structure. For example, the solutions of a canonical Hamiltonian system, i.e., a system with an even dimensional phase space $M = R^{2n}$, $n \geq 1$, and structure matrix

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}, \tag{8}$$

preserve volume in phase space. This is a consequence of the fact that the flow (or time-t-map) $\phi_t : M \to M$ of a canonical Hamiltonian system is a Poisson map, i.e., satisfies

$$(D_x\phi_t(x))J(D_x\phi_t(x))^T = J \tag{9}$$

for all t and all $x \in M$. Recent results on the numerical integration of Hamiltonian systems suggest (see Sanz-Serna & Calvo (1994) for an overview) that it is advantageous to to use a so called symplectic integrator. Let

$$x_{n+1} = \psi_{\Delta t}(x_n)\,, \tag{10}$$

denote a one-step method of order $p \geq 1$ and step-size Δt, i.e., $\phi_{t=\Delta t}(x) - \psi_{\Delta t}(x) = \mathcal{O}(\Delta t^{p+1})$. A symplectic (or Poisson) integrator is one for which the time step $\psi_{\Delta t}$ is a Poisson map. It can be shown that symplectic integrators not only conserve volume but also automatically preserve energy over exponentially long periods of time, i.e.,

$$|E(t_n) - E(0)| \;=\; O(\Delta t) \quad \text{for} \quad t_n \leq c_1 e^{c_2/\Delta t}\,, \tag{11}$$

$\Delta t > 0$ the step-size, c_1, c_2 appropriate constants, $t_n = n\Delta t$.

Example. Let us return to the Kepler problem. We compare the numerical discretization of the equations (4)–(5) by the implicit Euler method

$$r_{n+1} \;=\; r_n + \Delta t p_{n+1}/m \tag{12}$$

$$p_{n+1} \;=\; p_n + \Delta t \frac{M^2}{m r_{n+1}^3} - \Delta t \frac{1}{r_{n+1}^2} \tag{13}$$

and the following modified Euler method

$$r_{n+1} \;=\; r_n + \Delta t p_n/m \tag{14}$$

$$p_{n+1} \;=\; p_n + \Delta t \frac{M^2}{m r_{n+1}^3} - \Delta t \frac{1}{r_{n+1}^2} \tag{15}$$

which is a combination of explicit and implicit Euler and which turns out to be symplectic. The evolution of energy over time can be found in Fig. 1 ($M = m = 1$, $\Delta t = 0.05$, $(r(0), p(0)) = (2, 0)$). Obviously, the implicit Euler method "dissipates" energy while the symplectic Euler method preserves energy up to small fluctuations. $\diamond$

Matter becomes more delicate when one starts looking at Hamiltonian PDEs such as, e.g., the nonlinear wave equation (see the following section). Now a spatial discretization has to be found that maintains the Hamiltonian structure of the problem, i.e., the finite dimensional truncation should again be Hamiltonian. Some of the involved problems and implications will be discussed in Sec. 3 for the transmission line equations. This will also shed some light on the question which of the ladder circuits might be more appropriate as a circuit model of transmission lines (see Entenmann & Nossek (1991) and Reibiger (1995)). Before, we will provide a quick primer into Hamiltonian PDEs in Sec. 2.

2 Hamiltonian PDEs

For Hamiltonian PDEs, the phase space M typically consists of sets of smooth functions on a finite dimensional space Z, e.g., with periodic boundary condi-

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Fig. 1. Drift in energy for the implicit Euler method (solid line) and the symplectic Euler method (dash-dotted line)

tions, i.e., $Z = T^n$. We denote elements of M by $w : Z \to R^m$, $x \in Z = T^n$. The Hamiltonian $H : M \to R$ is a functional on M, and the Poisson bracket can be written as

$$\{F, G\}(w) \quad = \quad \int_Z \frac{\delta F}{\delta w} J \frac{\delta G}{\delta w} dx \tag{16}$$

where $\delta F / \delta w$ is the variational, or Gateaux, derivative defined by

$$\left(\frac{d}{d\epsilon} F(w + \epsilon v) \right)_{\epsilon = 0} \quad = \quad \int_Z \frac{\delta F}{\delta w} v dx\,. \tag{17}$$

We consider here only the case of constant J (the only requirement on J is then skew-adjointness).

Example. For the nonlinear wave equation, we have $w(x) = (q(x), p(x)) \in R^2$, $x \in S^1$,

$$J = \left[\begin{array}{cc} 0 & 1 \\ -1 & 1 \end{array} \right], \tag{18}$$

and the Hamiltonian

$$H(q, p) \quad = \quad \frac{1}{2} \int \left(p^2 + q_x^2 + V(q) \right) \, dx\,. \tag{19}$$

The equations of motion are

$$\partial_t q \quad = \quad p \tag{20}$$

$$\partial_t p \quad = \quad \partial_{xx} q - V'(q)\,. \tag{21}$$

$\diamond$

Example. The Korteweg-de Vries equation: J is the Gardner-Zajharov-Faddeev operator ∂_x and $H = \int(-w^3 + 1/2w_x^2)\,dx$:

$$\partial_t w \quad = \quad \partial_x(-3w^2 - w_{xx})\,. \tag{22}$$

$\diamond$

Example. The charge driven Maxwell equations:

$$\partial_t \mathbf{E} \quad = \quad +\mathrm{curl}\,\mathbf{B}\,, \tag{23}$$
$$\partial_t \mathbf{B} \quad = \quad -\mathrm{curl}\,\mathbf{E} \tag{24}$$

restricted to

$$\mathrm{div}\,\mathbf{E} \quad = \quad \rho\,, \tag{25}$$
$$\mathrm{div}\,\mathbf{B} \quad = \quad 0\,. \tag{26}$$

Here the Hamiltonian is

$$H(\mathbf{E},\mathbf{B}) \quad = \quad \frac{1}{2}\int \left(||\mathbf{E}||^2 + ||\mathbf{B}||^2\right)\, d^3x \tag{27}$$

and the Pauli-Born-Infeld bracket (see Pauli (1933/1981), Born & Infeld (1935), and Marsden & Ratiu (1994)) is given by

$$\{F,G\} \quad = \quad \int \left(\frac{\delta F}{\delta \mathbf{E}} \cdot \mathrm{curl}\,\frac{\delta G}{\delta \mathbf{B}} - \frac{\delta F}{\delta \mathbf{B}} \cdot \mathrm{curl}\,\frac{\delta G}{\delta \mathbf{E}}\right)\, d^3x \tag{28}$$

which yields the structure operator

$$J \quad = \quad \left[\begin{array}{cc} 0 & \mathrm{curl} \\ -\mathrm{curl} & 0 \end{array}\right]. \tag{29}$$

Note that (provided we assume periodic boundary conditions) the phase space M is given by $M = \{(\mathbf{E},\mathbf{B}) \in C^r(T^3, R^3 \times R^3) : \mathrm{div}\,\mathbf{B} = 0,\, \mathrm{div}\,\mathbf{E} = \rho\}$. $\diamond$

Let us consider now the finite dimensional truncation of Hamiltonian PDEs by finite differences. In a first step, the infinite dimensional phase space M is replaced by a finite dimensional space $\mathcal{M} = R^N$ with $N \gg 1$ appropriately chosen. Based on this, we need a finite dimensional approximation of the energy functional H and a finite dimensional truncation $\mathcal{J}$ of the operator J compatible with the phase space $\mathcal{M}$. The important point is that one has to do all this such that the truncated system is still Hamiltonian. Let us demonstrate this for the Korteweg-de Vries equation:

Let D be a finite difference approximation of ∂_x, i.e.

$$D\mathbf{w} \quad \approx \quad \partial_x w \tag{30}$$

where $\mathbf{w} \in \mathcal{M} = R^N$ such that its components w_i, $i = 1, \ldots, N$, satisfy $w_i = w(x_i)$, $\{x_i\}$ equidistributed mesh points in $Z = S^1$. Then the energy functional H becomes

$$\mathcal{H} \quad = \quad \sum_{i=1}^{N} w_i^3 + \frac{1}{2}\mathbf{w}^T D^T D\mathbf{w}\,. \tag{31}$$

Next we have to find an approximation to $J = \partial_x$ that maintains skew-adjointness. This is, for example, satisfied for the central difference approximation of ∂_x. Let us denote the corresponding operator by

$$D_1 \quad = \quad \frac{1}{2\Delta x}\begin{bmatrix} 0 & -1 & & & & & 1 \\ \cdot & \cdot & \cdot & & & & \\ & \cdot & \cdot & \cdot & & & \\ & & 1 & 0 & -1 & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & \cdot & \cdot & \cdot \\ -1 & & & & & 1 & 0 \end{bmatrix}, \tag{32}$$

Δx the mesh-size of the equidistributed grid $\{x_i\}$. Then $\mathcal{J} = D_1$ with $\mathcal{J} = -\mathcal{J}^T$ and a finite dimensional Hamiltonian truncation of the Korteweg-de Vries equation is given by

$$\frac{d}{dt}\mathbf{w} \quad = \quad D_1\left(\sum_{i=1}^{N} 3w_i^2\mathbf{e}_i + D_1^T D_1\mathbf{w}\right) \tag{33}$$

where $\mathbf{e}_i$ is the i-th basis vector in $\mathcal{M} = R^N$. For a throughout discussion of the numerical discretization of Hamiltonian PDEs see McLachlan (1994).

3 Transmission line equations

The fundamental differential equations for the analysis of the lossness transmission line is given by

$$\partial_t i \quad = \quad -\frac{1}{L}\partial_x u \tag{34}$$

$$\partial_t u \quad = \quad -\frac{1}{C}\partial_x i \tag{35}$$

These equations are known as the transmission line equations (in the older literature also as the telegraphist's equations). Here L is the distributed inductance per unit length and C is the distributed capacitance per unit length. Eqs. (34)–(35) can be combined to form a (linear) wave equation for either of the variables u or i. For example, one can differentiate (34) partially with respect to x and (35) with respect to t. Eliminating partial derivatives with respect to i, this yields

$$\partial_{tt} u \quad = \quad c^2 \partial_{xx} u \tag{36}$$

where

$$c = (LC)^{-1/2} \tag{37}$$

is the speed of light.

To close the transmission line equations, one has to impose appropriate boundary conditions at $x = 0$ and $x = L$, L the length of the transmission line. For simplicity, we assume here periodic boundary conditions, i.e.,

$$u(\cdot, 0) \; = \; u(\cdot, L) \quad \text{and} \quad i(\cdot, 0) \; = \; i(\cdot, L)\,. \tag{38}$$

The transmission line equations can now be provided with a Hamiltonian structure. Namely: The phase space M is given by $M = C^r(S^1, R^2)$, the space of r-times differentiable functions from the unit circle (assuming, for simplicity, $L = 1$) into R^2. The energy functional is

$$H(u,i) \;\; = \;\; \int \left(\frac{1}{2L} u^2 + \frac{1}{2C} i^2 \right) \, dx \tag{39}$$

and the Poisson bracket is defined by

$$\{F, G\} \;\; = \;\; \int \left(-\frac{\partial F}{\partial u} \partial_x \frac{\partial G}{\partial i} - \frac{\partial F}{\partial i} \partial_x \frac{\partial G}{\partial u} \right) \, dx\,, \tag{40}$$

i.e.,

$$J \;\; = \;\; \left[\begin{array}{cc} 0 & -\partial_x \\ -\partial_x & 0 \end{array} \right]. \tag{41}$$

Let us now discuss the finite dimensional truncation of the transmission line equations. We follow the notations of Sec. 2. The finite dimensional phase space $\mathcal{M}$ is given by $\mathcal{M} = R^N \times R^N$ and the energy functional becomes

$$\mathcal{H} \;\; = \;\; \sum_{i=1}^{N} \left(\frac{1}{2L} u_i^2 + \frac{1}{2C} i_i^2 \right). \tag{42}$$

The truncation of the operator J is a bit more tricky. As stressed before, we have to maintain skew-adjointness. Take any finite difference approximation of ∂_x. Denote the corresponding operator by D. Then the adjoint operator $-D^T$ is also an approximation to ∂_x. This suggests to approximate J by

$$\mathcal{J} \;\; = \;\; \left[\begin{array}{cc} 0 & -D \\ D^T & 0 \end{array} \right] \tag{43}$$

and $\mathcal{J} = -\mathcal{J}^T$ is automatically satisfied. For the central difference approximation (32) of ∂_x, we even have $D_1 = -D_1^T$, and $\mathcal{J}$ simplifies to

$$\mathcal{J} \;\; = \;\; \left[\begin{array}{cc} 0 & -D_1 \\ -D_1 & 0 \end{array} \right]. \tag{44}$$

A non-skew-adjoint discretization of ∂_x is given by

$$D_2 = \frac{1}{\Delta x}\begin{bmatrix} 1 & -1 & & & & & 0 \\ & \ddots & \ddots & & & & \\ & & \ddots & \ddots & & & \\ & & & 0 & 1 & -1 & \\ & & & & \ddots & \ddots & \\ & & & & & \ddots & \ddots \\ -1 & & & & & 0 & 1 \end{bmatrix}. \tag{45}$$

Note that this discretization together with (43) preserves the Hamiltonian character of the transmission line equations. But it leads to different approximations of the corresponding linear wave equations in the variables u and i and thus eliminates the symmetry between the two variables. Note also that

$$\mathcal{J} = \begin{bmatrix} 0 & -D_2 \\ -D_2 & 0 \end{bmatrix} \tag{46}$$

destroys the Hamiltonian character of the transmission line equations ($\mathcal{J}$ is not skew-adjoint) and the energy function $\mathcal{H}$ is no longer a first integral of the corresponding truncated equations.

Entenmann & Nossek (1991) and Paul et al. (1992) also considered the discretization of the transmission line equations by the trapezoidal rule. This yields

$$A\frac{d}{dt}\mathbf{i} = -\frac{1}{L}D_2\mathbf{u}\,, \tag{47}$$

$$A\frac{d}{dt}\mathbf{u} = -\frac{1}{C}D_2\mathbf{i} \tag{48}$$

with D_2 as in (45) and A given by

$$A = \frac{1}{2}\begin{bmatrix} 1 & 1 & & & & & 0 \\ & \ddots & \ddots & & & & \\ & & \ddots & \ddots & & & \\ & & & 0 & 1 & 1 & \\ & & & & \ddots & \ddots & \\ & & & & & \ddots & \ddots \\ 1 & & & & & 0 & 1 \end{bmatrix}. \tag{49}$$

Upon introducing $D_3 := A^{-1}D_2$, this can be rewritten as

$$\frac{d}{dt}\mathbf{i} = -\frac{1}{L}D_3\mathbf{u}\,, \tag{50}$$

$$\frac{d}{dt}\mathbf{u} = -\frac{1}{C}D_3\mathbf{i}\,. \tag{51}$$

Note that D_3 is skew-adjoint and thus leads to a Hamiltonian discretization but, unlike the skew-adjoint central difference operator D_1, D_3 is no longer a banded matrix.

Some of the finite dimensional truncations can also be recasted in form of LC-ladder/lattice networks. While D_2 in (43) leads to the classical low-pass LC-ladder network, the trapazoidal rule, i.e. D_3 in (43), results in a symmetric all-pass LC-lattice network (Entenmann & Nossek (1991) and Paul et al. (1992)). We are not aware of any LC-ladder/lattice network interpretation of the central difference approximation D_1 in (44) (D_1 does not correspond to a one-step discretization).

In a circuit simulator, LC-ladder/lattice networks should be simulated using a symplectic integrator such as the symplectic Euler method or the implicit midpoint rule (see Sanz-Serna & Calvo (1994)).

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References

Born, M. and Infeld, L., On the quantization of the new field theory, *Proc. Roy. Soc. London, A 150*, 141, 1935.

Entenmann, W. and Nossek, J. A., Zur Problematik der Leitungsersatzschaltungen, in: *ITG-Diskussionssitzung Entwicklung von Analogschaltungen mit CAE-Methoden*, Paderborn, 1991.

Marsden, J. E. and Ratiu, T. S., *Introduction to mechanics and symmetry*. Springer-Verlag, New York, 1994.

McLachlan, R. I., Symplectic integration of Hamiltonian wave equations, *Numer. Math.*, *66*, 465–492, 1994.

Olver, P. J., *Applications of Lie groups to differential equations*. Springer-Verlag, New York, 1988.

Paul, S., Hüper, K., Entenmann, W., and Nossek, J.A., Modelling Nonlinear Wave Equations by Transmission Lines, Report No. TUM-LNS-TR-92-3, Technical University Munich, 1992.

Pauli, W., *General principles of quantum mechanics*. Reprinted in English translation by Springer-Verlag, New York, 1933/1981.

Reibiger, A., Zur Netzwerkinterpretation von Randwerkaufgaben der Elektrotechnik, *Kleinheubacher Berichte*, *38*, 327–335, 1994.

Sanz-Serna, J. M. and Calvo, M. P., *Numerical Hamiltonian Problems*. Chapman & Hall, London, 1994.