BIRATIONAL GEOMETRY OF TERMINAL QUARTIC 3-FOLDS. I

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1. Introduction

1.1. **Abstract.** In this paper, we study the birational geometry of certain examples of mildly singular quartic 3-folds. A quartic 3-fold is a special case of a Fano variety, that is, a variety X with ample anticanonical sheaf $\mathcal{O}_X(-K_X)$. Nonsingular Fano 3-folds have been studied quite extensively. From the point of view of birational geometry they basically fall within two classes: either X is "close to being rational", and then it has very many biregularly distinct birational models as a Fano 3-fold, or, at the other extreme, X has a unique model and it is often even true that every birational selfmap of X is biregular. In this paper we construct examples of singular quartic 3-folds with *exactly two* birational models as Fano 3-folds; the other model is a complete intersection $Y_{3,4} \subset \mathbb{P}(1,1,1,1,2,2)$ of a quartic and a cubic in weighted projective space $\mathbb{P}(1^4, 2^2)$. These are the first examples to show this type of behavior. After a brief introduction to singularities and Fano 3-folds, we give a first precise statement of our main theorem. The rest of the introduction is not logically necessary to understand the results or their statements. We describe the more general context of Fano 3-folds and Mori fiber spaces, and we outline a program of research on 3-folds which brings together birational geometry, classification theory and commutative algebra.

1.2. **Singular quartic 3-folds.** We study quartic 3-folds having a unique singular point $P \in X$ analytically equivalent to $xy+z^3+t^3=0$. Choosing coordinates $x_0, ...x_4$ in $\mathbb{P}^4$ such that P is the point $(1,0,...,0)$, we may write the equation of X as

$$F = x_0^2 x_1 x_2 + x_0 a_3 + b_4$$

where a_3, b_4 are homogeneous forms of the indicated degree in the variables $x_1, ..., x_4$. We also assume that a_3, b_4 are general, where we take "general" in the sense of "outside a Zariski closed set" in moduli. Two properties of X are crucial to us.

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The first is that the singularity $xy+z^3+t^3=0$ is *terminal.* We need not enter into the precise details of the definition; the point of terminal singularities is that certain manipulations with the canonical class and discrepancy, familiar from the nonsingular case, still hold. The most important examples are isolated hypersurface singularities of the form $xy+f(z,t)=0$, and quotients of $\mathbb{C}^3$ by the diagonal action of $\mathbb{Z}/r\mathbb{Z}$ with weights $(1,a,-a)$ (when a is coprime with r). The reader can look into [YPG] for an accessible introduction to terminal singularities.

The second property is that X is $\mathbb{Q}$-factorial. This is an important and subtle (and much misunderstood) concept of Mori theory. It means by definition that every Weil divisor of X is $\mathbb{Q}$-Cartier; this is a property of the Zariski topology of X and not of the analytic type of its singularities. If X has hypersurface terminal singularities, X is $\mathbb{Q}$-factorial if and only if it is factorial, that is all its (Zariski) local rings are UFDs. In the case of a Fano 3-fold X, it is easy to see that $\mathbb{Q}$-factorial is equivalent to $\dim H^2(X)=\dim H_4(X)$, i.e., in the case of a quartic 3-fold with terminal singularities, the 4th integral homology group $H_4(X,\mathbb{Z})$ is generated by the class of a hyperplane section. This property is often tricky to verify in practice; see below for additional comments on this. For a very gentle introduction to Mori theory, we recommend the Foreword to [CR].

1.3. **Fano 3-folds.** A quartic 3-fold is a particular case of a Fano 3-fold. There are 16 deformation families of nonsingular Fano 3-folds with $\dim H^2=1$; with only one exception they were known classically. Most of these varieties can be shown to be rational, and sometimes unirational, by means of classical constructions.

Mori theory requires that we look at Fano 3-folds with terminal singularities with $\dim H^2=1$ and $\mathbb{Q}$-factorial; this is indeed one of the possible end products of the minimal model program. In this paper, we use the terminology "Fano 3-fold" in this more general sense implied by Mori theory.

There is at present no complete classification of Fano 3-folds, but several hundred families are known, see [IF] and Altınok [Al] for some lists. For instance, there are 95 families of Fano 3-fold weighted hypersurfaces; it is conjectured in [CPR], and proved in many cases, that every quasismooth member of one of these families is the unique model as a Fano 3-fold in its birational class. There are also 85 families of Fano 3-fold codimension 2 weighted complete intersections of which $Y_{3,4}$ is an example [IF], 69 codimension 3 Pfaffians and more than 100 codimension 4 families [Al].

Not only there are many more singular Fano 3-folds than nonsingular ones; they have a much richer variety of behavior from the point of view of birational geometry. In this paper we begin to study some of the simplest new phenomena.

1.4. **Main result, crude statement.** This is our main result.

Theorem 1.1. *Let $X = X_4 \subset \mathbb{P}^4$ be a quartic 3-fold, with a singularity $P \in X$ analytically equivalent to $xy + z^3 + t^3 = 0$, but otherwise general (in particular, nonsingular outside P). Then*

1. *Let Y be a Fano 3-fold (according to our conventions, this includes that Y has terminal singularities, is $\mathbb{Q}$-factorial and has $\dim H^2X = 1$). If Y is birational to X, then* either*:*
   (a) *$Y \cong X$ is biregular to X,* or
   (b) *$Y = Y_{3,4} \subset \mathbb{P}(1,1,1,1,2,2)$ is a quasismooth complete intersection of a quartic and a cubic in weighted projective space $\mathbb{P}(1^4, 2^2)$.*
2. *Let Y be a 3-fold admitting either a morphism $Y \to S$ to a curve S with typical fibre a rational surface, or $Y \to S$ to a surface S with typical fibre $\mathbb{P}^1$. Then Y is not birational to X.*

We could make the generality requirements on X explicit if we wanted; the precise conditions are $2.2(a-b)$, and the condition (a) at the end of § 6 (and 7) "the discriminant surface contains no lines". Note that X is indeed factorial: even though the *analytic* singularity $xy+z^3+t^3 = 0$ has a nontrivial class group (it is equivalent to $xy + tz(t + z) = 0$, so that $x = t = 0$, for example, is not locally the divisor of a function), nevertheless when it appears on X it is *algebraically* factorial.

In Section 2 we exhibit an explicit birational map of X_4 to a quasismooth $Y_{3,4} \subset \mathbb{P}(1^4, 2^2)$, the complete intersection of a cubic and a quartic in the weighted 5-dimensional projective space $\mathbb{P}(1, 1, 1, 1, 2, 2)$. A striking feature of this construction is that $Y_{3,4}$ is general; in fact any quasismooth $Y_{3,4}$ is birational to a quartic X_4 of the special kind considered.

In the rest of the introduction, we outline a program of research in birational geometry generalizing the results of this paper in various directions.

1.5. **Birational geometry, classification theory and commutative algebra.** It is not difficult, and fun, for someone experienced in the use of the known lists of Fano 3-folds, and aware of certain constructions of graded rings, to generate many examples of birational maps between Fano 3-folds. In particular many Fano 3-fold codimension 2 weighted complete intersections are birational to special Fano

3-fold hypersurfaces, and we conjecture that a statement analogous to our main theorem holds for a lot of them. To name but a few, a general $Y_{6,7} \subset \mathbb{P}(1,1,2,3,3,4)$ is birational to a special $X_7 \subset \mathbb{P}(1,1,1,2,3)$ with a singular point $y^2 + z^2 + x_1^6 + x_2^6$, a general $Y_{14,15} \subset \mathbb{P}(1,2,5,6,7,9)$ is birational to a special $X_{15} \subset \mathbb{P}(1,1,2,5,7)$ with a singular point $u^2 + z^2 y + y^7 + x^{14}$, etc. It is remarkable that a significant part of the list of Fano weighted complete intersections can be generated in this way, starting from singular hypersurfaces.

This is a fairly general phenomenon. When trying to classify Fano 3-folds, the problem is often to construct a variety Y with a given Hilbert function. Usually Y has high codimension; in the absence of a structure theory of Gorenstein rings, one method to construct Y starts by studying a suitable projection $Y \dashrightarrow X$ to a Fano X in smaller codimension (the work of Fano, and then Iskovskikh, is an example of this). The classification of Fano 3-folds involves the study of the geometry of special members of some families (like our special singular quartics), as well as general members of more complicated families (like our codimension 2 complete intersections); the two points of view match like the pieces of a gigantic jigsaw puzzle.

Miles Reid calls the map $X \dashrightarrow Y$ an "unprojection". Given X satisfying certain properties, the problem is to construct Y. There are at present a handful known constructions of this kind, leading to formats for codimension 4 Gorenstein rings which are often a good substitute for the still missing general structure theory.

The ideas here are due to Miles Reid, see for example [R]; for these and other issues not touched upon in this introduction, we also refer to [R2].

There is here an interplay of different problem areas: methods of unprojection were first discovered in birational geometry [CPR] 7.3, then applied to the construction of Fano 3-folds [Al], and formats of Gorenstein rings [Pa]. Commutative algebra in turn clarifies birational geometry.

1.6. **The Sarkisov category.**

Definition 1.2.
1. The *Sarkisov category* is the category whose objects are Mori fibre spaces and whose morphisms birational maps (regardless of the fibre structure).
2. Let $X \to S$ and $X' \to S'$ be Mori fibre spaces. A morphism in the Sarkisov category, that is, a birational map $f\colon X \dashrightarrow X'$, is

square if it fits into a commutative square

$$\begin{tikzcd} X \arrow[r, dashed, "f"] \arrow[d] & X' \arrow[d] \\ S \arrow[r, dashed, "g"] & S' \end{tikzcd}$$

where g is a birational map (which thus identifies the function field L of S with that of S') and, in addition, the induced birational map of generic fibers $f_L\colon X_L \dashrightarrow X'_L$ is biregular. In this case, we say that $X \to S$ and $X' \to S'$ are *square birational*, or *square equivalent*.

3. A *Sarkisov isomorphism* is a birational map $f\colon X \dashrightarrow X'$ which is biregular and square.
4. If X is an algebraic variety, we define the *pliability* of X to be the set

$$\mathcal{P}(X) = \bigl\{\text{Mfs } Y \to T \mid Y \text{ is birational to } X\bigr\}/\text{square equivalence}.$$

We say that X is *birationally rigid* if $\mathcal{P}(X)$ consists of one element.

To say that X is rigid means that it has an essentially unique model as a Mori fibre space; this implies in particular that X is nonrational, but it is much more precise than that. For example, it is known that a nonsingular quartic 3-fold is birationally rigid [IM], [Pu2], [Co2]; a quartic 3-fold which is nonsingular except for a single ordinary node is also birationally rigid [Pu], [Co2].

1.7. **The main result restated.** We restate our main theorem in the language of the Sarkisov category:

Theorem 1.3. *Let $X = X_4 \subset \mathbb{P}^4$ be a quartic 3-fold, with a singularity $P \in X$ analytically equivalent to $xy + z^3 + t^3 = 0$, but otherwise general (in particular X is nonsingular outside P). Then $\mathcal{P}(X)$ consists of two elements.*

1.8. **Strict Mori fibre spaces.** Our focus in this paper is Fano 3-folds. It is natural, and eventually necessary, to study similar problems in the more general context of strict Mori fibre spaces. For example Grinenko [Gr] looks at a "double singular quadric", i.e., Z is a special $Z_{2,4} \subset \mathbb{P}(1^5, 2)$, where the degree 2 equation is the cone $x_1x_2 + x_3x_4 = 0$ over a 2-dimensional quadric. Here Z has two fibre structures of Del Pezzo surfaces of degree 2, corresponding to the rulings of the (2-dimensional) quadric, and it is shown that $\mathcal{P}(Z)$ consists of these two Mfs.

Similar examples are a general $Y_{3,3} \subset \mathbb{P}(1^5, 2)$ birational to a cubic Del Pezzo fibration, but also a general $Y_{4,4} \subset \mathbb{P}(1^3, 2^3)$ birational to a Del Pezzo fibration of degree 2, a general $Y_{6,6} \subset \mathbb{P}(1, 1, 2, 2, 3, 4)$ birational to a Del Pezzo fibration of degree 1, etc. We may expect that these and many similar examples will be studied extensively in the near future.

1.9. **Pliability and rationality.** Traditionally, we like to think of Fano 3-folds as being "close to rational". We are now confronted with a view of 3-fold birational geometry of great richness, on a scale much larger than accessible with the methods of calculation and theoretical framework prior to Mori theory.

The notion of pliability is more flexible; a case division in terms of the various possibilities for $\mathcal{P}(X)$ allows to individuate a wider spectrum of behavior ranging from birationally rigid to rational.

1.10. **Our starting point.** Our starting point and eventual goal is a uniform study of the pliability of (singular) quartic 3-folds. We hope that we will soon be able to settle the following

Conjecture 1.4. *Let $X = X_4 \subset \mathbb{P}^4$ be a quartic 3-fold satisfying the following conditions*

1. *X has isolated singular points, that are locally analytically of the form $x^2 + y^2 + z^2 + t^n = 0$, for some positive integer n (dependent on the point in question),*
2. *X is factorial (this is equivalent to $\mathbb{Q}$-factorial, X is a hypersurface).*

Then X is birationally rigid.

Note the most striking feature of this conjecture: we do not restrict the number of singular points, although we insist that X must be factorial. We have seen that this is a subtle condition equivalent to $\dim H_4 = 1$ i.e. the integral homology group $H_4(X, \mathbb{Z})$ is generated by the class of a hyperplane section.

For example, if X has an ordinary node as its unique singular point, then X is automatically factorial. On the other hand, if X has only ordinary nodes as singularities, X is $\mathbb{Q}$-factorial if and only if the nodes impose independent linear conditions on cubics. Indeed if $\tilde{X}$ is the blowup of the nodes, $H^1(\tilde{X}, \Omega^2_{\tilde{X}})$ arises from residuation of 3-forms

$$\frac{P\Omega}{F^2}$$

on $\mathbb{P}^4$ with a pole of order two along $X = \{F = 0\}$. Here $\Omega = \sum x_i\, dx_0 \cdots \widehat{dx_i} \cdots dx_4$ and P is a cubic containing all the nodes of X, see e.g. [Cl].

Consider a quartic 3-fold Z containing the plane $x_0 = x_1 = 0$. The equation of Z can be written in the form $x_0a_3 + x_1b_3 = 0$ and, in general, Z has 9 ordinary nodes $x_0 = x_1 = a_3 = b_3 = 0$. The linear system $|a_3, b_3|$ defines a map to $\mathbb{P}^1$; blowing up the base locus gives a Mori fibre space $Z \to \mathbb{P}^1$ with fibers cubic Del Pezzo surfaces. However, our conjecture does not apply to Z. Indeed Z is not $\mathbb{Q}$-factorial: the plane $\{x_0 = x_1 = 0\} \subset Z$ is not a Cartier divisor. Thus Z is not a Mori fibre space; it doesn't even make sense to say that it is rigid. (Note in passing: introducing the ratio $y = a_3/x_1 = b_3/x_0$, gives a birational map $Z \dashrightarrow Y_{3,3} \subset \mathbb{P}(1^5, 2)$ to a Fano 3-fold $Y_{3,3}$, the complete intersection of two cubics in $\mathbb{P}(1^5, 2)$, a Mori fibre space birational to Z. In the language of the Sarkisov program, Z is the *midpoint* of a *link* $X \dashrightarrow Y_{3,3}$.)

However, a quartic 3-fold with 9 nodes is factorial in general, and our conjecture predicts that then it is birationally rigid.

The factoriality of projective hypersurfaces is the subject of a lovely paper by C. Ciliberto and V. Di Gennaro [CDG].

1.11. **Pliability and deformations.** We hope that the notion of pliability will be helpful in other ways too. It is not known how rationality and, especially, unirationality behave in families. It is suspected that rationality is not stable under deformations. For example, a general nonsingular quartic 4-fold X in $\mathbb{P}^5$ is probably nonrational, whereas X is rational if it contains two skew planes. The situation is even worse with unirationality, because there is no known way (other than trivial reasons) to show that a variety is not unirational; every judgment about the behavior of this notion is at present pure guesswork. We can hope that $\mathcal{P}(X)$ is often a finite union of algebraic varieties, and that it is reasonably well behaved under deformations, for example, that it is an upper semi-continuous function of X. For example, consider again a quartic X containing a plane; then X has 9 nodes but it is not factorial. It is impossible to deform X to a quartic X' with 9 nodes in general position (by what has been said, $\dim H_4X = 2$ and $\dim H_4X' = 1$, hence X is not diffeomorphic to X'). This seems to suggest that birationally rigid is quite robust under deformations.

The crucial point here is not so much whether this is literally true or not; indeed it would be foolish to try to legislate over a large body of still largely unexplored examples. The point is that we now have the technology to test these ideas on substantial examples.

1.12. **Mori theory grows down.** Mori theory has enjoyed an initial phase of tremendous abstract development, which continues today with higher dimensional flips and abundance. Our main interest instead lies in the program of explicit birational geometry of 3-folds as described in the Foreword to [CR]. The aim is to treat 3-folds as explicitly as possible. An important step in this development is to work out explicit description for the steps of the minimal model program, divisorial contractions and flips, and the links of the Sarkisov program.

1.13. **Extremal divisorial contractions.** An extremal divisorial contraction is a birational extremal contraction $f\colon E \subset Z \to P \in X$, in the Mori category, which contracts a divisor E. This is the 3-fold analogue of the contraction of a -1 curve on a surface. Despite some remarkable recent progress in special cases [Kw], we still don't know an explicit classification of 3-fold extremal divisorial contractions. We prove in Section 3 that, if $P \in X$ is the singularity $xy + z^3 + t^3 = 0$, and f contracts E to the origin, then f is the weighted blowup with weights $(2, 1, 1, 1)$ or $(1, 2, 1, 1)$. This classification is a key point in the proof of the main theorem. In the proof we combine, and refine, two methods which have been instrumental in the solution of similar problems. One is the connectedness theorem of Shokurov, which is the key ingredient in the classification of divisorial contractions in the case $P \in X$ is the ordinary node $xy + zt = 0$ [Co2]. The other is a fairly simple minded multiplicity calculation which is sufficient to establish the case $P \in X$ a terminal quotient singularity $1/r(a, -a, 1)$ [Ka]. Our proof is not easy and presupposes a good understanding of these other (simpler) cases. We advise the reader who is not an expert to study the simpler cases first, or else just skim through the proof.

1.14. **Methods.** Our proof uses the Sarkisov program [Co1] and builds on and refines the methods of [CPR], [Co2]. The refinements are quite subtle and we expect that the nonspecialist reader will find it difficult to follow the details of the proof. Because of this, we have made an effort to keep most of the discussion accessible to a general audience. We hope to have successfully swept the most technical parts of the proof under the final two Sections 6 and 7. These demand a great deal of motivation, and experience, on the part of the reader.

In short, the new elements are

- the partial classification of divisorial contractions, which improves on previous work, as already explained;
- a use of test surfaces and especially inequalities arising from the theory of log surfaces, in combination with Shokurov's "inversion

of adjunction", more efficient than for example in the proof of rigidity of a general $Y_{2,3} \subset \mathbb{P}^5$ given in [Co2];

- the general organization of the exclusion of curves as centers in Section 7, which circumvents the need of having to treat a large number of particular cases.

1.15. **Organization and contents.** The paper is organized as follows. In Section 2 we construct the birational map $X_4 \dashrightarrow Y_{3,4}$. We explain the construction at length within a narrative context and in much greater generality than needed for the treatment of our example. Our aim is to equip the reader with the knowledge to do many calculations of this type.

Section 3 is a short survey of what is known about the classification of 3-fold divisorial contractions. The material here is interesting in its own right; we tried to be as self-contained as possible, except for the proof of our own Theorem 3.8.

In Section 4 we outline the proof of the main theorem. This is copied from [CPR] 3.1, 3.2, 3.3, with very few words changed.

In Section 5 we survey the technique for excluding maximal centers, and indicate our own improvements for later use; in the final Sections 6 and 7 we get down to the gruesome details of the classification of maximal singularities on X_4 and $Y_{3,4}$, namely we show that, other than those untwisted in Section 2, there are no other maximal singularities. The results here combine to give a proof of the main theorem.

1.16. **Acknowledgements.** We like to thank Miles Reid for following this project with interest and a huge amount of advise and teaching. In particular he helped us to understand the algebra of our map $X_4 \dashrightarrow Y_{3,4}$. We also like to thank János Kollár for persuading us to rewrite the introduction, and the referee for reading the manuscript very thoroughly.

2. Birational maps

In this section we explain several constructions of birational maps. First of all we recall the definition of link of the Sarkisov program, in a context which is sufficient for our purposes. We continue with an informal discussion leading to a link $X_4 \dashrightarrow Y_{3,4}$; this is how we first discovered it. Then we give a much shorter, more efficient construction, which generalizes to many other cases. This corresponds to "Type I" of [CPR] and was explained to us by M. Reid. Finally, if $P \in L \subset X$ is a line on X, we construct a link $X \dashrightarrow X$ centered on L; this presents some topical intricacies which we treat quickly, because they are very similar to "Type II" of [CPR], 4.11 and 7.3.

2.1. **Links.**

Definition 2.1. A *Sarkisov link of Type II* between two Fano 3-folds X and Y is a birational map $f\colon X \dashrightarrow Y$ that factorizes as

$$\begin{tikzcd} V \arrow[r, dashed] \arrow[d] & V' \arrow[d] \\ X \arrow[r, dashed, "f"] & Y \end{tikzcd}$$

where

(a) $V \to X$ and $V' \to Y$ are extremal divisorial contractions in the Mori category, and
(b) $V \dashrightarrow V'$ is a composite of inverse flips, flops and flips (in that order), and in particular, is an isomorphism in codimension 1.

Usually (always in this paper) the map $V \dashrightarrow V'$ is a flop (flip, inverse flip):

$$\begin{tikzcd} & V \arrow[rr, dashed] \arrow[dl] \arrow[dr] & & V' \arrow[dl] \arrow[dr] & \\ X & & Z & & Y \end{tikzcd}$$

In this case we say that Z is the *midpoint* of the link

2.2. **Constructions, first approach.** Fix a quartic 3-fold $X = X_4 \subset \mathbb{P}^4$ with a singular point $P \in X$, locally analytically equivalent to the origin in the hypersurface

$$\{xy + z^3 + t^3 = 0\} \subset \mathbb{C}^3$$

Changing coordinates, we may then write the equation of X as:

$$F = x_0^2 x_1 x_2 + x_0 a_3 + b_4 = 0 \tag{1}$$

where a_3, b_4 are homogeneous polynomials of degrees $3, 4$ in the variables x_1, x_2, x_3, x_4. In what follows we always assume that X is general, in the sense that

(a) X has only one singular point $P = (1 : 0 : 0 : 0 : 0)$,
(b) $a_3(0, 0, x_3, x_4) = b_4(0, 0, x_3, x_4) = 0$ only if $x_3 = x_4 = 0$ (this is a genericity condition involving the lines $P \in L \subset X$ passing through P, see below).

We begin with some heuristic considerations that lead to the construction of a birational map $X_4 \dashrightarrow Y_{3,4}$. Theorem 2.3 states that this map is a link of the Sarkisov program.

Lemma 2.2. *Let $X = \{xy + z^3 + t^3 = 0\} \subset \mathbb{C}^4$ be a 3-fold germ. Let $U \to \mathbb{C}^4$ be the weighted blowup with weights $(2, 1, 1, 1)$, $V \subset U$ the inverse image of X and $E \subset V$ the exceptional divisor. Then*

(1) *$E \subset V \to 0 \in X$ is an extremal divisorial contraction in the Mori category with discrepancy $a_E(K_X) = 1$,*
(2) *$E_{|E} = \mathcal{O}(-1)$, where $\mathcal{O}_E(1)$ denotes the tautological sheaf under the obvious embedding*

$$E = \{xy + z^3 + t^3 = 0\} \subset \mathbb{P}(2, 1^3).$$

In particular, $E^3 = 3/2$. □

Later in Section 3, we prove that any divisorial contraction $E \subset V \to P \in X$, contracting the exceptional divisor E to P, is isomorphic to a weighted blowup with weights either $(2, 1, 1, 1)$ or $(1, 2, 1, 1)$.

Let $X = X_4$ be a quartic 3-fold as in equation 1, and let $E \subset V \to P \in X$ be the weighted blowup assigning weights $2, 1, 1, 1$ to the coordinates x_1, x_2, x_3, x_4. We now play a 2-ray game (see [Co2], pp. 269–272) starting with the configuration $V \to X$. We do this to determine if a link of the Sarkisov program exists originating from this blowup. The heuristic calculations based on the Hilbert function are also explained in [CPR], 7.2. Denoting $A = -K_X$ and $B = -K_V$, we have

$$B^3 = (A - E)^3 = A^3 - E^3 = 4 - \frac{3}{2} = \frac{5}{2}$$

Note that $N^1 V = H^2(V, \mathbb{R}) = \mathbb{R}^2$ is 2-dimensional, so $\overline{\mathrm{NE}}\, V$ has two (pseudo)extremal rays; we denote the one, corresponding to the curves contracted by $V \to X$, R_{old}, the other R_{new}. To perform the 2-ray game, the first step is to locate R_{new} and determine its nature. Since $B^3 > 0$, we guess that B is nef and that R_{new} is a flop (in fact, with a little experience, this is easy to see: the functions x_1, x_2, x_3, x_4 all vanish on E hence define sections of B, their common base locus is the union of the (proper transforms of the) 24 lines $P \in L_i \subset X$ given by $x_1 x_2 = a_3 = b_4 = 0$. These divide in two groups of 12; those lying on $x_1 = 0$ have $B \cdot L_i = 0$ and support the new ray, while those on $x_2 = 0$ have $B \cdot L_i = 1/2$ and are thus not extremal), then V is a weak Fano 3-fold (that is, $-K_V$ is nef and big) with anticanonical model

$$Z = \operatorname{Proj} \oplus_{n \geq 0} H^0(V, nB)$$

Quite generally if Z is a Fano 3-fold with virtual singularities $1/r_i(a_i, -a_i, 1)$, we can write

$$\begin{aligned} -K_Z^3 &= 2g-2+\sum \frac{a_i(r_i-a_i)}{r_i} \\ h^0(Z,-K_Z) &= g+2 \end{aligned}$$

(this uses the Riemann-Roch formula of Fletcher and Reid from [YPG]; it is natural, by analogy with the classical case, to say that g is the *genus* of Z). In fact these numerical data determine the whole Hilbert function of Z

$$h^0(Z,-nK_Z) = -\frac{1}{12}n(n+1)(2n+1)K_Z^3 + (2n+1) - \sum_{Q\in \operatorname{Sing} Z} \ell_Q(n+1)$$

where the sum is taken over the virtual singularities $Q \in Z$ and the local contribution is given by the formula

$$\ell_Q(n) = \sum_{k=1}^{n-1} \frac{\overline{ka}(r-\overline{ka})}{2r}$$

(see [YPG] for details, explanations and examples).

Now in our case Z must be a Fano 3-fold with a singularity $1/2(1,1,1)$ and genus 2. This determines the Hilbert function of Z uniquely, and we could try to use it to determine Z, as explained in [IF] or, more extensively, in [R]. However, it is easy to look first if we can spot Z in the list [IF] of (weighted) hypersurfaces or codimension 2 complete intersections. We easily find that $Z = Z_5 \subset \mathbb{P}(1,1,1,1,2)$ has the correct numerical invariants. Let us try and construct a birational map $X \dashrightarrow \mathbb{P}(1,1,1,1,2)$ with image a variety of degree 5. We look for a section $y \in H^0(V, 2B)$ where $B = A - E$; it is easy to see that

$$y = x_0x_1$$

gives such a section, indeed by construction x_1 vanishes twice along E. Finally

$$x_1F = y^2x_2 + ya_3 + x_1b_4 = 0$$

gives the equation of Z. Next, the 2-ray game instructs us to flop the curves contracted by the morphism $V \to Z$, that is, the proper transforms of the 12 lines $P \in L_i \subset X$ given by $x_1 = a_3 = b_4 = 0$. Let $t\colon V \dashrightarrow V'$ be the flop of these lines. Again $\overline{\mathrm{NE}}\, V' = R'_{\mathrm{old}} + R'_{\mathrm{new}}$ is a 2-dimensional cone, with R'_{old} corresponding to the curves just flopped. As before, we now wish to locate and determine the structure of R'_{new}. Before we do this explicitly, we want to make a general remark. First, V' is uniruled hence $K_{V'}$ is not nef. We know $K_{V'} \cdot R'_{\mathrm{old}} = 0$, therefore

$K_{V'} \cdot R'_{\mathrm{new}} < 0$. This means that the 2-ray game from now on is an ordinary minimal model program for V'; in particular the existence of this minimal model program guarantees that the 2-ray game ends in a link of the Sarkisov program.

We can hope that the contraction of R'_{new} is a divisorial contraction $S' \subset V' \to Y$, landing in a new Fano 3-fold Y and completing a link $X \dashrightarrow Y$ of the Sarkisov program. To take this further, let us look for the exceptional surface S', or rather its transform $S \subset X$. We locate S in X as a *special surface* [CPR], i.e. $S = \{f = 0\} \cap X \subset X$ where $f \in k[x_0, ..., x_4]$ is a homogeneous function with highest slope

$$\mu_P f = \frac{\mathrm{mult}_P f}{\deg f}$$

We already own $f = x_1$ with slope $\mu_P x_1 = 2$ and indeed it is easy to check that the proper transform $S' \subset V'$ of $X \cap \{x_1 = 0\}$ is a $\mathbb{P}^2$ with normal bundle $\mathcal{O}(-2)$. Putting $A' = -K_Y$, we have that $B' = A' - (1/2)S'$ hence

$$A'^3 = B'^3 + \frac{4}{8} = \frac{5}{2} + \frac{1}{2} = 3$$

As before, this determines the Hilbert function of Y, a Fano 3-fold of genus $g = 2$ and singularities $2 \times 1/2(1,1,1)$. We find Y in the list of Fano 3-fold codimension 2 weighted complete intersections

$$Y = Y_{3,4} \subset \mathbb{P}(1^4, 2^2)$$

All that remains to do now is to exhibit the map $Z \dashrightarrow Y \subset \mathbb{P}(1^4, 2^2)$ and the equations of Y. To do this observe that the strict transform of the cubic $(x_0x_1x_2 + a_3 = 0)_{|X}$ is in $|3B'|$. The rational map is defined as follows

$$Z \ni Q \mapsto \big(x_1(Q), ..., x_4(Q), y_1(Q), y_2(Q)\big) \in \mathbb{P}(1^4, 2^2)$$

where $y_1 = y = x_0x_1$, and

$$y_2 = -\frac{x_0x_1x_2 + a_3}{x_1}$$

Finally, it is easy to verify (exercise!) that Z maps birationally onto its image Y, with equations

$$\begin{cases} y_1y_2 + b_4(x_1, x_2, x_3, x_4) & = 0 \\ y_1x_2 + y_2x_1 + a_3(x_1, x_2, x_3, x_4) & = 0 \end{cases}$$

We have proved:

Theorem 2.3. *The diagram*

$$\begin{tikzcd} & V \arrow[dl] \arrow[dr] \arrow[rr, dashed] & & V' \arrow[dl] \arrow[dr] & \\ X_4 & & Z & & Y_{3,4} \end{tikzcd} \qquad \square$$

is a Sarkisov link of Type II

Remark 2.4. *Note that there are really two links $X_4 \dashrightarrow Y_{3,4}$, corresponding to the two weighted blowups of the singularity $xy + z^3 + t^3$, with weights $(2,1,1,1)$ and $(1,2,1,1)$. Note that both links go to the* same *variety $Y_{3,4}$*

Exercise 2.5. Construct the following links (recall the Introduction, 1.5).

(a) Take $X = X_7 \subset \mathbb{P}(1,1,1,2,3)$ with a singular point at $P = (0,0,1,0,0)$ of the form $y^2 + z^2 + x_1^6 + x_2^6$, then $X_7 \dashrightarrow Y_{6,7} \subset \mathbb{P}(1,1,2,3,3,4)$ with midpoint $Z_9 \subset \mathbb{P}(1,1,2,3,3)$.
(b) Take $X = X_{15} \subset \mathbb{P}(1,1,2,5,7)$ with a singular point at $P = (0,1,0,0,0)$ of the form $u^2 + z^2y + y^7 + x^{14}$, then $X_{15} \dashrightarrow Y_{14,15} \subset \mathbb{P}(1,2,5,6,7,9)$ with midpoint $Z_{20} \subset \mathbb{P}(1,2,5,6,7)$.

2.3. **Constructions, second approach.** We briefly discuss a much more concise description of the link $X_4 \dashrightarrow Y_{3,4}$ which also points out to a very large number of similar examples. Everything here was suggested by Miles Reid.

The idea is to describe the link starting from the midpoint Z, rather than either of the ends X, Y. In our example Z is a Fano hypersurface in weighted projective space; it is special because it is not $\mathbb{Q}$-factorial, this is just an expression of the fact that Z is not $\mathbb{Q}$-factorial, that is, it is not in the Mori category.

Start with a Fano 3-fold hypersurface $Z = Z_d \subset \mathbb{P}(1, a_1, a_2, a_3, a_4)$ containing a surface

$$\xi = \eta = 0$$

Here ξ, η are homogeneous functions of the coordinates of degrees $\deg \xi < \deg \eta$. Usually one takes ξ, η to be two coordinate functions, but not always. The equation of Z can be written as

$$F = a\eta - b\xi = 0$$

Assume that Z is quasismooth outside the "nodes" $a = b = \xi = \eta = 0$, which is often the case. Then we obtain two small partial resolution of Z, by considering the two ratios $y = \eta/\xi = b/a$ or $z = b/\eta = a/\xi$.

For example, if $\xi = x_{i_1}$ and $\eta = x_{i_2}$ with $i_1 < i_2$, then the first ratio gives the hypersurface

$$X = \{ya(...x_{i_1}, ...x_{i_1}y, ...) = b(...x_{i_1}, ...x_{i_1}y, ...)\}$$

while the second ratio gives, in general, the codimension 2 complete intersection

$$Y : \begin{cases} z\eta = b \\ z\xi = a \end{cases}$$

This construction explains all the quadratic involutions of [CPR], our $X_4 \dashrightarrow Y_{3,4}$ with midpoint $Z_5 = \{a_3y + b_4x_1 = 0\}$, and many more links involving complete intersections $Y_{d_1,d_2} \subset \mathbb{P}(a_0, ..., a_5)$.

Exercise 2.6. Study the following cases

(a) Z is a quartic 3-fold and $\xi = f_2$, $\eta = g_2$ are two quadrics,
(b) Z is a quartic 3-fold and $\xi = x_1$, $\eta = g_2$,
(c) the examples of Exercise 2.5

2.4. **Links centered on lines.** We next construct involutions $\tau_L : X \dashrightarrow X$ and show that they also are Sarkisov links of Type II. These are very similar to the involutions in [CPR] pp. 198 foll. "Elliptic involutions".

Let $L \subset X$ be a line passing through the singular point $P \in X$. Choosing coordinates so that $L = (x_2 = x_3 = x_4 = 0)$, the equation of X can be written as

$$F = x_2x_0^2x_1 + a_1x_0x_1^2 + b_1x_1^3 + a_2x_0x_1 + b_2x_1^2 + a_3x_0 + b_3x_1 + b_4 = 0,$$

where a_i, b_i are homogeneous polynomial of degree i in $\mathbb{C}[x_2, x_3, x_4]$. It is easy to understand how a birational map $X \dashrightarrow X$ arises in this context: the generic fibre of the projection from L is an elliptic curve with a section, corresponding to the singular point $P \in X$. The birational map $X \dashrightarrow X$ is the reflection given by the group law on the elliptic curve. Our aim is to show that this birational selfmap is a link of the Sarkisov program. We follow closely the treatment of elliptic involutions in [CPR].

We eliminate both variables x_0 and x_1 at once, replacing them by more complicated terms

$$y = x_2x_0^2 + a_1x_0x_1 + b_1x_1^2 + \cdots \quad \text{and} \quad z = x_2x_0y + \cdots .$$

These are designed to be plurianticanonical on V, where $E \subset V \to L \subset X$ is the (unique) extremal divisorial contraction which blows up the generic point of L. In other words, y, z vanish enough times on the exceptional divisor E of $V \to X$, and it turns out that, together with the other coordinates x_2, x_3, x_4, they generate the anticanonical ring of V, and satisfy a relation of the form

$$z^2 + Azy + Bz = x_2y^3 + Cy^2 + Dy + E$$

with $A, B, C, D, E \in k[x_2, x_3, x_4]$. This equation defines the midpoint of the link, which is a (weak) Fano hypersurface $Z_{10} \subset \mathbb{P}(1, 1, 1, 3, 5)$ having a biregular involution i_Z coming from interchanging the two roots of the quadratic equation.

The form of the equation makes clear that the argument depends at some level on the fact that the fibers of the rational map to $\mathbb{P}^2$ given by x_2, x_3, x_4 are birationally elliptic curves with a section.

Define

$$y = x_2x_0^2 + a_1x_0x_1 + b_1x_1^2 + a_2x_0 + b_2x_1 + b_3$$

so that

$$F = yx_1 + a_3x_0 + b_4 \tag{2}$$

From the last equation, it is clear that the divisor of zeros of y on V is $\geq 3E$ or, equivalently, $\operatorname{mult}_L y_{|X} = 3$. Next comes the tricky step. Multiply F by x_2x_0, substitute for $x_2x_0^2$ in terms of y:

$$\begin{aligned} x_2x_0F &= x_2x_0x_1y + a_3x_2x_0^2 + b_4x_2x_0 \\ &= x_2x_0x_1y + a_3(y - a_1x_0x_1 - b_1x_1^2 - \cdots) + b_4x_2x_0 \end{aligned}$$

Collecting the terms divisible by x_1 we then get

$$\begin{aligned} x_2x_0F = x_1(x_2x_0y - a_1a_3x_0 - a_3b_1x_1 - a_3b_2) \\ + a_3y - a_2a_3x_0 - a_3b_3 + b_4x_2x_0 \end{aligned}$$

Set now

$$z = x_2x_0y - a_1a_3x_0 - a_3b_1x_1 - a_3b_2 \tag{3}$$

so that

$$x_2x_0F = zx_1 + (x_2b_4 - a_2a_3)x_0 + a_3(y - b_3) \tag{4}$$

Again the last equation makes it manifest that the divisor of zeros of z on V is $\geq 5E$ or, equivalently, $\operatorname{mult}_L z_{|X} = 5$.

In order to eliminate x_0, x_1 in favor of y, z, note that we can view (2–4) as inhomogeneous linear relations in x_0, x_1 with coefficients in $k[x_2, x_3, x_4, y, z]$:

$$\begin{array}{llccccccc} (2) & F = & yx_1 & + & a_3x_0 & + & b_4 & = & 0, \\ (3) & x_2x_0F = & zx_1 & + & (x_2b_4 - a_2a_3)x_0 & + & a_3(y - b_3) & = & 0, \\ (4) & \text{definition of } z\text{:} & a_3b_1x_1 & + & (a_1a_3 - x_2y)x_0 & + & z + a_3b_2 & = & 0. \end{array}$$

The equation relating y and z can be easily expressed in the following determinantal *format*

$$\frac{1}{a_3}\det\begin{pmatrix} a_3 & y & b_4 \\ x_2b_4 - a_2a_3 & z & a_3(y-b_3) \\ a_1a_3 - x_2y & a_3b_1 & z + a_3b_2 \end{pmatrix} = 0.$$

The equation of Z is quadratic in the last variable z, so that Z is a 2-to-1 cover of $\mathbb{P}(1^3, 3)$, which gives a (biregular!) involution of $i_L\colon Z \to Z$ by interchanging the sheets. The involution $\tau_L : X \dashrightarrow X$, as in [CPR], is the composite

$$\begin{tikzcd}
 & V \arrow[dl, "f"'] \arrow[dr, "g"] & & & V \arrow[dl, "g"'] \arrow[dr, "f"] & \\
X & & Z \arrow[r, "i_L"] & Z & & X.
\end{tikzcd}$$

Theorem 2.7. *The map $\tau_L\colon X \dashrightarrow X$ just constructed is a link of the Sarkisov program.*

Proof. We have to show that $V \to Z$ contracts a finite number of curves. The verification is tedious, but similar to [CPR], pp. 200–201. ☐

Exercise 2.8. Let X be a quartic 3-fold, and $L \subset X$ a line on X. If X has one or more singular points along L, we get an elliptic fibration and a link centered on L. It is very tricky, and amusing, to determine the structure of this link. As an exercise, prove that:

(a) if X has one node on L, then $X \dashrightarrow X$ with midpoint $Z_{12} \subset \mathbb{P}(1,1,1,4,6)$,
(b) if X has two nodes on L, then $X \dashrightarrow X$ with midpoint $Z_8 \subset \mathbb{P}(1,1,1,2,4)$

3. Divisorial contractions

We survey the known results on the classification of 3-fold divisorial contractions. Our main goal is to classify divisorial contractions contracting a divisor to the singularity $xy + z^3 + t^3 = 0$.

Definition 3.1. Let $P \in X$ be the germ of a 3-fold terminal singularity. A *divisorial contraction* is a proper birational morphism $f\colon Y \to X$ such that

(1) Y has terminal singularities,
(2) the exceptional set of f is an irreducible divisor $E \subset Y$,
(3) $-K_Y$ is relatively ample for f.

An *extremal* divisorial contraction $f\colon Y \to X$ is an extremal divisorial contraction in the Mori category. In other words, Y has $\mathbb{Q}$-factorial terminal singularities, f is the contraction of an extremal ray R of $\overline{\mathrm{NE}}\,Y$ satisfying $K_Y \cdot R < 0$, and the exceptional set $\operatorname{Exc} f = E \subset Y$ is a divisor in Y. Its image $\Gamma = f(E)$ is a closed point or a curve of X, and we usually write $f\colon (E \subset Y) \to (\Gamma \subset X)$. Here X is not necessarily a germ, but $Y \to X$ is a divisorial contraction in the above sense above the germ around any point $P \in \Gamma$. Viewed from X, we also say that f is an *extremal extraction*, or that it *extracts* the valuation $v = v_E$ of $k(X)$ from its *center* $\Gamma = C(X, v_E) \subset X$.

The classification of 3-fold divisorial contractions is now known in several important special cases:

Theorem 3.2 (Kawamata [Ka]). *Let*

$$P \in X \cong \frac{1}{r}(1, a, r-a) \quad \text{(with } r \geq 2 \text{ and } a \text{ coprime to } r)$$

be the germ of a 3-fold terminal quotient singularity, and

$$f\colon (E \subset Y) \to (\Gamma \subset X)$$

a divisorial contraction such that $P \in \Gamma$. Then $\Gamma = P$ and f is the weighted blowup with weights $(1, a, r-a)$. □

Corollary 3.3. *Suppose that X is a 3-fold with only terminal quotient singularities. If a curve $\Gamma \subset X$ is the center of a divisorial extraction $f\colon (E \subset Y) \to (\Gamma \subset X)$ then $\Gamma \subset \operatorname{NonSing} X$ (and f is the blowup of I_Γ over the generic point of Γ).*

Proof. For if Γ passed through a terminal quotient point P, Theorem 3.2 would imply that $\Gamma = P$, a contradiction. □

The next Corollary is a very useful characterization of terminal singularities of pairs in terms of multiplicity.

Corollary 3.4. *Let*

$$P \in X \cong \frac{1}{r}(1, a, r-a) \quad \text{(with } r \geq 2 \text{ and } a \text{ coprime to } r)$$

be the germ of a 3-fold terminal quotient singularity, and let $\mathcal{H}$ be a linear system (not necessarily mobile) on X. Let

$$f\colon (E \subset Y) \to (P \in X)$$

be the blow up with weights $(1, a, r-a)$, and $\delta = \operatorname{mult}_E \mathcal{H}$. Then

$$K + \frac{1}{n}\mathcal{H}$$

is terminal, if and only if $\delta < n/r$.

Proof. If $(X, (1/n)\mathcal{H})$ is not terminal, by [Co1] §2, there is a divisorial contraction $f\colon (E \subset Y) \to (P \in X)$, extracting a divisor E for which $\operatorname{mult}_E \mathcal{H} \geq n a_E$. By 3.2, f is the weighted blow up we are speaking of, so we get that $\delta \geq n/r$. Vice-versa if $(X, \mathcal{H})$ is terminal, it is part of the definition that $\delta < n/r$. □

Another known case is in [Co2]:

Theorem 3.5. *Let $P \in X$ be a 3-fold germ analytically isomorphic to an ordinary node*

$$xy + zt = 0$$

and $f\colon (E \subset Y) \to (P \in X)$ a divisorial contraction; assume in addition that $f(E) = P$. Then f is the blow up of the maximal ideal at P. □

The following Corollary is similar to 3.4 but slightly weaker, essentially because curve maximal centers passing through an ordinary node do exist.

Corollary 3.6. *Let $P \in X$ be a 3-fold germ analytically isomorphic to an ordinary node*

$$xy + zt = 0$$

and $\mathcal{H}$ a linear system (not necessarily mobile) on X. Let

$$f\colon (E \subset Y) \to (P \in X)$$

be the blow up of the maximal ideal at P, and $d = \operatorname{mult}_E \mathcal{H}$. Assume that there exists a valuation F, with center $C_F X = P$, and $\operatorname{mult}_F \mathcal{H} > n a_F$. Then $d > n$.

Proof. It is awkward to try to prove this by the same method as Corollary 3.4, because of possible curve maximal centers. Fortunately, you can check that the proof of Theorem 3.5 in [Co2] pg. 282, proves the statement. □

Before stating our main result in this section, we mention the following very nice result of Kawakita [Kw]

Theorem 3.7. *Let $(E \subset Y) \xrightarrow{f} P \in X$ be a 3-fold divisorial contraction. Assume $P \in X$ is a nonsingular point and $f(E) = P$. In suitable analytic coordinates on X, f is a weighted blow up; the weights are of the form $(1, a, b)$ with $\operatorname{hcf}(a, b) = 1$.* □

Our main result is the following:

Theorem 3.8. *Let $x \in X$ be a 3-fold germ analytically isomorphic to*
$$xy + z^3 + t^3 = 0$$
and $p\colon (E \subset Y) \to (x \in X)$ a divisorial contraction. Then p is the weighted blow up with weights $(2,1,1,1)$ or $(1,2,1,1)$.

Before starting the proof, which occupies the rest of the section, we like to make a few comments.

Our proof is a systematic integration of the ideas of [Ka] and [Co2], §3.2 and is not difficult in principle. However, it is rather complicated and it requires a great deal of notation. The main idea is to apply the connectedness theorem of Shokurov to many different morphisms. The reason all these morphisms exist is that the (analytic) class group $\mathrm{Cl}(X,x)$ is large. We use this manoeuvre many times and without much explanation; [Co2], Theorem 3.10 shows the technique at work in a much simpler situation.

It is certain that the method can be applied to other singularities, for instance $xy + t^n + z^n$ and similar cases with relatively large local class group. On the other hand, we have not been able to make the method work when the class group is small; for instance, we have had little success so far with $xy + z^2 + t^3$ (which is factorial). We have not attempted to determine how far the method can be pushed.

On the other hand a large part of the paper of Kawakita [Kw] holds for arbitrary cDV points $x \in X$. He was able recently to extend his method to the classification of divisorial contractions to $xy + z^2 + t^n$ [Kw2], and even general cA_n singularities [Kw3]. This includes our result as a special case.

Proof. We begin by describing the general setup for the proof.
General setup. Let n be a sufficiently large and divisible positive integer; fix a finite dimensional very ample linear system
$$\mathcal{H}^Y \subset |-nK^Y|$$
Note that we write K^Y to signify the canonical class of Y, as opposed to the usual notation K_Y. We use the notation K^Y throughout this proof. Denote $\mathcal{H} = p(\mathcal{H}^Y)$ the image of $\mathcal{H}^Y$ in X, so that
$$K^Y + \frac{1}{n}\mathcal{H}^Y = p^*\Big(K^X + \frac{1}{n}\mathcal{H}\Big)$$
By construction
$$\mathrm{mult}_E(\mathcal{H}) = na_E(K_X)$$
while
$$\mathrm{mult}_\nu(\mathcal{H}) < na_\nu(K_X)$$

for all valuations $\nu \neq E$.

Description. We summarize in the following diagram, and explain below, our notation for the various spaces and morphisms which we use in the course of the proof

$$\begin{tikzcd}
 & U \arrow[dl, "q_1"'] \arrow[d, "h"] \arrow[dr, "q_2"] & \\
Y_1 \arrow[dr, "p_1"'] & Z \arrow[d, "g"] & Y_2 \arrow[dl, "p_2"] \\
 & X &
\end{tikzcd}$$

We now introduce in detail the various spaces and morphisms. Please draw your own picture, and do your own calculations to justify the description we give; otherwise you will not follow the proof.

(a). Denote $p_1 \colon E_1 \subset Y_1 \to x \in X$ the weighted blow up with weights $(2,1,1,1)$ and exceptional divisor $E_1 \subset Y_1$. Similarly denote $p_2 \colon E_2 \subset Y_2 \to x \in X$ the weighted blow up with weights $(1,2,1,1)$. It is easy to check, for instance by performing the blowing up explicitly, that Y_i has a singular point $y_i \in Y_i$ of type $1/2(1,1,1)$ and is elsewhere nonsingular.

(b). Denote $g \colon Z \to X$ the blow up of the maximal ideal at $x \in X$ with exceptional divisors E_1, E_2. The abuse of notation means to suggest, for instance, that the rational map $Z \dashrightarrow Y_1$ (not a morphism!) is an isomorphism at the generic point of E_1, and contracts E_2. Thus the notation "E_1" denotes the "same" divisor in two different varieties Y_1 and Z. We let the context decide which is meant; however, when we wish to be precise about the ambient variety, we write for instance E_1^Z meaning the divisor E_1 on the variety Z. We do this for other varieties and divisors, as well. This is justified since many of the quantities we are interested in, such as discrepancies and multiplicities, depend only on the divisor, not on the ambient variety.

(c). It is easy to check that $E_i^Z \cong \mathbb{P}^2$, and E_1^Z, E_2^Z intersect in a line $B = E_1^Z \cap E_2^Z$. Also, Z itself is nonsingular apart from three distinct ordinary nodes

$$z_j \in B \subset E_1^Z + E_2^Z \subset Z,$$

$j \in \{1,2,3\}$, each looking like

$$\text{origin} \in z\text{-axis} \subset (t=0) \subset (xy + zt = 0)$$

(d). Denote $h : U \to Z$ the blow up of the three z_js; it has three exceptional divisors $F_j \subset U$, all isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ with normal bundles $N_{F_j}U \cong \mathcal{O}(-1,-1)$. It is easy to check that U is nonsingular. We also denote $q_i\colon U \to Y_i$ the obvious morphisms and

$$f = g \circ h\colon U \to X$$

(e). It is important to understand that the maps $Z \dashrightarrow Y_i$ are not morphisms. Indeed for instance we can resolve the map $Z \dashrightarrow Y_1$ by a diagram

$$\begin{tikzcd} & V_1 \arrow[dl] \arrow[dr] & \\ Z \arrow[rr, dashed] & & Y_1 \end{tikzcd}$$

where $V_1 \to Z$ is a small resolution of all the $z_j \in Z$; $E_1^{V_1} \to E_1^Z$ blows up all the three points $z_j \in E_1^Z$, introducing exceptional curves $\Gamma_j^1 = C_{F_j}(V_1)$, while $E_2^{V_1} \to E_2^Z$ is an isomorphism. The morphism $V_1 \to Y_1$ contracts $E_2^{V_1} \cong \mathbb{P}^2$, with normal bundle $N_{E_2}V_1 = \mathcal{O}(-2)$, to a singular point $1/2(1,1,1)$. The images of the Γ_j^1 are three lines $C_j^1 = C_{F_j}Y_1$ passing through $y_1 \in Y_1$. We use these lines later in the proof. Similar remarks and notation apply to the map $Z \dashrightarrow Y_2$.

Main division into cases. Now we start with our given divisorial contraction $E \subset Y \to x \in X$ and we want to show that $E = E_1$ or $E = E_2$; assuming the contrary we will derive a contradiction. The proof divides out in cases, depending on the position of the center $C_E(Z)$ of E on Z:

$$\text{Cases:}\begin{cases} (1) & C_E(Z) \not\subset B = E_1^Z \cap E_2^Z \\ (2.1) & C_E(Z) \subset B, \text{but } C_E(Z) \neq z_j \\ (2.2) & C_E(Z) = z_1 \end{cases}$$

In all cases, we define b_i, c_j by the formula

$$f^*\mathcal{H} = \mathcal{H}^U + \sum b_i E_i^U + \sum c_j F_j$$

Another way to say this is that $b_i = \operatorname{mult}_{E_i}\mathcal{H}$ and $c_j = \operatorname{mult}_{F_j}\mathcal{H}$ are the multiplicities of $\mathcal{H}$ along the E_is and F_js. The assumption that $E \neq E_1, E_2$ means that $b_1, b_2 < n$.

We treat each case separately.

Case 1. Let $S \subset X$ be a generic surface through $x \in X$. It is easy to compute

$$\begin{aligned} f^*K^X =& K^U - \sum E_i^U - 2\sum F_j \\ f^*S =& S^U + \sum E_i^U + \sum F_j \end{aligned}$$

hence

$$f^*\Big(K^X+S+\frac{1}{n}\mathcal{H}\Big)=K^U+S^U+\frac{1}{n}\mathcal{H}^U \\ +\sum\frac{b_i}{n}E_i^U+\sum\Big(\frac{c_j}{n}-1\Big)F_j$$

which is another way to say, for instance, that $a_{E_i}(K^X+S+(1/n)\mathcal{H})=b_i/n$, and similarly for the discrepancies of the F_j.

Suppose now that $C_E Z\in E_1$, say. We apply Shokurov connectedness theorem, [Co2] §3.2 and 3.3 and especially Corollary 3.5, to the morphism $f\colon U\to X$ and the divisor

$$K^U+S^U+D^U=K^U+S^U+\frac{1}{n}\mathcal{H}^U+\sum\frac{b_i}{n}E_i^U+\sum\Big(\frac{c_j}{n}-1\Big)F_j,$$

(where the formula defines D^U) and we conclude that there is a "line" $L_1\subset E_1^U$ (by this we mean that L_1 maps to a line under the morphism $E_1^U\to E_1^Z\cong\mathbb{P}^2$) such that

$$L_1\subset LC\big(U,K^U+S^U+D^U\big)$$

One of the three z_is, say z_1, does not lie on the line L_1. It is easy to construct a contraction $W_1\to X$ having F_1 as its unique exceptional divisor (W_1 has canonical but not terminal singularities so this is not a divisorial contraction in the Mori category). In fact there is a morphism $\varphi\colon U\to W_1$, resulting from a free linear system Δ^U on U, which in turn is the proper transform of a linear system Δ on X such that

$$f^*\Delta=\Delta^U+\sum E_i+2F_1+\sum_{j=2,3}F_j$$

Define now

$$\delta=\max\Big\{0,\frac{1}{2}\Big(1-\frac{c_1}{n}\Big)\Big\}$$

Then

$$f^*\Big(K^X+S+\frac{1}{n}\mathcal{H}+\delta\Delta\Big)=K^U+S^U+\frac{1}{n}\mathcal{H}^U+\delta\Delta^U+ \\ \sum\Big(\frac{b_i}{n}+\delta\Big)E_i+pF_1+\sum_{j=2,3}\Big(\frac{c_j}{n}+\delta-1\Big)F_j$$

where $p=(c_1/n)-1+2\delta\geq 0$. We now apply Shokurov connectedness to the morphism φ and the divisor

$$K^U+S^U+D_1^U=K^U+S^U+\frac{1}{n}\mathcal{H}^U+\delta\Delta^U+ \sum\Big(\frac{b_i}{n}+\delta\Big)E_i+pF_1+\sum_{j=2,3}\Big(\frac{c_j}{n}+\delta-1\Big)F_j$$

(where the formula defines D_1^U). It is important here to understand that the divisor F_1 is not contracted by $\varphi\colon U\to W_1$; the reason we can apply Shokurov connectedness is that $p\geq 0$. This is why we introduced δ in the first place; it would have been tempting to run the argument with $\delta=0$. Now E_1^U is contracted by φ; the fibers of the rational map $\varphi_{|E_1^Z}\colon E_1^Z\cong\mathbb{P}^2\dashrightarrow W_1$ are the lines trough z_1. Furthermore

$$L_1\cup S^U\subset LC\big(U,K^U+S^U+D_1^U\big)$$

Since $C_{F_1}(Y_1)$ is a curve, $a_{F_1}(K^{Y_1})=1$, and a simple calculation then shows that $c_1=b_1+\operatorname{mult}_{F_1}\mathcal{H}^{Y_1}\geq b_1$, therefore

$$\frac{b_1}{n}+\delta<1.$$

This implies that $LC(U,K^U+S^U+D_1^U)$ is not connected in the neighborhood of a general fiber of $\varphi_{|E_1^U}\colon E_1^U\to W_1$. The contradiction finishes the proof in Case 1.

Case 2. Let us define δ_1, δ_2 by the formulas

$$q_1^*\mathcal{H}^{Y_1}=\mathcal{H}^U+\delta_1E_2^U+(\text{other})$$
$$q_2^*\mathcal{H}^{Y_2}=\mathcal{H}^U+\delta_2E_1^U+(\text{other})$$

where "(other)" means a combination of the F_js. Another way to define these numbers would have been to set

$$\delta_1=\operatorname{mult}_{E_2}\mathcal{H}^{Y_1}$$
$$\delta_2=\operatorname{mult}_{E_1}\mathcal{H}^{Y_2}$$

It is convenient (and, ultimately, straightforward) for us to calculate δ_1 and δ_2 in terms of b_1, b_2, in fact we only need the

Claim.

$$\delta_1+\delta_2=\frac{b_1+b_2}{2}$$

To prove the claim, note

$$q_1^*E_1 = E_1^U + \frac{1}{2}E_2^U + (\text{other})$$
$$q_2^*E_2 = E_2^U + \frac{1}{2}E_1^U + (\text{other})$$

therefore

$$f^*\mathcal{H} = q_1^*p_1^*\mathcal{H} = q_1^*(\mathcal{H}^{Y_1} + b_1E_1)$$
$$= \mathcal{H}^U + b_1E_1^U + \Big(\delta_1 + \frac{b_1}{2}\Big)E_2^U + (\text{other})$$

from which we conclude $b_2 = \delta_1 + b_1/2$. Similarly, $b_1 = \delta_2 + b_2/2$ and the claim follows.

Subcase 2.1. Assume that $C_E(Z) \subset B$ but is not one of the z_js. This implies that $C_E(Y_i) = y_i \in Y_i$ is the unique singular point. The divisor

$$p_i^*\Big(K^X + \frac{1}{n}\mathcal{H}\Big) = K^{Y_i} + \frac{1}{n}\mathcal{H}^{Y_i} - \Big(1 - \frac{b_i}{n}\Big)E_i$$

is *strictly canonical*, that is, canonical but not terminal, in a neighborhood of $y_i \in Y_i$ (the valuation ν corresponding to the exceptional divisor of the divisorial contraction which we have been studying all this time, has discrepancy $= 0$). Because $b_i < n$, the divisor

$$K^{Y_i} + \frac{1}{n}\mathcal{H}^{Y_i}$$

is not canonical in a neighborhood of $y_i \in Y_i$. By Corollary 3.4, both δ_1, δ_2 are $> n/2$ hence $(b_1 + b_2)/2 > n$ and either b_1 or $b_2 > n$, a contradiction which concludes this case.

Subcase 2.2. Assume now that $C_E(Z)$ is one of the z_js, say z_1. The proof just given breaks down because typically in this case the center $C_E(Y_i)$ of E on Y_i is not the singular point $y_i \in Y_i$. Therefore we argue directly on Z. It is important to be aware that the divisors E_1^Z and E_2^Z are not $\mathbb{Q}$-Cartier at z_1, but the sum $E_1^Z + E_2^Z$ is. Consider the divisor

$$D^Z = \frac{1}{n}\mathcal{H}^Z + \frac{b_2 - b_1}{n}E_2^Z$$

on Z. Note

$$g^*\Big(K^X + \frac{1}{n}\mathcal{H}\Big) = K^Z + \frac{1}{n}\mathcal{H}^Z + \Big(\frac{b_1}{n} - 1\Big)E_1^Z + \Big(\frac{b_2}{n} - 1\Big)E_2^Z$$
$$= K^Z + D^Z + \Big(\frac{b_1}{n} - 1\Big)(E_1^Z + E_2^Z)$$

hence $K^Z + D^Z$ is $\mathbb{Q}$-Cartier but not canonical at $C_EZ = z_1 \in Z$, hence assuming as we may that $b_2 \geq b_1$ so that $D^Z \geq 0$, and using

Corollary 3.6, we have

$$d = \operatorname{mult}_{F_1} D^Z > 1$$

We now calculate $K^X + (1/n)\mathcal{H}$ in two different ways. On one hand

$$f^*\Big(K^X + \frac{1}{n}\mathcal{H}\Big) = K^U + \frac{1}{n}\mathcal{H}^U + \Big(d + \frac{b_1}{n} - 2\Big)F_1 + (\text{other})$$

while, on the other hand, using $f = p_i q_i$:

$$\begin{aligned} q_i^* p_i^*\Big(K^X + \frac{1}{n}\mathcal{H}\Big) &= q_i^*\Big(K^{Y_i} + \frac{1}{n}\mathcal{H}^{Y_i} + \frac{b_i}{n}E_i\Big) \\ &= K^U + \frac{1}{n}\mathcal{H}^U + \frac{b_i}{n}E_i^U + \Big(\frac{\kappa_i}{n} + \frac{b_i}{n} - 2\Big)F_1 + (\text{other}) \end{aligned}$$

where $\kappa_i = \operatorname{mult}_{F_1} \mathcal{H}^{Y_i}$; hence $\kappa_1/n = d > 1$ gives

$$\kappa_1 > n$$

Similarly $\kappa_2/n + b_2/n - 2 = d + b_1/n - 2$ gives

$$\kappa_2 + b_2 - b_1 > n$$

Now recall the curves $y_i \in C^i_j = C_{F_j} Y_i \subset Y_i$; by definition $\delta_1 = \operatorname{mult}_{E_2} \mathcal{H}^{Y_1}$, and similarly for δ_2, while $\kappa_i = \operatorname{mult}_{C^i_1} \mathcal{H}^{Y_i}$. It follows from Theorem 3.2 that $\kappa_i/2 \leq \delta_i$, hence combining inequalities we get

$$\frac{b_1 + b_2}{2} = \delta_1 + \delta_2 \geq \frac{\kappa_1 + \kappa_2}{2} > n + \frac{b_1 - b_2}{2}$$

and from this we conclude $b_2 > n$, a contradiction. □

4. Plan of proof of the main theorem

In this Section we give a more precise statement of the main theorem 1.1 and an outline of the proof. This is almost word by word as [CPR] §3. The proof is a formal consequence of the machinery and definitions of the Sarkisov program, and the classification of maximal singularities on X_4 and $Y_{3,4}$. We state the relevant results here, and prove them in sections 6 and 7. Our precise statement is

Theorem 4.1. *Let $X = X_4 \subset \mathbb{P}^4$ be a quartic 3-fold as above. In other words, X has a singular point $P \in X$ of the form $xy + z^3 + t^3 = 0$, and is otherwise general. Let V/T be an arbitrary Mori fibre space and $\varphi\colon X \dashrightarrow V$ a birational map. Then φ is a composition of the following birational maps:*

(a) *an involution $\tau_L\colon X \dashrightarrow X$, centered on a line $P \in L \subset X$ as in Theorem 2.7,*
(b) *one of the two links $X \dashrightarrow Y_{3,4}$ as in Theorem 2.3,*
(c) *The inverse of (b).*

In particular, V is isomorphic to either X_4, or $Y_{3,4}$, and $\mathcal{P}(X) = \{X_4, Y_{3,4}\}$.

Before giving an outline of the proof, we quickly recall some basic notions from the Sarkisov program. We refer to [CPR] Chapter 3 for more details and discussion on this material.

Definition 4.2 (canonical threshold)**.** X is a variety, $\mathcal{H}$ a mobile linear system, and $\widetilde{X} \to X$ a resolution with exceptional divisors E_i. As usual, we write

$$K_{\tilde{X}} = K_X + \sum a_i E_i,$$
$$\widetilde{\mathcal{H}} = \mathcal{H} - \sum m_i E_i,$$

to define the *discrepancies* $a_i = a_{E_i}(K_X)$ of the exceptional divisors E_i and their *multiplicities* m_i in the base locus of $\mathcal{H}$. For $\lambda \in \mathbb{Q}$, we say that $K_X + \lambda\mathcal{H}$ is *canonical* if all $\lambda m_i \leq a_i$, so that $K_{\tilde{X}} + \lambda\widetilde{\mathcal{H}} - (K_X + \lambda\mathcal{H})$ is effective (≥ 0). Then we define the *canonical threshold* to be

$$\begin{aligned} c = c(X, \mathcal{H}) &= \max\bigl\{\lambda \bigm| K + \lambda\mathcal{H} \text{ is canonical}\bigr\} \\ &= \min_{E_i}\bigl\{a_i/m_i\bigr\}. \end{aligned}$$

This is well defined, independently of the resolution $\widetilde{X}$. In all the cases we're interested in, $K_X + (1/n)\mathcal{H} = 0$ and $c < 1/n$.

Definition 4.3 (maximal singularity)**.** Now suppose that $K_X + (1/n)\mathcal{H} = 0$ and $K_X + (1/n)\mathcal{H}$ is not canonical, so that $c < 1/n$. We make the following definitions:

(1) a *weak maximal singularity* of $\mathcal{H}$ is a valuation v_E of $k(X)$ for which $m_E(\mathcal{H}) \geq n a_E(K_X)$;
(2) a *maximal singularity* is an extremal extraction $Z \to X$ in the Mori category (see Definition 3.1) having exceptional divisor E with $c = a_E(K_X)/m_E(\mathcal{H})$.

In either case, the image of E in X, or the center $C(X, v_E)$ of the valuation v_E, is called the *center* of the maximal singularity E.

Definition 4.4 (degree of φ)**.** Suppose that X is a Fano 3-fold with the property that $A = -K_X$ generates the Weil divisor class group: $\operatorname{WCl} X = \mathbb{Z}\cdot A$ (this holds in our case). Let $\varphi\colon X \dashrightarrow V$ be a birational map to a given Mori fibre space $V \to T$, and fix a very ample linear system $\mathcal{H}_V$ on V; write $\mathcal{H} = \mathcal{H}_X$ for the birational transform $\varphi_*^{-1}(\mathcal{H}_V)$.

The *degree* of φ, relative to the given V and $\mathcal{H}_V$, is the natural number $n = \deg\varphi$ defined by $\mathcal{H} = nA$, or equivalently $K_X + (1/n)\mathcal{H} = 0$.

Definition 4.5 (untwisting)**.** Let $\varphi\colon X \dashrightarrow V$ be a birational map as above, and $f\colon X \dashrightarrow X'$ a Sarkisov link of Type II. We say that f *untwists* φ if $\varphi' = \varphi \circ f^{-1}\colon X' \dashrightarrow V$ has degree smaller than φ.

Remark 4.6. *The Sarkisov program factorizes an arbitrary birational map between Mori fibers spaces as a chain of more general types of links, using a more complicated inductive framework. See* [Co1]*, Definition 3.4 for the general definition of a* Sarkisov link $f\colon X \dashrightarrow X'$*, and* [Co1]*, Definition 5.1 for the* Sarkisov degree *of a birational map* $\varphi\colon X \dashrightarrow Y$ *between Mori fibre spaces. The above Definitions are special cases that are sufficient for our purposes in this paper. We can get away with this because we start from our quartic* $X = X_4$*, and we only ever perform untwistings that either return to* X*, or to the Fano 3-fold* $Y_{3,4}$

Lemma 4.7. *Let* X*,* V/T *be as before and* $\varphi : X \dashrightarrow V$ *a birational map. If* $E \subset Z \to X$ *is a maximal singularity, any Type II link* $X \dashrightarrow X'$ *(as in Definition 2.1), starting with the extraction* $Z \to X$*, untwists* φ*.*

Proof. [CPR], Lemma 4.2. ☐

The following classification of maximal singularities on X_4 and $Y_{3,4}$ implies our main theorem 4.1.

Theorem 4.8. *Let* $X = X_4 \subset \mathbb{P}^4$ *be a quartic 3-fold as in the assumption of Theorem 1.1, and* E *a maximal singularity of* X*. Either:*

1. $E \subset Z \to P \in X$ *is one of the blow ups with weights* $(2,1,1,1)$ *or* $(1,2,1,1)$ *described above, or*
2. *the center* $C(E,X) = L$ *is a line* $P \in L \subset X$ *and* E *is generically the blow up of the ideal of* L *in* X*.*

Theorem 4.9. *Let* E *be a maximal singularity on a general* $Y_{3,4} \subset \mathbb{P}(1^4, 2^2)$*, then* $E \subset Z \to P \in Y$ *is the standard blow up of either one of the two singular points of* Y *on the line* $x_1 = x_2 = x_3 = x_4 = 0$ *in* $\mathbb{P}(1^4, 2^2)$*.*

This theorems summarize the conclusions of a whole series of calculations carried out for 4.8 in Section 6 and for 4.9 in Section 7.

Proof that 4.8 and 4.9 imply Theorem 4.1. This is standard, and is the same as the proof in [CPR]. If X is Fano and $V \to T$ a Mori fibre space, a birational map $\varphi\colon X \dashrightarrow V$ is defined by a mobile linear system $\mathcal{H}$. By the Norther-Fano-Iskovskikh inequalities [Co1], Theorem 4.2, if φ is not an isomorphism then $\mathcal{H}$ has a maximal center P or C, hence a maximal singularity $E \subset Z \to P$ or C by [Co1], Proposition 2.10. By

Theorem 4.8 and 4.9, there is a birational map $i\colon X \dashrightarrow X'$, where either $X' = X$ or $X' = Y$, that is a Sarkisov link, and by Lemma 4.7 untwists the maximal center P or C, so that $\varphi \circ i$ has smaller degree. Thus after a number of steps, either $X \cong V$ or $Y \cong V$. □

5. Excluding

Let W be a center on a Fano 3-fold X; that is, $W = P \in X$ or $W = \Gamma \subset X$ is either a point or a curve on X. Eventually in the next two Sections, we take $X = X_4 \subset \mathbb{P}^4$ our special singular quartic 3-fold, or $X = Y_{3,4} \subset \mathbb{P}(1^4, 2^2)$, but here we keep the discussion general. We are concerned with the problem of "excluding W", that is, to prove that W is not a maximal center for *any* linear system $\mathcal{H}$ on X. In this Section we explain our general strategy for doing this.

5.1. **Reduction to a surface problem.** The first step is to reduce to a surface question.

5.1.1. *The starting point.*

(a) We assume by contradiction that W is a maximal center: there is a mobile linear system $\mathcal{H} \subset |\mathcal{O}_X(n)|$ on X, and a valuation E with center $C_E X = W$ and $m_E\mathcal{H} > na_E K_X$.

(b) We select a *test linear system* $\mathcal{T}$ on X with $W \subset \operatorname{Bs}\mathcal{T}$ contained in the base locus of $\mathcal{T}$. Often we take $\mathcal{T} = |I_W(1)|$, but this does not always work. In the simplest cases, but not in all cases, $W = \operatorname{Bs}\mathcal{T}$. The choice of the test system is often delicate.

5.1.2. *The strategy.* We work with a general member $S \in \mathcal{T}$; the argument is slightly different according to whether the center W is a curve or a point.

The center is a curve: the assumption means that $\operatorname{mult}_W \mathcal{H} = m > n$, so we have

$$\mathcal{H}_{|S} = \mathcal{L} + m'W + (\text{other}) \subset |\mathcal{O}_S(n)|$$

where $\mathcal{L}$ is the mobile part of $\mathcal{H}_{|S}$. In general $m' \geq m$ but in most applications below $m' = m$. We concentrate on showing that the mobile system $\mathcal{L}$ can not exist. The idea of course is simple: a non empty linear subsystem in $\mathcal{O}_S(n)$ is unlikely to have a fixed part as large as mW, $m > n$.

The center is a point: by construction

$$K + S + \frac{1}{n}\mathcal{H}$$

is not log canonical in a neighborhood of W. By Shokurov's inversion of adjunction, see [FA] 17.7

$$K_S + \frac{1}{n}\mathcal{H}_{|S}$$

is also not log canonical. Here the method works better if $\mathcal{H}_{|S}$ is mobile but in general we have to allow $\mathcal{H}_{|S} = \mathcal{L}+F$ with nonempty fixed part F. We try to reach a contradiction by choosing general members L_1, L_2 in $\mathcal{L}$ and calculating the intersection number $L_1 \cdot L_2$ on S. Theorem 5.1 states that, if $K_S + (1/n)(\mathcal{L} + F)$ is not log canonical at P, then the local intersection number $(L_1 \cdot L_2)_P$ at P is large; the contradiction happens when it is too large. As before, the idea is very simple, even crude: two curves in $\mathcal{O}_S(n)$ can not intersect in too many points.

5.2. **Linear system on surfaces.** The following theorem is very useful in the study of linear system on surfaces:

Theorem 5.1. *Suppose that $P \in \Delta_1 + \Delta_2 \subset S$ is the analytic germ of a normal crossing curve on a nonsingular surface. Let $\mathcal{L}$ be a mobile linear system on S and denote $\mathcal{L}^2$ the local intersection multiplicity $(L_1 \cdot L_2)_P$ at P of two general members L_1, $L_2 \in \mathcal{L}$. Fix rational numbers $a_1, a_2 \geq 0$ and suppose that*

$$K_S + (1-a_1)\Delta_1 + (1-a_2)\Delta_2 + \frac{1}{m}\mathcal{L}$$

is not log canonical for some $m > 0$.

1. *If either $a_1 \leq 1$ or $a_2 \leq 1$ then*

$$\mathcal{L}^2 > 4a_1a_2m^2.$$

2. *If both $a_i > 1$ then*

$$\mathcal{L}^2 > 4(a_1 + a_2 - 1)m^2.$$

Proof. In [Co2], Theorem 3.1. □

5.3. **The surface problem.** We summarize the general set up for the surface problem in very broad terms.

5.3.1. *The set up.* The initial set up is as follows.

(a) A polarized surface S, with polarizing (integral Weil, often Cartier) divisor $A = \mathcal{O}_S(1)$. In practice, S is often a K3 surface with DuVal singularities.

(b) A configuration $\{\Gamma_i\}$ of curves $\Gamma_i \subset S$. Usually Γ_i is a -2 curve on the minimal resolution. The intersection matrix $a_{ij} = \Gamma_i \cdot \Gamma_j$ is known in principle but it may be impractical to calculate exactly, because there are many Γ_i or the geometry of the configuration maybe itself complicated or contain a fair number of different degenerations. Some geometric information is easily accessible, for instance the Γ_i are linearly independent in $H_2(S)$, any proper subset of $\{\Gamma_i\}$ is contractible on S, $\overline{\mathrm{NE}}\, S = \sum \mathbb{Q}_+[\Gamma_i]$ etc.

(c) $A = \sum b_i\Gamma_i$ with b_i small; often all $b_i = 1$, sometimes some $b_i = 2$.

(d) We assume a nef $\mathbb{Q}$-divisor L on S given by a formula

$$A = L + \sum \gamma_i\Gamma_i$$

with the $\gamma_i \geq 0$. In the notation of the previous subsection, this is $A = (1/n)\mathcal{H}_{|S} = (1/n)\mathcal{L} + (1/n)F$, that is $L = (1/n)\mathcal{L}$ and $\sum \gamma_i\Gamma_i = (1/n)F$.

5.3.2. *The goal.* The aim is slightly different according to whether W is a curve or a point.

When W is a curve: the aim is to show that all $\gamma_i \leq 1$.

When W is a point: assuming an inequality of the form

$$L^2 > 4(2 - \gamma_0)(1 - \gamma_1),$$

the aim is to show that L cannot exist.

6. Centers on X_4

We fix $X = X_4 \subset \mathbb{P}^4$, with a singular point $P \in X$, satisfying all the assumptions of Theorem 1.1. Our main goal in this section is to prove the following:

Theorem 6.1. *A curve $\Gamma \subset X$, other than a line $P \in L \subset X$, can not be a maximal center.*

Proof. Let Γ be a curve and assume that Γ is a maximal center for $\mathcal{H} \subset |\mathcal{O}_X(n)|$. This implies that $m = \operatorname{mult}_\Gamma \mathcal{H} > n$. In the proof, we reach a contradiction in several steps:

Step 1: A raw argument shows that $\deg\Gamma \leq 3$.

Step 2: Γ can not be a space curve.

Step 3: Γ can not be a plane curve.

Step 1. Choosing general members H_1, H_2 of $\mathcal{H}$ and intersecting with a general hyperplane section S we obtain

$$4n^2 = H_1 \cdot H_2 \cdot S > m^2 \deg\Gamma.$$

This implies that $\deg\Gamma \leq 3$.

Step 2: space curves. If Γ is a space curve, then by Step 1 it must be a rational normal curve of degree 3, contained in a hyperplane $\Pi \cong \mathbb{P}^3 \subset \mathbb{P}^4$. Let $S \in |I_{\Gamma,X}(2)|$ be a general quadric vanishing on Γ,$\mathcal{L}$ the mobile part of $\mathcal{H}_{|S}$; write

$$A = \mathcal{O}_S(1) = \frac{1}{n}\mathcal{H}_{|S} = L + \gamma\Gamma,$$

where $L = (1/n)\mathcal{L}$ is nef. Note that, because I_Γ is cut out by quadrics,

$$\operatorname{mult}_\Gamma \mathcal{H} = \operatorname{mult}_\Gamma \mathcal{H}_{|S} = n\gamma > n.$$

We reach a contradiction by showing that $\gamma \leq 1$. For simplicity we treat two separate cases, namely:

Case 2.1: $P \notin \Gamma$,
Case 2.2: $P \in \Gamma$.

Case 2.1. Here we assume that $P \notin \Gamma$. It follows that S is nonsingular and $\Gamma^2 = -5$ (all calculations of intersection numbers are performed on S). Indeed it is easy to see that $S = S_{2,4} \subset \mathbb{P}^4$ is a nonsingular complete intersection of a quadric and a quartic, therefore $K_S = \mathcal{O}_S(1)$. Then:

$$-2 = \deg K_\Gamma = \Gamma \cdot (K_S + \Gamma) = 3 + \Gamma^2$$

shows that $\Gamma^2 = -5$. A simple calculation then gives:

$$0 \leq L^2 = (A - \gamma\Gamma)^2 = 8 - 6\gamma - 5\gamma^2$$

This shows that $\gamma \leq 4/5 < 1$ and finishes the proof in this case. Note that we only need $\gamma \leq 1$; the additional room in the argument, is what ultimately makes it possible to treat the next Case 2.2 essentially by the same method.

Case 2.2. Now we assume that $P \in \Gamma$. Write as in Section 3 $\varphi\colon U \to X$ the resolution of singularities of $P \in X$, constructed in the proof of Theorem 3.8. Using that $I_{\Gamma,\mathbb{P}^4}$ is generated by quadrics, and that a general quadric $Q \in |I_{\Gamma,\mathbb{P}^4}(2)|$ is nonsingular, it is easy to see that the proper transform S^U must itself be nonsingular.

Denote $\psi = \varphi_{|S^U} : S^U \to S$ and write

$$\psi^*\Gamma = \Gamma^U + \sum \gamma_i\Gamma_i + \sum g_j\Delta_j,$$

where $\Gamma_i = E^U_{i|S^U}$ and $\Delta_j = F_{j|S^U}$ are (-2)-curves (here, following the notation of the proof of Theorem 3.8, E^U_1, E^U_2 and F^U_1, F^U_2, F^U_3 are the exceptional divisors of $U \to X$).

There are now two subcases (up to relabelling the exceptional divisors), depending on how the curve Γ “sits” in the singularity $P \in X$. The crucial observation is that, because Γ is a nonsingular curve, Γ^U

intersect transversally a unique exceptional divisor. The cases are as follows:

Subcase 2.2.1: The proper transform Γ^U intersects E_1^U. In this case, S^U meets E_1^U, E_2^U and is disjoint from all the F_js. Here $P \in S$ is an A_2-singularity, and $(\gamma_1, \gamma_2, g_1, g_2, g_3) = (2/3, 1/3, 0, 0, 0)$.

Subcase 2.2.2: The proper transform Γ^U intersects F_1^U. In this case, S^U meets E_1^U, E_2^U, F_1^U and is disjoint from F_2 and F_3. Here $P \in S$ is an A_3-singularity, and $(\gamma_1, \gamma_2, g_1, g_2, g_3) = (1/2, 1/2, 1, 0, 0)$.

We claim that in both subcases

$$\Gamma^2 \leq -4$$

Indeed it is easy to see, as in Case 2.1 ($P \notin S$), that $\Gamma^U \cdot \Gamma^U = -5$, and by the projection formula

$$\Gamma \cdot \Gamma = \Gamma^U \cdot \psi^* \Gamma = \begin{cases} \Gamma^U \cdot \Gamma^U + 2/3 = -5 + 2/3 \\ \Gamma^U \cdot \Gamma^U + 1 = -5 + 1 \end{cases}$$

in the two Subcases 2.2.1 and 2.2.2, respectively. Finally

$$0 \leq L^2 = 8 - 6\gamma + \Gamma^2\gamma^2 \leq 8 - 6\gamma - 4\gamma^2$$

implies $\gamma < 1$, a contradiction which concludes Step 2.

Step 3: plane curves. Here we assume that Γ is a plane curve of degree d (by Step 1, $d \leq 3$), other than a line passing through P. Here too, as in Step 2, it is helpful and convenient to treat two cases, namely:

Case 3.1: $P \notin \Gamma$ and $1 \leq d \leq 3$.

Case 3.2: $P \in \Gamma$ and $2 \leq d \leq 3$.

Case 3.1. We first deal with the easy case $P \notin \Gamma$ (following well known "ancient" methods of Iskovskikh and Manin). Consider as usual a general element $S \in |I_{\Gamma,X}(d)|$, denote $\mathcal{L}$ the mobile part of $\mathcal{H}_{|S}$; write

$$A = \mathcal{O}_S(1) = \frac{1}{n}\mathcal{H}_{|S} = L + \gamma\Gamma,$$

where $L = (1/n)\mathcal{L}$ is nef. We aim to show that $\gamma \leq 1$. It is easy to see that $S = S_{d,4} \subset \mathbb{P}^4$ is a nonsingular complete intersection of a quartic with a hypersurface of degree $d \leq 3$, therefore $K_S = \mathcal{O}_S(d-1)$. If $d \leq 2$, then $p_a\Gamma = 0$ and:

$$-2 = \deg K_\Gamma = (\Gamma \cdot K_S + \Gamma) = d(d-1) + \Gamma^2$$

shows that $\Gamma^2 = -2 - d(d-1)$. A simple calculation gives:

$$\begin{aligned} 0 \leq L^2 = (A - \gamma\Gamma)^2 &= A^2 - 2A \cdot \Gamma\gamma + \Gamma^2\gamma^2 \\ &= 4d - 2d\gamma - d(d-1)\gamma^2 - 2\gamma^2 \end{aligned}$$

which implies that $\gamma \leq 1$. The proof is similar when $d = 3$: $\Gamma^2 = -6$, and then $0 \leq L^2 = (A - \gamma\Gamma)^2 = A^2 - 2A \cdot \Gamma\gamma + \Gamma^2\gamma^2 = 12 - 6\gamma - 6\gamma^2$ and again $\gamma \leq 1$.

These calculations finish Case 3.1 $P \notin \Gamma$.

CASE 3.2. From now on we assume that $P \in \Gamma$, Γ not a line. In this case, restriction to a general element of the test linear system $|I_\Gamma(d)|$ does not lead to a contradiction; it is necessary to use a different test system.

Denote $\Pi \subset \mathbb{P}^4$ the plane spanned by Γ, let S_1, S_2 be general hyperplane sections of X containing Γ.

We work with the "test system" $|S_1, S_2|$, even though Γ is usually only a component of its base locus $C = S_1 \cap S_2 = \operatorname{Bs}|S_1, S_2| = X \cap \Pi$. We are assuming that X is general, hence in particular it is terminal and $\mathbb{Q}$-factorial. This implies that Π can not be contained in X, and C is a *curve*. Unfortunately, we have to divide the proof in several cases according to what C is. In the end each case is not very different or harder than any of the other cases, but we could not find a unified presentation. The cases are as follows:

(a) $C = \text{cubic} + \text{line}$,
(b) $C = \text{conic} + 2\,\text{lines}$,
(c) $C = 2\,\text{conics}$,
(d) $C = \text{conic} + \text{double line}$,
(e) $C = \text{double conic}$.

We now treat Cases (a)–(c); at the end we will show that Cases (d) and (e) do not happen (at least assuming, as we do, that X is general), in other words, we will show that $C = \Pi \cap X$ is always *reduced* when X is general. Before treating each case individually, we make some general comments and fix the notation for the whole argument.

Assuming for now that C is reduced, we restrict to S_1 and write

$$\begin{aligned} A = (1/n)\mathcal{H}_{|S_1} &= L + \gamma\Gamma + \sum \gamma_i \Gamma_i \\ S_{2|S_1} &= C = \Gamma + \sum \Gamma_i \end{aligned}$$

Our technique consists in selecting a "most favorable" component of C, calculating an intersection number on S_1 using that L is nef, and finally get that $\gamma \leq 1$. When C is reduced, it is clear that if W is a component of C, $\operatorname{mult}_W \mathcal{H} = \operatorname{mult}_W \mathcal{H}_{|S_1}$; in particular $\operatorname{mult}_\Gamma \mathcal{H} = \operatorname{mult}_\Gamma \mathcal{H}_{|S_1} = n\gamma$, and also $\operatorname{mult}_{\Gamma_i} \mathcal{H} = n\gamma_i$. Because Γ is a *maximal* singularity, $\gamma \geq \gamma_i$, hence possibly after relabelling components of C, we can assume that:

$$\gamma \geq \gamma_2 \geq \gamma_1.$$

(ignore the term γ_2 if no curve Γ_2 is present). Consider now the effective $\mathbb{Q}$-divisor

$$(A-\gamma_1 S_2)_{|S_1} = L + (\gamma-\gamma_1)\Gamma + (\gamma_2-\gamma_1)\Gamma_2.$$

In Cases (a) and (b), Γ_1 is a line and

$$(1-\gamma_1) = (A-\gamma_1 S_2)\cdot\Gamma_1 \geq (\gamma-\gamma_1)\Gamma\cdot\Gamma_1.$$

We now show that $\Gamma\cdot\Gamma_1 \geq 1$ (on S_1); together with the last displayed equation this implies that $\gamma \leq 1$ and finishes the proof in Cases (a) and (b). Note that this is intuitively almost obvious: for example in Case (a) C is the plane union of a cubic and a line, and we expect these to intersect in 3 points (when we only need one!). The problem with saying this is, of course, that the set theoretic intersection $\Gamma\cap\Gamma_1$ can be all concentrated on the singular point $P\in X$. We now study this situation more carefully.

Note first that S_1 is nonsingular outside P. This follows easily from the fact that the base locus C of $|S_1, S_2|$ is a reduced curve with only *planar* singularities, and X itself is nonsingular outside of P (this is all familiar and easy: if $f: Y\to X$ is the blow up of X along C, then Y has isolated singularities outside $f^{-1}(P)$).

By our generality assumption 2.2(b), and using the notation of the proof of Theorem 3.8, we have that $\Gamma_1^Z\cap E_1^Z\cap E_2^Z=\emptyset$. Also, $S^Z_{1|E_i}$ is nonsingular away from $E_1^Z\cap E_2^Z$. Therefore *either* the set theoretic intersection $\Gamma\cap\Gamma_1$ contains a nonsingular point of S_1, *or* $\Gamma^Z\cap\Gamma_1^Z$ contains a nonsingular point of S_1^Z. In both cases this point contributes with an integer value ≥ 1 to the intersection number $\Gamma\cdot\Gamma_1$, hence our claim that $\Gamma\cdot\Gamma_1\geq 1$. This finishes the proof in Cases (a), (b).

In Case (c), Γ and Γ_1 are both conics and

$$2(1-\gamma_1) = (\mathcal{H}-\gamma_1 S_2)\cdot\Gamma_1 \geq (\gamma-\gamma_1)\Gamma\cdot\Gamma_1.$$

It is easy to see that Γ^Z and Γ_1^Z intersect in at least 2 nonsingular points on S_1^Z, and from this conclude that, in this case also, $\gamma\leq 1$ (the details are very similar to Cases (a) and (b) and left to the reader).

It remains to show that Cases (d), (e) can not occur, that is, $C=\Pi\cap X$ is always reduced when X is general.

Claim. If X is general, every plane section of X is reduced.

This is a fairly easy exercise. In coordinates X is given by

$$x_0^2x_1x_2 + x_0a_3 + b_4 = 0$$

Where $a_3=a_3(x_1,\dots,x_4)$ and $b_4=b_4(x_1,\dots,x_4)$ are a homogeneous cubic and quartic not involving x_0. The singular point $P\in X$ is of course the coordinate point $(1,0,0,0,0)$. Consider the projection $\pi: X \dashrightarrow \mathbb{P}^3$ on $\mathbb{P}^3$ with homogeneous coordinates $x_1,\dots,x_4$; it is a generically 2-to-1 map, which is to say that the equation of X is *quadratic* in the variable x_0. Now $\Pi = \pi^{-1}\ell$ for a unique line $\ell \subset \mathbb{P}^3$ and it is almost immediate that the hyperplane section $\Pi \cap X$ is nonreduced if, and only if, either one of the following happens:

(a) The line ℓ is contained in the discriminant surface $x_1x_2b_4 - a_3^2 = 0$. It is very easy to see that, for a general choice of a, b, this surface contains no lines.

(b) The line ℓ is contained in the plane $x_1 = x_2$ *and* a_3, b_4, when restricted to ℓ, both have a double root at $x_1 = x_2 = 0$. In any event, this is ruled out by condition 2.2(b).

$\square$

7. Centers on $Y_{3,4}$

In this section we study maximal centers on $Y = Y_{3,4}$. We show first that no curve on Y can be a maximal center, Theorem 7.1, then that a nonsingular point can not be a maximal center, Theorem 7.2.

Theorem 7.1. *No curve on Y can be a maximal center.*

Proof. We can choose weighted projective coordinates $(x_1, x_2, x_3, x_4, y_1, y_2)$ such that the equations of Y are as follows:

$$Y : \begin{cases} y_1y_2 + b_4(x_1, x_2, x_3, x_4) & = 0 \\ y_1x_1 + y_2x_2 + a_3(x_1, x_2, x_3, x_4) & = 0 \end{cases}$$

To understand the proof, it helps to know some explicit features of the geometry of Y. To begin with, Y is nonsingular apart from two $\mathbb{Z}/2\mathbb{Z}$-points q_1, q_2 at $(0,0,0,0,1,0)$ and $(0,0,0,0,0,1)$. Denote $\rho_i : Y \dashrightarrow \mathbb{P}(1^4, 2)$ the projection from $q_i \in Y$; it can be useful to know that the image of ρ_1, for example, is the hypersurface $x_2y_2^2 + a_3y_2 - x_1b_4 = 0$, as can be readily calculated eliminating the variable y_1 from the equations of Y. Also, denote $\pi : \mathbb{P}(1,1,1,1,2,2) \dashrightarrow \mathbb{P}^3$ the projection on the coordinates of degree 1.

The curves of Y, contracted by ρ_1, are the 12 curves ℓ with $\deg \mathcal{O}_\ell(1) = 1/2$ given by $x_1 = a_3 = b_4 = 0$. Similarly, the curves of Y, contracted by ρ_2, are the 12 curves ℓ with $\deg \mathcal{O}_\ell(1) = 1/2$ given by $x_2 = a_3 = b_4 = 0$. Finally, the curves contracted by π are the 24 just mentioned, plus the 3 curves C with $\deg \mathcal{O}_C(1) = 1$ given by $x_1 = x_2 = a_3 = 0$; under the generality assumption 2.2(b) these are irreducible.

Assume that the curve $\Gamma = C_X(E)$ is the center of a maximal singularity E of a mobile linear system $\mathcal{H} \subset |\mathcal{O}(n)|$. By Corollary 3.3, Γ is contained in the nonsingular locus of X. Denote $d = \deg \mathcal{O}_\Gamma(1)$, $m = \operatorname{mult}_\Gamma \mathcal{H} > n$.

Choosing general members H_1, H_2 of $\mathcal{H}$ and intersecting with a general hyperplane section S we obtain

$$3n^2 = H_1 \cdot H_2 \cdot S \geq m^2 d.$$

This implies that $d \leq 2$. We treat the two cases $d = 2$, $d = 1$ separately.
CASE $d = 2$. Here $\pi(\Gamma)$ is either a line or a conic in $\mathbb{P}^3$. In either case Γ is a nonsingular rational curve and Γ is defined scheme theoretically by the quartics (with the natural embedding of $\mathbb{P}(1^4, 2^2)$ in $\mathbb{P}^{11}$ the curve Γ is a normal quartic). Let $S \in |I_{\Gamma,Y}(4)|$ be a general member; write as usual

$$A = \frac{1}{n}\mathcal{H}_{|S} = L + \gamma\Gamma,$$

(with $L = (1/n)\mathcal{L}$ nef...). We easily calculate on S that $\Gamma^2 = -8$, and

$$0 \leq L^2 = 12 - 4\gamma - 8\gamma^2.$$

This implies that $\gamma \leq 1$ and finishes this case.
CASE $d = 1$. Here $\pi(\Gamma) \subset \mathbb{P}^3$ is a line, Γ is a nonsingular rational curve. Denote S_1, S_2 two general members of the pencil $|I_{\Gamma,Y}(1)|$, $C = \operatorname{Bs}|S_1 \cap S_2|$ the base locus. Denoting $\Pi = \pi^{-1}\pi\Gamma \cong \mathbb{P}(1,1,2,2)$, we can also say that $C = \Pi \cap Y$.

In the end we will show that C is reduced; for now let us assume it. We restrict to S_1 and write

$$\begin{aligned} A = (1/n)\mathcal{H}_{|S_1} = &\quad L + \gamma\Gamma + \textstyle\sum_{i=1}^r \gamma_i\Gamma_i \\ S_{2|S_1} = &\qquad C = \Gamma + \textstyle\sum \Gamma_i \end{aligned}$$

When C is reduced, it is clear that if W is a component of C, $\operatorname{mult}_W \mathcal{H} = \operatorname{mult}_W \mathcal{H}_{|S_1}$; in particular $\operatorname{mult}_\Gamma \mathcal{H} = \operatorname{mult}_\Gamma \mathcal{H}_{|S_1} = n\gamma$, and also $\operatorname{mult}_{\Gamma_i} \mathcal{H} = n\gamma_i$. Because Γ is a *maximal* singularity, $\gamma \geq \gamma_i$ for all i, hence possibly after relabelling components of C, we can assume that $\gamma \geq \gamma_r \geq ... \geq \gamma_1$. Consider the effective $\mathbb{Q}$-divisor

$$M = (A - \gamma_1 S_2)_{|S_1} = L + (\gamma - \gamma_1)\Gamma + \sum_{i>1}(\gamma_i - \gamma_1)\Gamma_i.$$

We calculate the intersection product, on S_1, with Γ_1:

$$M \cdot \Gamma_1 = (1 - \gamma_1)\deg \mathcal{O}_{\Gamma_1}(1) \geq (\gamma - \gamma_1)\Gamma \cdot \Gamma_1$$

Note that here $1/2 \leq \deg \mathcal{O}_{\Gamma_1}(1) \leq 2$ is a half-integer. It is completely elementary to check that $\Gamma \cdot \Gamma_1 \geq \deg \mathcal{O}_{\Gamma_1}(1)$ (in doing this, it helps to note that S_1 is nonsingular outside q_1, q_2). Together with the last displayed equation this implies that $\gamma \leq 1$ and finishes the proof. It remains to show that C is reduced.

Claim. If X is general C is reduced.

This is a fairly easy exercise; the situation corresponds exactly to the quartic X_4 as treated in the proof of Theorem 6.1. In short, it is easy to see that $C = \Pi \cap Y$ is nonreduced if, and only if, either one of the following happens:

(a) The line $\pi\Gamma$ is contained in the discriminant surface $x_1x_2b_4 - a_3^2 = 0$. It is clear that, for a general choice of a, b, this surface contains no lines.

(b) The line $\pi\Gamma$ is contained in the plane $x_1 = x_2$ *and* a_3, b_4, when restricted to $\pi\Gamma$, both have a double root at $x_1 = x_2 = 0$. In any even this possibility is certainly ruled out by condition 2.2(b).

$\square$

Theorem 7.2. *Let $x \in Y$ be a nonsingular point. Then x is not a maximal center.*

Proof. Let $x \in Y$ be a nonsingular point, $B = \operatorname{Bs}|I_x(1)|$. If $\dim B = 0$ consider a general element $S \in |I_x(1)|$. Then $\mathcal{H}_{|S}$ is mobile and

$$3 = \frac{1}{n^2}\mathcal{H}^2 \cdot S < 4.$$

This is enough, by Theorem 5.1, to conclude that x is not a center.

Let us now worry about the case $\dim B = 1$; this can only happen when $x \in B$ is a curve contracted by π, and as we have already noted at the start of the proof of Theorem 7.1, it is a consequence of the generality assumption 2.2(b) that B is irreducible. If $\deg \o_B(1) = 1$ then write $\mathcal{H}_{|S} = \mathcal{L} + mB$ where $\mathcal{L}$ is the mobile part; dividing by n be obtain

$$A = \frac{1}{n}\mathcal{H}_{|S} = L + cB$$

where $L = (1/n)\mathcal{L}$, and $c = m/n$. Note that B is a rational curve on S passing trough 2 simple double points. Therefore $(B \cdot B)_S = -1$ and, computing the self intersection of L, we get

$$L^2 = 3 - c^2 - 2c \leq 4(1 - c).$$

Again by Theorem 5.1 we exclude x as a center.

If $\deg \o_B(1)B = 1/2$ then the above arguments on the surface S are not enough to exclude it as a maximal center. This time we need to consider the linear system $\Delta = |I_x^{\otimes 2}(2)|$.

It is easy to check that:

Claim If $D \in \Delta$ is general, then

(a) D has a simple double point at x,

(b) D is nonsingular along $B \setminus (\operatorname{Sing}(Y) \cup \{x\})$,

(c) D has a singularity of type $1/4(1,-1)$ at $B\cap \operatorname{Sing}(Y)$.

Let $\nu : Y' \to Y$ the blow up of x with exceptional divisor E. Write $\nu^*D = D' + bE$ and $F = E_{|D'}$. D has a double point at x thus $F \subset E$ is a conic. By Shokurov connectedness there is a line $\ell \subset E$ such that

$$\ell \subset LC\Big(Y', K_{Y'} + \frac{1}{n}\mathcal{H}' + \Big(\frac{b}{n} - 1\Big)E + D'\Big).$$

Therefore, for the generic D, by inversion of adjunction $K'_D+((1/n)\mathcal{H}'+(b/n+1)E)_{|D'}$ is not LC at two distinct points p_1 and p_2. We want to use this fact to derive a numerical constraint on $\mathcal{H}$. To do this let us first compute the intersection matrix on D':

$$(F\cdot F)_{D'} = -2,\ \ (B'\cdot B')_{D'} = -\frac{7}{4},\ \ (F\cdot B')_{D'} = 1,$$

(The only nontrivial product is $(B'\cdot B')_{D'} = -2 - 1/2 + 3/4$, by the adjunction formula with correction coming from the different). Write

$$A = \Big(\nu^*\frac{1}{n}\mathcal{H}\Big)_{|D'} = \frac{1}{n}\mathcal{L} + \beta F + \alpha B',$$

where $\mathcal{L}$ is the mobile part. We have

$$(\mathcal{L}/n)^2 = 6 - 2\beta^2 - \frac{7}{4}\alpha^2 - \alpha + 2\beta\alpha. \tag{5}$$

To find a lower bound for $(\mathcal{L}/n)^2$ recall that $K'_D+(\mathcal{H}'/n+(b/n-1)E)_{|D'}$ is not log canonical at p_1 and p_2 therefore by Theorem 5.1, we always have

$$(\mathcal{L}/n)^2 > 4(2-\beta) + 4(2-\beta)(1-\alpha) = 16 - 8\beta - 8\alpha + 4\beta\alpha.$$

Combining with equation 5 yields

$$0 > 2\beta^2 - 2\beta(4-\alpha) - 7\alpha + \frac{7}{4}\alpha^2 + 10,$$

and the discriminant of this quadratic equation with respect to β is

$$\begin{aligned}\Delta/4 &= 16 + \alpha^2 - 8\alpha + 14\alpha - \frac{7}{2}\alpha^2 - 20\\ &= -\frac{5}{2}\alpha^2 + 6\alpha - 4 = -\frac{5}{2}\big(\alpha - \frac{6}{5}\big)^2 - \frac{4}{5} < 0.\end{aligned}$$

This inequality shows that x cannot be a center of maximal singularities and concludes the proof of the Theorem.

$\square$

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