

Motivic decomposition and intersection Chow groups II

Alessio Corti and Masaki Hanamura

This paper is a continuation of [CH], where we formulated the motivic analogue of the decomposition theorem in [BBD]. The decomposition theorem says, if X, S are quasi-projective complex algebraic varieties with X smooth, and $p : X \rightarrow S$ is a projective map, then the direct image of the constant sheaf $Rp_*\mathbb{Q}_X$ is a direct sum of intersection complexes (of local systems on smooth locally closed subvarieties of S) with shifts. The motivic analogue is a conjectural statement that the decomposition be lifted to a decomposition in a suitably defined motivic category. In [CH] we defined the category of Chow motives over S , and showed that the existence of the motivic decomposition in this category is a consequence of the conjectures of Grothendieck and of Bloch-Beilinson-Murre.

If the map $p : X \rightarrow S$ is a resolution of singularities, one of the direct summands of $Rp_*\mathbb{Q}_X$ is the intersection complex $IC_S = IC_S(\mathbb{Q})$ of S . The motivic decomposition has a direct summand corresponding to the intersection complex. We will call it the *motivic intersection complex* of S . The Chow group of this object we call the *intersection Chow group* of S , and denote it by $ICH^r(S)$.

The content of this paper is as follows.

(1) In §3 we give an account of this theory, *under the conjectures of Grothendieck and of Bloch-Beilinson-Murre*. The definition of intersection Chow group $ICH^r(S)$ of a quasi-projective variety S rests on the existence of the motivic decomposition for a desingularization $p : X \rightarrow S$. The group $ICH^r(S)$ is a canonical subquotient of the Chow group $CH^r(X)$. We then derive a formula (3.9) for the intersection Chow group in terms of the Chow groups of X and of the exceptional loci of p . These Chow groups have filtrations denoted $F_{\mathcal{S}}^{\bullet}$, which appear in the formula. The filtration has to do with the perverse Leray filtration on objects in the motivic category, which is defined using the motivic decomposition.

(2) In §4 we give an unconditional definition of intersection Chow group. Here “unconditional” means independent of any conjectures. For this we take the formula mentioned above, and turn it into the definition. We need to define the filtrations $F_{\mathcal{S}}^{\bullet}$ on Chow groups, without assuming the existence of motivic decomposition. This can be done, using cohomology realizations, as Shuji Saito did for the case $S = \text{Spec } k$. We can show the intersection Chow group is well-defined, independent of the choice of a resolution.

§4 was inspired by §3, but it is logically independent. We do not even need the motivic category of Chow motives in §4.

(3) There is an analogous formula for the intersection cohomology of S , in terms of the cohomology of the exceptional loci of a desingularization, see (2.4). This is discussed in §2.

In the summer of 1996 the second author had a chance to communicate the present work to Bob MacPherson. On that occasion we had conversations on the motivic analogue of the lifting theorem [BBFFK], which later developed into [Ha-2]. We would like to take the present

opportunity to thank Bob cordially for these discussions and for his profound influence on our work.

Throughout this paper we consider quasi-projective varieties over $k = \mathbb{C}$. The Chow group of a quasi-projective variety X , *tensored with* \mathbb{Q} , is denoted $\text{CH}(X)$.

Let $D_c^b(S) = D_c^b(S(\mathbb{C}), \mathbb{Q})$ denote the derived category of sheaves of \mathbb{Q} -vector spaces on $S(\mathbb{C})$ with cohomology sheaves bounded and constructible. The cohomology of X is always with \mathbb{Q} -coefficients: $H^i(X) = H^i(X, \mathbb{Q})$. For intersection complex and perverse sheaves, we always take the middle perversity and \mathbb{Q} as coefficients: $IH^i(X) = IH^i(X, \mathbb{Q})$. We refer to [GM-1], [BBD] and [Bo] for expositions on intersection complexes and perverse sheaves; see [CH] for additional information. $\text{Perv}(S)$ denotes the category of perverse sheaves on $S(\mathbb{C})$.

§1. Stratification of a projective map and the decomposition theorem.

(1.1) **Definition.** Let S be an irreducible quasi-projective variety over \mathbb{C} . An algebraic *Whitney stratification* $\mathcal{S} = \{S_\alpha\}$ of S is a filtration of S by closed sets

$$S = S_0 \supset S_1 \supset \cdots \supset S_\alpha \supset \cdots \supset S_{\dim S}$$

such that $S_\alpha - S_{\alpha+1}$ are smooth of codimension α (or empty) satisfying Whitney's conditions A and B (see [GM-2, Chap.I] for details).

Let X be a quasi-projective variety and $p : X \rightarrow S$ be a projective map. $p : X \rightarrow S$ is a *stratified map* over \mathcal{S} if there is a Whitney stratification Σ of X such that p is a stratified map with respect to Σ and \mathcal{S} ([GM-2, p.42]). In particular, p is a stratified fiber bundle over each stratum $S_\alpha^0 := S_\alpha - S_{\alpha+1}$.

Let $X_\alpha = p^{-1}(S_\alpha)$, $p_\alpha : X_\alpha \rightarrow S_\alpha$ the induced map, and $i_\alpha : S_\alpha \rightarrow S$, $k_\alpha : X_\alpha \rightarrow X$ be the closed immersions. Let $X_\alpha^0 = p^{-1}(S_\alpha^0)$ and $p_\alpha^0 : X_\alpha^0 \rightarrow S_\alpha^0$ be the induced map. For $\alpha = 0$, we will drop the subscript as follows: $S^0 = S_0^0 = S - S_1$, $X^0 = X_0^0 = X - X_1$, and $p^0 = p_0^0 : X^0 \rightarrow S^0$. We thus have a commutative diagram:

$$\begin{array}{ccccc} X & \xleftarrow{k_\alpha} & X_\alpha & \xleftarrow{\quad} & X_\alpha^0 \\ p \downarrow & & p_\alpha \downarrow & & p_\alpha^0 \downarrow \\ S & \xleftarrow{i_\alpha} & S_\alpha & \xleftarrow{\quad} & S_\alpha^0 \end{array}$$

Given a projective map $p : X \rightarrow S$, there is a Whitney stratification \mathcal{S} on S over which p is stratified.

(1.2) **Definition.** Let X, S be quasi-projective varieties, with X *smooth*, and $p : X \rightarrow S$ a projective map. Let $\mathcal{S} = \{S_\alpha\}$ be a Whitney stratification of S over which p is stratified.

A *resolution* of $p : X \rightarrow S$ over \mathcal{S} is a collection $\{\pi_\alpha : \tilde{X}_\alpha \rightarrow X_\alpha\}$ consisting of smooth quasi-projective varieties \tilde{X}_α and *projective surjective* maps π_α , for $\alpha \geq 1$.

Let $\iota_\alpha = k_\alpha \circ \pi_\alpha : \tilde{X}_\alpha \rightarrow X$ and $q_\alpha = p_\alpha \circ \pi_\alpha : \tilde{X}_\alpha \rightarrow S_\alpha$.

$$\begin{array}{ccccc} & & & & \tilde{X}_\alpha \\ & & & \swarrow & \downarrow \pi_\alpha \\ X & \xleftarrow{\quad} & X_\alpha & & \\ p \downarrow & & p_\alpha \downarrow & & \\ S & \xleftarrow{\quad} & S_\alpha & & \end{array}$$

Given a projective map $p : X \rightarrow S$ with smooth X , stratified over \mathcal{S} , there exists a resolution over \mathcal{S} .

Remark. We may require π_α be desingularizations, rather than projective surjective maps. For purposes of later sections, however, it is more convenient to allow projective surjective maps.

The following is known as the decomposition theorem, [BBD]. In the statement, ${}^p R^i p_* \mathbb{Q}_X = {}^p \mathcal{H}^i R^i p_* \mathbb{Q}_X$, where ${}^p \mathcal{H}^i$ is perverse cohomology, and $IC_{S_\alpha}(\mathcal{V})$ is the intersection complex of a local system \mathcal{V} .

(1.3) **Theorem.** *Let X be smooth and $p : X \rightarrow S$ a projective map, stratified over $\mathcal{S} = \{S_\alpha\}$. Then there is a non-canonical direct sum decomposition*

$$Rp_* \mathbb{Q}_X = \bigoplus_i {}^p R^i p_* \mathbb{Q}_X[-i]$$

and a unique direct sum decomposition

$${}^p R^i p_* \mathbb{Q}_X = \bigoplus_\alpha IC_{S_\alpha}(\mathcal{V}_\alpha^i)[\dim S_\alpha] ,$$

where \mathcal{V}_α^i is a local system on S_α^0 . One thus has a direct sum decomposition

$$Rp_* \mathbb{Q}_X = \bigoplus_{i,\alpha} IC_{S_\alpha}(\mathcal{V}_\alpha^i)[-i + \dim S_\alpha] .$$

Remark. One has $\mathcal{V}_0^i = R^{i-\dim S} p_*^0 \mathbb{Q}_{X^0}$.

(1.4) **Proposition.** *Let X be smooth and $p : X \rightarrow S$ a projective map, stratified over $\{S_\alpha\}$. Keep the notation in (1.1) and (1.3).*

(1) Let

$$k_\alpha^* : Rp_* \mathbb{Q}_X \rightarrow i_{\alpha*} Rp_{\alpha*} \mathbb{Q}_{X_\alpha}$$

be the map induced by k_α ; upon applying the functor ${}^p \mathcal{H}^i$, one has a map

$${}^p \mathcal{H}^i(k_\alpha^*) : {}^p R^i p_* \mathbb{Q}_X \rightarrow i_{\alpha*} {}^p R^i p_{\alpha*} \mathbb{Q}_{X_\alpha} .$$

The restriction of this to $\bigoplus_{S_\beta \subset S_\alpha} IC_{S_\beta}(\mathcal{V}_\beta^i)[\dim S_\beta]$,

$$\bigoplus_{S_\beta \subset S_\alpha} IC_{S_\beta}(\mathcal{V}_\beta^i)[\dim S_\beta] \rightarrow i_{\alpha*} {}^p R^i p_{\alpha*} \mathbb{Q}_{X_\alpha} ,$$

is a split injection.

(2) Let

$$k_{\alpha*} : i_{\alpha*} Rp_{\alpha*} D_{X_\alpha}[-2d] \rightarrow Rp_* \mathbb{Q}_X$$

be the map induced by k_α (D_{X_α} is the dualizing complex of X_α); upon applying ${}^p \mathcal{H}^i$, one has

$${}^p \mathcal{H}^i(k_{\alpha*}) : i_{\alpha*} {}^p \mathcal{H}^i(Rp_{\alpha*} D_{X_\alpha}[-2d]) \rightarrow {}^p R^i p_* \mathbb{Q}_X .$$

Composition of this with the quotient map to $\bigoplus_{S_\beta \subset S_\alpha} IC_{S_\beta}(\mathcal{V}_\beta^i)[\dim S_\beta]$,

$${}^p \mathcal{H}^i(k_{\alpha*}) : i_{\alpha*} {}^p \mathcal{H}^i(Rp_{\alpha*} D_{X_\alpha}[-2d]) \rightarrow \bigoplus_{S_\beta \subset S_\alpha} IC_{S_\beta}(\mathcal{V}_\beta^i)[\dim S_\beta] ,$$

is a split surjection.

Proof. (1) The map k_α^* is the adjunction map $Rp_*\mathbb{Q}_X \rightarrow i_{\alpha*}i_\alpha^*Rp_*\mathbb{Q}_X$, with identification $i_\alpha^*Rp_*\mathbb{Q}_X = Rp_{\alpha*}\mathbb{Q}_{X_\alpha}$. Take a decomposition $Rp_*\mathbb{Q}_X = \bigoplus IC_{S_\alpha}(\mathcal{V}_\alpha^i)[-i + \dim S_\alpha]$. Examine the adjunction map on each summand to obtain the proof.

(2) is dual to (1).

(1.5) **Proposition.** *Let X be smooth and $p : X \rightarrow S$ a projective map, stratified over $\{S_\alpha\}$, and $\{\pi_\alpha : \tilde{X}_\alpha \rightarrow X_\alpha\}$ its resolution over $\{S_\alpha\}$. Keep the notation in (1.2) and (1.3).*

(1) *Let $\iota_\alpha^* : Rp_*\mathbb{Q}_X \rightarrow i_{\alpha*}Rq_{\alpha*}\mathbb{Q}_{\tilde{X}_\alpha}$ be the map ι_α induces; applying ${}^p\mathcal{H}^i$, one has a map*

$${}^p\mathcal{H}^i(\iota_\alpha^*) : {}^pR^i p_*\mathbb{Q}_X \rightarrow i_{\alpha*} {}^pR^i q_{\alpha*}\mathbb{Q}_{\tilde{X}_\alpha} .$$

The restriction of this map to the direct summand $IC_{S_\alpha}(\mathcal{V}_\alpha^i)[\dim S_\alpha]$,

$${}^p\mathcal{H}^i(\iota_\alpha^*) : IC_{S_\alpha}(\mathcal{V}_\alpha^i)[\dim S_\alpha] \rightarrow i_{\alpha*} {}^pR^i q_{\alpha*}\mathbb{Q}_{\tilde{X}_\alpha} ,$$

is a split injection.

(2) *Let $i_{\alpha*} : i_{\alpha*}Rq_{\alpha*}D_{\tilde{X}_\alpha}[-2d] \rightarrow Rp_*\mathbb{Q}_X$ be the map ι_α induces; applying ${}^p\mathcal{H}^i$ one has*

$${}^p\mathcal{H}^i \iota_{\alpha*} : i_{\alpha*} {}^p\mathcal{H}^i Rq_{\alpha*}D_{\tilde{X}_\alpha}[-2d] \rightarrow {}^pR^i p_*\mathbb{Q}_X .$$

The composition of this with the quotient map to $IC_{S_\alpha}(\mathcal{V}_\alpha^i)[\dim S_\alpha]$,

$${}^p\mathcal{H}^i(\iota_{\alpha*}) : i_{\alpha*} {}^p\mathcal{H}^i R\tilde{p}_{\alpha*}D_{\tilde{X}_\alpha}[-2d] \rightarrow IC_{S_\alpha}(\mathcal{V}_\alpha^i)[\dim S_\alpha] ,$$

is a split surjection.

Proof. (1) The maps $\tilde{X}_\alpha \rightarrow X_\alpha \rightarrow X$ induce the maps

$$Rp_*\mathbb{Q}_X \rightarrow i_{\alpha*}Rp_{\alpha*}\mathbb{Q}_{X_\alpha} \rightarrow i_{\alpha*}Rq_{\alpha*}\mathbb{Q}_{\tilde{X}_\alpha} .$$

Applying ${}^p\mathcal{H}^i$, one has

$${}^pR^i p_*\mathbb{Q}_X \rightarrow i_{\alpha*} {}^pR^i p_{\alpha*}\mathbb{Q}_{X_\alpha} \rightarrow i_{\alpha*} {}^pR^i q_{\alpha*}\mathbb{Q}_{\tilde{X}_\alpha} .$$

According to the theory of weights [BBD],[SaM], the three perverse sheaves have weight filtrations; ${}^pR^i p_*\mathbb{Q}_X$ and $R^i q_{\alpha*}\mathbb{Q}_{\tilde{X}_\alpha}$ are of pure weight i , and $R^i p_{\alpha*}\mathbb{Q}_{X_\alpha}$ is of weight $\leq i$ since p_α is proper. Taking Gr_i^W , one has maps in a semi-simple abelian category.

$${}^pR^i p_*\mathbb{Q}_X \rightarrow i_{\alpha*} \text{Gr}_i^W {}^pR^i p_{\alpha*}\mathbb{Q}_{X_\alpha} \rightarrow i_{\alpha*} {}^pR^i q_{\alpha*}\mathbb{Q}_{\tilde{X}_\alpha} .$$

By Proposition (1.4), the map $IC_{S_\alpha}(\mathcal{V}_\alpha^i)[\dim S_\alpha] \rightarrow i_{\alpha*} \text{Gr}_i^W {}^pR^i p_{\alpha*}\mathbb{Q}_{X_\alpha}$ is a split injection.

One has only to show $i_{\alpha*} \text{Gr}_i^W {}^pR^i p_{\alpha*}\mathbb{Q}_{X_\alpha} \rightarrow i_{\alpha*} {}^pR^i q_{\alpha*}\mathbb{Q}_{\tilde{X}_\alpha}$ is a split injection when restricted to S_α^0 ; equivalently, letting $(X_\alpha)_s = p_\alpha^{-1}(s)$ and $(\tilde{X}_\alpha)_s = q_\alpha^{-1}(s)$ for $s \in S_\alpha^0$, the map

$$\text{Gr}_i^W H^{i-\dim S_\alpha}((X_\alpha)_s) \rightarrow H^{i-\dim S_\alpha}((\tilde{X}_\alpha)_s)$$

is an injection. This is [De, part II, Proposition 8.2.5].

(2) Dual to (1).

§2. Intersection cohomology of projective maps.

In this section X, S are quasi-projective varieties, X is smooth, and $p : X \rightarrow S$ a projective map.

(2.1) Recall $D_c^b(S)$ is the bounded derived category of constructible sheaves. There is the perverse t -structure on this, in particular the functors ${}^p\tau_{\leq}$ and ${}^p\tau_{\geq}$. For simplicity, denote them by τ_{\leq} and τ_{\geq} .

For the object $Rp_*\mathbb{Q}_X$, Theorem (1.3) implies $\tau_{\leq k}Rp_*\mathbb{Q}_X$ is a subobject, and non-canonically a direct summand. One thus has a filtration by subobjects.

This filtration induces a filtration on $H^a(X)$, as follows. Let

$$F_S^\nu H^a(X) := \text{Hom}_{D_c^b(S)}(\mathbb{Q}_S, \tau_{\leq -\nu}(Rp_*\mathbb{Q}_X[a])) ,$$

it is a subspace of $\text{Hom}_{D_c^b(S)}(\mathbb{Q}_S, Rp_*\mathbb{Q}_X[a]) = H^a(X)$, with a non-canonical splitting. The decreasing filtration F_S^\bullet on $H^a(X)$ thus defined has the following properties.

- (1) $F_S^{-\dim S} H^a(X) = H^a(X)$.
- (2) For ν large enough, $F_S^\nu H^a(X) = 0$.
- (3) The graded pieces in the filtration are

$$\text{Gr}_{F_S}^\nu H^a(X) = F_S^\nu / F_S^{\nu+1} H^a(X) = \text{Hom}(\mathbb{Q}_S, {}^pR^{a-\nu}p_*\mathbb{Q}_X[\nu]) .$$

If p is stratified over $\{S_\alpha\}$, with the notation in Theorem (1.3) this is equal to

$$\text{Hom}(\mathbb{Q}_S, \bigoplus_\alpha IC_{S_\alpha}(\mathcal{V}_\alpha^{a-\nu})[\dim S_\alpha][\nu]) = \bigoplus_\alpha IH^{\nu+\dim S_\alpha}(S_\alpha, \mathcal{V}_\alpha^{a-\nu}) .$$

(2.2) Assume p is stratified over $\{S_\alpha\}$, and $\{\pi_\alpha : \tilde{X}_\alpha \rightarrow X_\alpha\}$ its resolution over $\{S_\alpha\}$. For each $\alpha \geq 1$ the map ι_α induces maps

$$\iota_\alpha^* : H^a(X) \rightarrow H^a(\tilde{X}_\alpha)$$

and

$$\iota_{\alpha*} : H_{2\dim X - a}^{BM}(\tilde{X}_\alpha) \rightarrow H^a(X) .$$

Here H_*^{BM} denotes Borel-Moore homology. Taking graded pieces for F_S^\bullet , one has maps

$$\iota_\alpha^* : \text{Gr}_{F_S}^\nu H^a(X) \rightarrow \text{Gr}_{F_S}^\nu H^a(\tilde{X}_\alpha) \quad \text{and} \quad \iota_{\alpha*} : \text{Gr}_{F_S}^\nu H_{2\dim X - a}^{BM}(\tilde{X}_\alpha) \rightarrow \text{Gr}_{F_S}^\nu H^a(X) .$$

The next Proposition follows from Proposition (1.5).

(2.3) **Proposition.** (1) The kernel of the map $\sum_{\alpha \geq 1} \iota_\alpha^* : \text{Gr}_{F_S}^\nu H^a(X) \rightarrow \bigoplus_{\alpha \geq 1} \text{Gr}_{F_S}^\nu H^a(\tilde{X}_\alpha)$ is equal to $IH^{\nu+\dim S}(S, \mathcal{V}_0^{a-\nu})$.

(2) The image of the map $\sum_{\alpha \geq 1} \iota_{\alpha*} : \bigoplus_{\alpha \geq 1} \text{Gr}_{F_S}^\nu H_{2\dim X - a}^{BM}(\tilde{X}_\alpha) \rightarrow \text{Gr}_{F_S}^\nu H^a(X)$ is equal to $\bigoplus_{\alpha \geq 1} IH^{\nu+\dim S_\alpha}(S_\alpha, \mathcal{V}_\alpha^{a-\nu})$.

In the case p is birational, we can describe the intersection cohomology of S in terms of the cohomology of X and $\{\tilde{X}_\alpha\}$, and the filtrations F_S^\bullet .

(2.4) **Theorem.** If $p : X \rightarrow S$ is a birational map, $d = \dim S$,

$$IH^a(S) = \frac{\bigcap_{\alpha \geq 1} (\iota_\alpha^*)^{-1} F_S^{a-d+1} H^a(\tilde{X}_\alpha)}{\sum_{\alpha \geq 1} \iota_{\alpha*} F_S^{a-d+1} H_{2d-a}^{BM}(\tilde{X}_\alpha)} .$$

We omit the proof, which is similar to the proof of an analogous formula for the intersection Chow group, Theorem (3.9).

§3. The intersection Chow group (under hypotheses).

(3.1) Let S be a quasi-projective variety over $k = \mathbb{C}$. Denote by $(Smooth/k, Proj/S)$ the category of smooth quasi-projective varieties X equipped with projective maps to S , $p : X \rightarrow S$.

For X, Y in $(Smooth/k, Proj/S)$, $CH_a(X \times_S Y)$ denotes the rational Chow group of dimension a of the variety $X \times_S Y$. An element of this group is a *relative correspondence* from X to Y .

If X, Y, Z are in $(Smooth/k, Proj/S)$, with Y equi-dimensional, we have a map, the *composition* of correspondences,

$$CH_a(X \times_S Y) \otimes CH_b(Y \times_S Z) \rightarrow CH_{a+b-\dim Y}(X \times_S Z)$$

which sends $u \otimes v$ to $v \circ u$, see [CH] for the definition. The composition is associative. In particular if X has connected components X_i , $\bigoplus_i CH_{\dim X_i}(X \times_S X_i)$ is a ring with the composition as multiplication. The identity element is the class of the diagonal $\Delta_X = id$.

Let $CHM(S)$ be the pseudo-abelian category of Chow motives over S , defined in [CH]. It has the following properties.

(1) An object of $CHM(S)$ is of the form

$$(X, r, P) = (X/S, r, P)$$

where X is a smooth variety over k with a projective (not necessarily smooth) map $p : X \rightarrow S$, $r \in \mathbb{Z}$, and if X has connected components X_i ,

$$P \in \bigoplus_i CH_{\dim X_i}(X \times_S X_i)$$

such that $P \circ P = P$. If (Y, s, Q) is another object, Y_j the components of Y , then

$$\text{Hom}((X, r, P), (Y, s, Q)) = Q \circ \left(\bigoplus_j CH_{\dim Y_j - s + r}(X \times_S Y_j) \right) \circ P .$$

Composition of morphisms is induced from the composition of relative correspondences.

Denote by $M = (X, r, P) \mapsto M(n) = (X, r + n, P)$ the ‘‘Tate twist’’ functor.

(2) There is a functor $h : (Smooth/k, Proj/S)^{opp} \rightarrow CHM(S)$, which sends $p : X \rightarrow S$ to the object $h(X/S) = (X/S, 0, id)$. Note $h(X/S)(n) = (X/S, n, id)$.

If X and Y are objects of $(Smooth/k, Proj/S)$ and $f : X \rightarrow Y$ is a map over S , there corresponds a morphism

$$f^* : h(Y/S) \rightarrow h(X/S) .$$

If X, Y are equidimensional, there corresponds

$$f_* : h(X/S) \rightarrow h(Y/S)(\dim Y - \dim X) .$$

(3) There is a functor

$$CH^0(S, -) : CHM(S) \rightarrow Vect_{\mathbb{Q}}$$

(the target is the category of \mathbb{Q} -vector spaces) such that $\mathrm{CH}^0(S, (X, r, P)) = \mathrm{CH}^0((X, r, P)) = P_* \mathrm{CH}^r(X)$.

Define $\mathrm{CH}^t(S, -) : \mathrm{CHM}(S) \rightarrow \mathrm{Vect}_{\mathbb{Q}}$ by $\mathrm{CH}^t(S, K) = \mathrm{CH}^t(K) = \mathrm{CH}^0(K(t))$. Note $\mathrm{CH}^r(h(X/S)) = \mathrm{CH}^0(h(X/S)(r)) = \mathrm{CH}^r(X)$.

(4) There is the realization functor

$$\rho : \mathrm{CHM}(S) \rightarrow D_c^b(S)$$

such that on objects

$$(X, r, P) \mapsto P_* R p_* \mathbb{Q}_X[2r] .$$

Here $P_* := \rho(P) \in \mathrm{End}_{D_c^b(S)}(R p_* \mathbb{Q}_X)$ is a projector, and $P_* R p_* \mathbb{Q}_X$ is its image, which exists since $D_c^b(S)$ is pseudo-abelian.

(3.2) For $p : X \rightarrow S$ in $(\mathrm{Smooth}/k, \mathrm{Proj}/S)$ and $r \in \mathbb{Z}$, let

$${}^p \mathcal{H}^*(X/S, r) := \bigoplus_i {}^p R^{i+2r} p_* \mathbb{Q}_X ,$$

called the *total perverse cohomology*, a graded perverse sheaf (grading by i). Denote the category of graded perverse sheaves by $gr \mathrm{Perv}(S)$. One has a map

$$\mathrm{Hom}_{\mathrm{CHM}(S)}((X/S, r, id), (Y/S, s, id)) \rightarrow \mathrm{Hom}_{gr \mathrm{Perv}(S)}({}^p \mathcal{H}^*(X/S, r), {}^p \mathcal{H}^*(Y/S, s)) ,$$

obtained using the functor ρ and perverse cohomology. The image of this map is denoted by $\mathrm{Hom}_{gr \mathrm{Perv}(S)}({}^p \mathcal{H}^*(X/S, r), {}^p \mathcal{H}^*(Y/S, s))_{alg}$. It is proved in [CH] that this group is closed under composition.

The pseudo-abelian category of Grothendieck motives over S , denoted by $\mathcal{M}(S)$ has objects $(X/S, r, p)$ where X/S is in $(\mathrm{Smooth}/k, \mathrm{Proj}/S)$, and $p \in \mathrm{End}({}^p \mathcal{H}^*(X/S, r))_{alg}$ is an idempotent. Morphisms are defined by

$$\mathrm{Hom}((X, r, p), (Y, s, q)) = q \circ \mathrm{Hom}({}^p \mathcal{H}^*(X/S, r), {}^p \mathcal{H}^*(Y/S, s))_{alg} \circ p .$$

There is a canonical full functor $\mathrm{cano} : \mathrm{CHM}(S) \rightarrow \mathcal{M}(S)$ and a faithful realization functor $\rho : \mathcal{M}(S) \rightarrow gr \mathrm{Perv}(S)$. The following diagram commutes.

$$\begin{array}{ccc} \mathrm{CHM}(S) & \xrightarrow{\mathrm{cano}} & \mathcal{M}(S) \\ \rho \downarrow & & \downarrow \rho \\ D_c^b(S) & \xrightarrow{{}^p \mathcal{H}^*} & gr \mathrm{Perv}(S) \end{array}$$

Here ${}^p \mathcal{H}^* = \bigoplus_i {}^p \mathcal{H}^i$ is the total perverse cohomology functor.

(3.3) **Theorem.** [CH, §7] *Assume the conjecture of Grothendieck and the conjecture of Bloch-Beilinson-Murre. Let $p : X \rightarrow S$ be as before. Let $\{S_\alpha\}$ be a Whitney stratification of S over which p is stratified. Then:*

(1) *There are local systems \mathcal{V}_α^j on $S_\alpha - S_{\alpha+1}$, non-canonical direct sum decomposition in $\mathrm{CHM}(S)$*

$$h(X/S) = \bigoplus_{j, \alpha} h_\alpha^j(X/S)$$

and isomorphisms

$$\rho(h_\alpha^j(X/S)) \cong IC_{S_\alpha}(\mathcal{V}_\alpha^j)[-j + \dim S_\alpha]$$

in $D_c^b(S)$.

(2) For each i , the sum $\bigoplus_{j \leq i, \alpha} h_\alpha^j(X/S)$ is a well-defined subobject of $h(X/S)$ (independent of the decomposition).

(3) The category $\mathcal{M}(S)$ is semi-simple abelian, and the functor $\rho : \mathcal{M}(S) \rightarrow \text{Perv}(S)$ is exact and faithful.

(4) For $i \in \mathbb{Z}$, let $CHM(S)^i$ (resp. $\mathcal{M}(S)^i$) be the full subcategory of $CHM(S)$ (resp. $\mathcal{M}(S)$) consisting of objects with realizations of pure perverse degree i . Then the canonical functor $\text{cano} : CHM(S)^i \rightarrow \mathcal{M}(S)^i$ is an equivalence of categories.

For the rest of this section, we assume the conjecture of Grothendieck and the conjecture of Bloch-Beilinson-Murre.

(3.4) Let $p : X \rightarrow S$ be as above. Define a subobject ${}^p\tau_{\leq i}$ of $h(X/S)(r)$ by:

$$\tau_{\leq i}(h(X/S)(r)) := \bigoplus_{j \leq i+2r, \alpha} h_\alpha^j(X/S)(r)$$

the sum over (j, α) with $j \leq i + 2r$. This is a subobject with a non-canonical splitting. ${}^p\tau_{\leq i}$ gives an increasing filtration by subobjects. From now we write $\tau_{\leq i}$ for ${}^p\tau_{\leq i}$. The subquotients are

$$\tau_{\leq i}/\tau_{\leq i-1}(h(X/S)(r)) = \bigoplus_{\alpha} h_\alpha^{i+2r}(X/S)(r).$$

This decomposition is uniquely determined, independent of $\{S_\alpha\}$ (this follows from (3.3), (4)).

Correspondingly $\text{CH}^r(X) = \text{CH}^0(S, h(X/S)(r))$ has a decreasing filtration F_S^\bullet defined by

$$F_S^\nu \text{CH}^r(X) = \text{CH}^0(S, \tau_{\leq -\nu}(h(X/S)(r))) \subset \text{CH}^r(X).$$

Note $F_S^\nu \text{CH}^r(X) = \text{CH}^r(X)$ for ν small enough, and $F_S^\nu \text{CH}^r(X) = 0$ for ν large enough. We conjecture $\text{CH}^r(X) = F_S^{-\dim S} \text{CH}^r(X)$. The graded quotients are

$$\text{Gr}_{F_S}^\nu \text{CH}^r(X) = \text{CH}^0(S, \bigoplus_{\alpha} h_\alpha^{2r-\nu}(X/S)(r)) = \text{CH}^r(S, \bigoplus_{\alpha} h_\alpha^{2r-\nu}(X/S)).$$

Each piece $\text{CH}^r(S, h_\alpha^{2r-\nu}(X/S))$ is a direct summand of $\text{Gr}_{F_S}^\nu \text{CH}^r(X)$, in particular a subgroup. Thus one can write

$$\text{CH}^r(S, h_\alpha^{2r-\nu}(X/S)) = A/F_S^{\nu+1}$$

for a subgroup $A \subset F_S^\nu$. With a slight abuse of notation, we write $A = F_S^{\nu+1} + \text{CH}^r(S, h_\alpha^{2r-\nu}(X/S))$.

The filtration $\tau_{\leq i}$ is respected by morphisms in $CHM(S)$, see [CH, Theorem (7.4), (1)]. If $u : h(X/S)(r) \rightarrow h(Y/S)(s)$ is a morphism, there is a unique morphism $\tau_{\leq i} u : \tau_{\leq i} h(X/S)(r) \rightarrow \tau_{\leq i} h(Y/S)(s)$ such that the following diagram commutes.

$$\begin{array}{ccc} h(X/S)(r) & \xrightarrow{u} & h(Y/S)(s) \\ \uparrow & & \uparrow \\ \tau_{\leq i} h(X/S)(r) & \xrightarrow{\tau_{\leq i} u} & \tau_{\leq i} h(Y/S)(s) \end{array}$$

Thus one has induced morphisms $\tau_{\leq i}/\tau_{\leq i-1} h(X/S)(r) \rightarrow \tau_{\leq i}/\tau_{\leq i-1} h(Y/S)(s)$. This is the direct sum, for α , of morphisms $h_\alpha^{i+2r}(X/S)(r) \rightarrow h_\alpha^{i+2r}(Y/S)(s)$.

(3.5) **Definition.** Let X^0 and S^0 be smooth quasi-projective, and $p^0 : X^0 \rightarrow S^0$ be a smooth projective map. Let S be a quasi-projective variety and $S^0 \rightarrow S$ an open immersion.

Take a smooth variety X , an open immersion $X^0 \rightarrow X$ and a projective map $p : X \rightarrow S$ which extends p^0 . Let $h(X/S) = \bigoplus h_\alpha^j(X/S)$ be a decomposition as in Theorem (3.3). Define the *intersection Chow group* of the higher direct image $R^i p_* \mathbb{Q}_{X^0}$ to be

$$\text{ICH}^r(S, R^i p_* \mathbb{Q}_{X^0}) := \text{CH}^0(S, h_0^{i+\dim S}(X/S)(r)) .$$

The group depends on $p^0 : X^0 \rightarrow S^0$, S and i . As we show below, it does not depend on the choice of $p : X \rightarrow S$. One should take $R^i p_* \mathbb{Q}_{X^0}$ as a notation; the intersection Chow group is not determined by S and the local system $R^i p_* \mathbb{Q}_{X^0}$ alone.

(3.6) **Proposition.** (1) The object $h_0^j(X/S)$ is independent of the choice of $p : X \rightarrow S$, up to canonical isomorphism. Hence $\text{ICH}^r(S, R^i p_* \mathbb{Q}_{X^0})$ is well-defined.

(2) Let $S^1 \subset S^0$ be an open set, $X^1 = p^{-1}(S^1)$, and $p^1 : X^1 \rightarrow S^1$ the induced map. Then one has a canonical isomorphism $\text{ICH}^r(S, R^i p_* \mathbb{Q}_{X^0}) = \text{ICH}^r(S, R^i p_*^1 \mathbb{Q}_{X^1})$.

Proof. (1) More precisely if $p' : X' \rightarrow S$ is another extension of p^0 , there is an isomorphism

$$\iota(X, X') : h_0^j(X/S) \rightarrow h_0^j(X'/S) ;$$

If $X'' \rightarrow S$ is another such, the three isomorphisms satisfy the cocycle condition $\iota(X, X'') = \iota(X, X')\iota(X', X'')$.

To prove this, one is reduced to the case where there is a map $f : X' \rightarrow X$ over S , extending the identity on X^0 . Then $f^* : h(X/S) \rightarrow h(X'/S)$ induces an isomorphism $f^* : h_0^j(X/S) \rightarrow h_0^j(X'/S)$.

(2) This is obvious.

(3.7) **Definition.** Let S be an irreducible quasi-projective variety of dimension d , and $p : X \rightarrow S$ its desingularization (proper birational map from a smooth variety). Define the *intersection Chow group* by

$$\text{ICH}^r(S) := \text{CH}^0(S, h_0^d(X/S)) .$$

This is a special case of Definition (3.5), where one takes a smooth open set $S^0 \subset S$, $p^0 = id : X^0 = S^0 \rightarrow S^0$, and $i = 0$. By Proposition (3.6), the intersection Chow group is well-defined.

The map $\iota_\alpha : \tilde{X}_\alpha \rightarrow X$ induces maps $\iota_\alpha^* : \text{CH}^r(X) \rightarrow \text{CH}^r(\tilde{X}_\alpha)$ and $\iota_{\alpha*} : \text{CH}_{\dim X - r}(\tilde{X}_\alpha) \rightarrow \text{CH}^r(X)$, thus also maps between the graded pieces

$$\text{Gr}_{F_S}^\nu \text{CH}_{\dim X - r}(\tilde{X}_\alpha) \xrightarrow{\iota_{\alpha*}} \text{Gr}_{F_S}^\nu \text{CH}^r(X) \xrightarrow{\iota_\alpha^*} \text{Gr}_{F_S}^\nu \text{CH}^r(\tilde{X}_\alpha) .$$

Taking the sum over $\alpha \geq 1$, one has maps

$$\sum_{\alpha \geq 1} \iota_\alpha^* : \text{CH}^r(X) \longrightarrow \bigoplus_{\alpha \geq 1} \text{CH}^r(\tilde{X}_\alpha) ,$$

$$\sum_{\alpha \geq 1} \iota_{\alpha*} : \bigoplus_{\alpha \geq 1} \text{CH}_{\dim X - r}(\tilde{X}_\alpha) \rightarrow \text{CH}^r(X) ,$$

as well as the maps on graded pieces.

Proposition (1.5) and Theorem (3.3) imply:

(3.8) **Proposition.** (1) The kernel of the map $\sum_{\alpha \geq 1} \iota_\alpha^* : \mathrm{Gr}_{F_S}^\nu \mathrm{CH}^r(X) \longrightarrow \bigoplus_{\alpha \geq 1} \mathrm{Gr}_{F_S}^\nu \mathrm{CH}^r(\tilde{X}_\alpha)$ is equal to $\mathrm{CH}^r(h_0^{2r-\nu}(X/S))$.

(2) The image of the map $\sum_{\alpha \geq 1} \iota_{\alpha*} : \bigoplus_{\alpha \geq 1} \mathrm{Gr}_{F_S}^\nu \mathrm{CH}_{\dim X - r}(\tilde{X}_\alpha) \rightarrow \mathrm{Gr}_{F_S}^\nu \mathrm{CH}^r(X)$ is equal to $\mathrm{CH}^r(S, \bigoplus_{\alpha \geq 1} h_\alpha^{2r-\nu}(X/S))$.

(3.8.1) **Corollary.** One has

$$\begin{aligned} \mathrm{ICH}^r(S, R^i p_* \mathbb{Q}_X) &= \mathrm{Ker}[\mathrm{Gr}_{F_S}^{2r-i-\dim S} \mathrm{CH}^r(X) \rightarrow \bigoplus_{\alpha \geq 1} \mathrm{Gr}_{F_S}^{2r-i-\dim S} \mathrm{CH}^r(\tilde{X}_\alpha)] \\ &= \mathrm{Cok}[\bigoplus_{\alpha \geq 1} \mathrm{Gr}_{F_S}^{2r-i-\dim S} \mathrm{CH}_{\dim X - r}(\tilde{X}_\alpha) \rightarrow \mathrm{Gr}_{F_S}^{2r-i-\dim S} \mathrm{CH}^r(X)]. \end{aligned}$$

(3.9) **Theorem.** If $p : X \rightarrow S$ is a birational map, $d = \dim S$,

$$\mathrm{ICH}^r(S) = \frac{\bigcap_{\alpha \geq 1} (\iota_\alpha^*)^{-1} F_S^{2r-d+1} \mathrm{CH}^r(\tilde{X}_\alpha)}{\sum_{\alpha \geq 1} \iota_{\alpha*} F_S^{2r-d+1} \mathrm{CH}_{d-r}(\tilde{X}_\alpha)}.$$

Proof. The map $\sum_{\alpha \geq 1} \iota_\alpha^* : \mathrm{Gr}_{F_S}^\nu \mathrm{CH}^r(X) \longrightarrow \bigoplus_{\alpha \geq 1} \mathrm{Gr}_{F_S}^\nu \mathrm{CH}^r(\tilde{X}_\alpha)$ is injective for $\nu \neq 2r - d$, and has kernel equal to $\mathrm{ICH}^r(S)$ if $\nu = 2r - d$. The map $\sum_{\alpha \geq 1} \iota_{\alpha*} : \bigoplus_{\alpha \geq 1} \mathrm{Gr}_{F_S}^\nu \mathrm{CH}_{\dim X - r}(\tilde{X}_\alpha) \rightarrow \mathrm{Gr}_{F_S}^\nu \mathrm{CH}^r(X)$ is surjective for $\nu \neq 2r - d$ and has image equal to $\mathrm{CH}^r(S, \bigoplus_{\alpha \geq 1} h_\alpha^d(X/S))$ for $\nu = 2r - d$. So

$$\bigcap_{\alpha \geq 1} (\iota_\alpha^*)^{-1} F_S^{2r-d+1} \mathrm{CH}^r(\tilde{X}_\alpha) = F_S^{2r-d+1} \mathrm{CH}^r(X) + \mathrm{ICH}^r(S)$$

and

$$\sum_{\alpha \geq 1} \iota_{\alpha*} F_S^{2r-d+1} \mathrm{CH}_{d-r}(\tilde{X}_\alpha) = F_S^{2r-d+1} \mathrm{CH}^r(X),$$

from which the claim follows.

We note some properties of the filtration F_S^\bullet .

(3.10) **Proposition.** Let X be smooth and $p : X \rightarrow S$ a projective map. Let $S \hookrightarrow S'$ be a closed immersion of quasi-projective varieties. Then the filtration F_S^\bullet and $F_{S'}^\bullet$ on $\mathrm{CH}^r(X)$ coincide.

(3.11) **Proposition.** (1) Let X and Y be smooth varieties, projective over S , and $f : X \rightarrow Y$ be a projective surjective map. Then:

(1) The injection $f^* : \mathrm{CH}^r(Y) \rightarrow \mathrm{CH}^r(X)$ is strictly compatible with the filtrations F_S , namely $F_S^\nu \mathrm{CH}^r(Y) = (f^*)^{-1} F_S^\nu \mathrm{CH}^r(X)$.

(2) The surjection $f_* : \mathrm{CH}_s(X) \rightarrow \mathrm{CH}_s(Y)$ is strictly compatible with the filtrations F_S , namely $f_* F_S^\nu \mathrm{CH}^r(X) = F_S^\nu \mathrm{CH}^r(Y)$.

Proof. (1) Take a smooth subvariety $X' \subset X$ such that the restriction $f|_{X'} : X' \rightarrow Y$ is generically finite. Considering the composition of $f^* : \mathrm{CH}^r(Y) \rightarrow \mathrm{CH}^r(X)$ with the restriction $\mathrm{CH}^r(X) \rightarrow \mathrm{CH}^r(X')$, one is reduced to the case where f is generically finite.

In that case the map $f^* : h(Y/S) \rightarrow h(X/S)$ has a left inverse $(1/d)f_* : h(X/S) \rightarrow h(Y/S)$, where d is the degree of p . Twisting and taking $\tau_{\leq -\nu}$, one has

$$f^* : \tau_{\leq -\nu} h(Y/S)(r) \rightarrow \tau_{\leq -\nu} h(X/S)(r)$$

with left inverse $(1/d)f_*$. The claim follows.

(2) Similar to (1).

§4. Unconditional theory of the intersection Chow group.

(4.1) For a smooth projective variety X over k , Shuji Saito defined a filtration F^\bullet on the Chow group $\mathrm{CH}^r(X)$, [SaS-1, 2]. In a similar way, if X is a smooth variety with a projective map to S , one can define a filtration F_S^\bullet on the Chow group of X .

Let S be a quasi-projective variety, and X a smooth variety with a projective map $p : X \rightarrow S$. For another smooth variety W with a projective map $q : W \rightarrow S$, an element $\Gamma \in \mathrm{CH}_{\dim X - s}(W \times_S X)$ induces a map $\Gamma_* : \mathrm{CH}^{r-s}(W) \rightarrow \mathrm{CH}^r(X)$, see [CH]. The cycle class of Γ in Borel-Moore homology gives a map $\Gamma_* : Rq_*\mathbb{Q}_W[-2s] \rightarrow Rp_*\mathbb{Q}_X$; passing to perverse cohomology one has a map (for each ν)

$${}^p\mathcal{H}^{2r-\nu}\Gamma_* : {}^p\mathcal{H}^{2r-2s-\nu}Rq_*\mathbb{Q}_W \rightarrow {}^p\mathcal{H}^{2r-\nu}Rp_*\mathbb{Q}_X .$$

(Here ${}^p\mathcal{H}^*$ stands for perverse cohomology.)

We define a filtration F_S^\bullet on $\mathrm{CH}^r(X)$ as follows. Let $\mathrm{CH}^r(X) = F_S^{-\dim S} \mathrm{CH}^r(X)$. Assume F_S^ν has been defined. Define

$$F_S^{\nu+1} \mathrm{CH}^r(X) := \sum \mathrm{Image}[\Gamma_* : F_S^\nu \mathrm{CH}^{r-s}(W) \rightarrow \mathrm{CH}^r(X)]$$

where the sum is over $(q : W \rightarrow S, \Gamma \in \mathrm{CH}_{\dim X - s}(W \times_S X))$ satisfying the following condition: the map ${}^p\mathcal{H}^{2r-\nu}\Gamma_* : {}^p\mathcal{H}^{2r-2s-\nu}Rq_*\mathbb{Q}_W \rightarrow {}^p\mathcal{H}^{2r-\nu}Rp_*\mathbb{Q}_X$ is zero. One can show:

(4.2) **Proposition.** *The filtration F_S^\bullet on $\mathrm{CH}^r(X)$ has the following properties.*

(1) $\mathrm{CH}^r(X) = F_S^{-\dim S} \mathrm{CH}^r(X)$. For any $\Gamma \in \mathrm{CH}_{\dim X - s}(W \times_S X)$, the induced map $\Gamma_* : \mathrm{CH}^{r-s}(W) \rightarrow \mathrm{CH}^r(X)$ respects F_S^\bullet .

(2) If ${}^p\mathcal{H}^{2r-\nu}\Gamma_* : {}^p\mathcal{H}^{2r-2s-\nu}Rq_*\mathbb{Q}_W \rightarrow {}^p\mathcal{H}^{2r-\nu}Rp_*\mathbb{Q}_X$ is zero, then Γ_* sends $F_S^\nu \mathrm{CH}^{r-s}(W)$ to $F_S^{\nu+1} \mathrm{CH}^r(X)$.

(3) The filtration is the smallest one with properties (1) and (2).

(4.3) **Definition.** Let S be an irreducible quasi-projective variety of dimension d , and $p : X \rightarrow S$ a resolution of singularities. Take a Whitney stratification $\{S_\alpha\}$ of S and resolutions $\tilde{X}_\alpha \rightarrow X_\alpha$ so that $(p, \{\tilde{X}_\alpha \rightarrow X_\alpha\})$ is stratified over $\{S_\alpha\}$. Recall $\iota_\alpha : \tilde{X}_\alpha \rightarrow X$ are the induced maps, which give rise to maps $\iota_{\alpha*} : \mathrm{CH}_{d-r}(\tilde{X}_\alpha) \rightarrow \mathrm{CH}^r(X)$ and $\iota_\alpha^* : \mathrm{CH}^r(X) \rightarrow \mathrm{CH}^r(\tilde{X}_\alpha)$.

Define the *intersection Chow group* as a subquotient of the Chow group of X given by:

$$\mathrm{ICH}^r(S) := \frac{\bigcap_{\alpha \geq 1} (\iota_\alpha^*)^{-1} F_S^{2r-d+1} \mathrm{CH}^r(\tilde{X}_\alpha)}{\sum_{\alpha \geq 1} \iota_{\alpha*} F_S^{2r-d+1} \mathrm{CH}_{d-r}(\tilde{X}_\alpha)} .$$

(4.4) **Theorem.** *$\mathrm{ICH}^r(S)$ is well-defined (up to canonical isomorphism) independent of the choice of a desingularization $p : X \rightarrow S$, a stratification and a resolution.*

(4.5) **Proposition.** (1) *Let X and Y be smooth varieties, projective over S , and $f : X \rightarrow Y$ be a projective surjective map. Then:*

(1) *The injection $f^* : \mathrm{CH}^r(Y) \rightarrow \mathrm{CH}^r(X)$ is strictly compatible with the filtrations F_S , namely $F_S^\nu \mathrm{CH}^r(Y) = (f^*)^{-1} F_S^\nu \mathrm{CH}^r(X)$.*

(2) The surjection $f_* : \mathrm{CH}_s(X) \rightarrow \mathrm{CH}_s(Y)$ is strictly compatible with the filtrations F_S , namely $f_* F_S^\nu \mathrm{CH}^r(X) = F_S^\nu \mathrm{CH}^r(Y)$.

Proof. Take a smooth subvariety $X' \subset X$ that maps generically finitely onto Y . Let $i : X' \rightarrow X$ be the inclusion and $f' := f \circ i : X' \rightarrow Y$. For (1), suppose $\alpha \in \mathrm{CH}^r(Y)$ such that $f^* \alpha \in F_S^\nu \mathrm{CH}^r(X)$. Then $f'_* i^* f^* \alpha = d \alpha \in F_S^\nu \mathrm{CH}^r(Y)$ ($d = \deg f'$) by the functoriality of F_S with respect to pull-back and push-forward. The proof of (2) is similar.

In the rest of this section we give the proof of Theorem (4.4).

The definition depends on X , $\mathfrak{S} = \{S_\alpha\}$ and $\{\pi_\alpha : \tilde{X}_\alpha \rightarrow X_\alpha\}$. Let

$$N = \bigcap_{\alpha \geq 1} (\iota_\alpha^*)^{-1} F_S^{2r-d+1} \mathrm{CH}^r(\tilde{X}_\alpha) \quad \text{and} \quad D = \sum_{\alpha \geq 1} \iota_{\alpha*} F_S^{2r-d+1} \mathrm{CH}_{d-r}(\tilde{X}_\alpha)$$

be the subgroups of $\mathrm{CH}^r(X)$, which appear in (4.4). If $(X', \mathfrak{S}', \pi'_\alpha : \tilde{X}'_\alpha \rightarrow X'_\alpha)$ is another choice, the groups N', D' are similarly defined subgroups of $\mathrm{CH}^r(X')$. We must show there is a canonical isomorphism $N/D \cong N'/D'$.

(I) Assume $X = X'$, $\mathfrak{S} = \mathfrak{S}'$, and only $(\tilde{X}_\alpha \rightarrow X_\alpha)$ differs. One may assume there are projective surjective maps $g_\alpha : \tilde{X}'_\alpha \rightarrow \tilde{X}_\alpha$ over X_α . Then (4.5) shows $N = N'$ and $D = D'$.

(II) Assume $X = X'$ and \mathfrak{S} and \mathfrak{S}' differ. One may assume \mathfrak{S}' is a refinement of \mathfrak{S} .

Let $(p : X \rightarrow S, \{\pi_\alpha : \tilde{X}_\alpha \rightarrow X_\alpha\})$ be a resolution of p over \mathfrak{S} . Let $S_{\alpha i}$ be the irreducible components of S_α , $X_{\alpha i} = p^{-1}(S_{\alpha i})$, and $\tilde{X}_{\alpha i} = q_\alpha^{-1}(S_{\alpha i})$.

We construct a resolution of p over \mathfrak{S}' as follows. Let $S'_{\alpha j}$ be the irreducible components of S'_α , for $\alpha \geq 1$. Let $X'_{\alpha j} = p^{-1}(S'_{\alpha j})$.

If $S'_{\alpha j}$ is an irreducible component of S_α , say $S'_{\alpha j} = S_{\alpha i}$, let $\tilde{X}'_{\alpha j} = \tilde{X}_{\alpha i}$. If $S'_{\alpha j} \not\subset S_\alpha$ let S_β be such that $S'_{\alpha j} \subset S_\beta$ and $S'_{\alpha j} \not\subset S_{\beta+1}$. Take smooth $\tilde{X}'_{\alpha j}$ so that there are a projective surjective map $\tilde{X}'_{\alpha j} \rightarrow X'_{\alpha j}$ and a map $g : \tilde{X}'_{\alpha j} \rightarrow \tilde{X}_\beta$ over S , namely the following diagram commutes.

$$\begin{array}{ccc} \tilde{X}'_{\alpha j} & \xrightarrow{g} & \tilde{X}_\beta \\ \downarrow & & \downarrow \\ X'_{\alpha j} & \longrightarrow & X_\beta \\ \downarrow & & \downarrow \\ S'_{\alpha j} & \hookrightarrow & S_\beta \end{array}$$

Let $\tilde{X}'_\alpha = \coprod_j \tilde{X}'_{\alpha j}$, and $\pi'_\alpha : \tilde{X}'_\alpha \rightarrow X'_\alpha$ the induced map.

We now show $N = N'$. Clearly $N' \subset N$. The inclusion $N \subset N'$ follows from the existence of the maps g . Similarly one shows $D = D'$.

(III) Assume now X and X' are not equal. In view of the weak factorization theorem, one may assume X' is a blow-up of X along a smooth center.

Let $\mu : X' \rightarrow X$ be the blow-up of a smooth center $Z \subset X$. Assume the maps $X' \rightarrow X \rightarrow S$ are stratified over \mathfrak{S} . Let $D \subset N \subset \mathrm{CH}^r(X)$ and $D' \subset N' \subset \mathrm{CH}^r(X')$ be defined as above.

In the rest of this section we show: $\mu^*(N) \subset N'$, $\mu_*(N') \subset N$, $\mu^*(D) \subset D'$, $\mu_*(D') \subset D$. Letting $K = \mathrm{Ker} \mu_*$, $N' = N \oplus (K \cap N')$, $D' = D \oplus (K \cap D')$, and $K \cap N' = K \cap D'$. Hence

$$\mu^* : N/D \xrightarrow{\sim} N'/D' .$$

(4.6) Let S be a quasi-projective variety, Z a smooth variety with a projective map $Z \rightarrow S$, and $\pi : E \rightarrow Z$ a \mathbb{P}^n -bundle. Let $\xi \in \text{CH}^1(E)$ be the first Chern class of $\mathcal{O}_E(1)$. One has $\text{CH}^r(E) = \bigoplus_{0 \leq i \leq n-1} \text{CH}^{r-i}(Z) \cdot \xi^i$. One easily shows the following proposition. (From now we will not write ξ^i .)

Proposition. *The above decomposition is compatible with the filtrations F_S^\bullet , namely*

$$F_S^\bullet \text{CH}^r(E) = \bigoplus_{0 \leq i \leq n-1} F_S^\bullet \text{CH}^{r-i}(Z) .$$

(4.7) Let S be quasi-projective, X smooth, and $p : X \rightarrow S$ be a projective map. (We do not assume p is birational, although we are mainly interested in that case. Let $Z \subset X$ be a smooth subvariety and $\mu : X' \rightarrow X$ be the blow-up of a smooth center $Z \subset X$. Let E be the exceptional divisor. One has a commutative diagram with maps as labeled.

$$\begin{array}{ccc} E & \xrightarrow{j} & X' \\ g \downarrow & & \downarrow \mu \\ Z & \xrightarrow{i} & X \end{array}$$

If c is the codimension of Z , g is a \mathbb{P}^n -bundle with $n = d - c - 1$.

The kernel of the map $g_* : \text{CH}^{r-1}(E) \rightarrow \text{CH}^{r-n-1}(Z)$ is:

$$\text{Ker } g_* = \bigoplus_{0 \leq i \leq n-2} F_S^\bullet \text{CH}^{r-1-i}(Z) \subset \text{CH}^{r-1}(E) = \bigoplus_{0 \leq i \leq n-1} F_S^\bullet \text{CH}^{r-1-i}(Z) .$$

It has the filtration induced from F_S^\bullet on $\text{CH}^{r-1}(E)$.

One has an isomorphism

$$\text{Ker } g_* \oplus \text{CH}^r(X) \xrightarrow{\sim} \text{CH}^r(X') ,$$

which sends (α, x) to $j_*\alpha + \nu^*x$.

(4.8) **Proposition.** *The injective map $j^*j_* : \text{Ker } g_* \rightarrow \text{CH}^r(E)$ is strictly compatible with the filtrations F_S^\bullet .*

Proof. Easy to show by decomposing both groups to the sum of the Chow groups of Z .

We now assume $p : X \rightarrow S$ is birational, and Z is a union of strata in the stratification (S_α) . We have maps $\mu^* : \text{CH}^r(X) \rightarrow \text{CH}^r(X')$ and $\mu_* : \text{CH}^r(X') \rightarrow \text{CH}^r(X)$. One has $\mu_*\mu^* = id$, $\text{Ker } \mu_* = \text{Ker } g_*$, and $\text{CH}^r(X) \oplus \text{Ker } g_* \cong \text{CH}^r(X')$. $D \subset N \subset \text{CH}^r(X)$ and $D' \subset N' \subset \text{CH}^r(X')$ be defined as above.

We may assume, for each component $\tilde{X}_{\alpha i}$ of \tilde{X}_α , either one of the following conditions is satisfied.

- (a) $X_{\alpha i} \not\subset Z$, and $\iota_\alpha^{-1}Z \cap \tilde{X}_{\alpha i}$ is a normal crossing divisor of $\tilde{X}_{\alpha i}$,
- (b) $X_{\alpha i} \subset Z$.

One may take, as resolutions of $p \circ \mu : X' \rightarrow S$, $\tilde{X}'_\alpha = \amalg \tilde{X}'_{\alpha i}$ where

$$\tilde{X}'_{\alpha i} = \begin{cases} \tilde{X}_{\alpha i} & \text{in case (a)} \\ \tilde{X}_{\alpha i} \times_Z E & \text{in case (b)} \end{cases}$$

In case (b), $\tilde{X}'_{\alpha i}$ is a \mathbb{P}^n -bundle over $\tilde{X}_{\alpha i}$. Note also that there is some $X_{\alpha i}$ containing Z , so one can take a smooth variety Z' which fits into the following commutative diagram

$$\begin{array}{ccc} Z' & \rightarrow & \tilde{X}_{\alpha i} \\ h \downarrow & & \downarrow \\ Z & \rightarrow & X_{\alpha i} \end{array}$$

where $h : Z' \rightarrow Z$ is projective surjective. Let $\iota'_\alpha : \tilde{X}'_\alpha \rightarrow X'$ be the induced maps. It is now easy to show the following descriptions for N and N' .

(4.9) **Proposition.** (1) *One has*

$$N = \bigcap_{\text{type (a)}} (\iota_\alpha^*)^{-1} F_S^{2r-d+1} \text{CH}^r(\tilde{X}_{\alpha i}) \cap (i^*)^{-1} F_S^{2r-d+1} \text{CH}^r(Z).$$

Here the first intersection is over $\tilde{X}_{\alpha i}$ of type (a). The restriction of $\iota_\alpha : \tilde{X}_\alpha \rightarrow X$ to each component $\tilde{X}_{\alpha i}$ is still denoted ι_α .

(2) *Similarly,*

$$N' = \bigcap_{\text{type (a)}} (\iota'_\alpha)^{-1} F_S^{2r-d+1} \text{CH}^r(\tilde{X}_{\alpha i}) \cap (j^*)^{-1} F_S^{2r-d+1} \text{CH}^r(E).$$

(Recall for \tilde{X}_α of type (a), ι'_α is the map $\tilde{X}'_\alpha = \tilde{X}_\alpha \rightarrow X'$.)

Proof. (1) Let $z \in \text{CH}^r(X)$. For a component $\tilde{X}_{\alpha i}$ of type (b),

$$z|_Z \in F_S^{2r-d+1} \text{CH}^r(Z) \quad \text{implies} \quad z|_{\tilde{X}_{\alpha i}} \in F_S^{2r-d+1} \text{CH}^r(\tilde{X}_{\alpha i}).$$

Conversely assume $z|_{\tilde{X}_{\alpha i}} \in F_S^{2r-d+1} \text{CH}^r(\tilde{X}_{\alpha i})$ for each $\tilde{X}_{\alpha i}$. Take a component $X_{\alpha i}$ containing Z and $Z' \rightarrow Z$ as above. We have $z|_{Z'} \in F_S^{2r-d+1} \text{CH}^r(Z')$, hence $z|_Z \in F_S^{2r-d+1} \text{CH}^r(Z)$ by Proposition.

(2) Similar to (1).

(4.10) **Proposition.** (1) $\mu^*(N) \subset N'$. (2) $\mu_*(N') \subset N$. (3) $N' = N \oplus (K \cap N')$.

Proof. (1) Obvious.

(2) For $z' \in N'$,

$$\begin{aligned} i^* \mu_* z' &= g_* i^! z' \\ &= g_*(c(\mathcal{E}) \cdot j^* x') \end{aligned}$$

where $c(\mathcal{E})$ is the top Chern class of the excess bundle \mathcal{E} . Thus $i^* \mu_* z' \in F_S^{2r-d+1} \text{CH}^r(Z)$.

(3) Follows from (1), (2) and $\mu_* \mu^* = id$ on N .

The proofs of the two propositions below are similar.

(4.11) **Proposition.** (1) $D = \sum_{\text{type (a)}} \iota_{\alpha*} F_S^{2r-d+1} \text{CH}_{d-r}(\tilde{X}_{\alpha i}) + i_* F_S^{2r-d+1} \text{CH}_{d-r}(Z)$. The first sum is over $\tilde{X}_{\alpha i}$ of type (a).

(2) $D' = \sum_{\text{type (a)}} \iota'_{\alpha*} F_S^{2r-d+1} \text{CH}_{d-r}(\tilde{X}_{\alpha i}) + j_* F_S^{2r-d+1} \text{CH}_{d-r}(E)$.

(4.12) **Proposition.** (1) $\mu_*(D') \subset D$. (2) $\mu^*(D) \subset D'$. (3) $D' = D \oplus (K \cap D')$.

Proof. (1) Obvious.

(2) For $\alpha \in F_S^{2r-d+1} \text{CH}_{d-r}(Z)$,

$$\mu^* i_* \alpha = j_* (c(\mathcal{E}) \cdot g^* \alpha) \in j_* F_S^{2r-d+1} \text{CH}_{d-r}(E) .$$

(3) Follows from (1) and (2).

(4.13) **Proposition.** *We have $K \cap N' = K \cap D'$. Hence an isomorphism $\mu^* : N/D \xrightarrow{\sim} N'/D'$, the inverse being the map induced by μ_* .*

Proof. If $z \in K$, $z = j_* w$ for an element $w \in \text{Ker } g_*$. If $z \in K \cap N'$, Proposition (4.8) implies $w \in F_S^{2r-d+1} \text{CH}^{r-1}(E)$. Thus $z \in K \cap D'$.

(4.14) Let X be a quasi-projective variety, with a quasi-projective map p to S . One can define a unique filtration F_S^\bullet on the Chow group $\text{CH}_s(X)$ satisfying properties. The definition of the filtration and the verification of the properties are similar to the case $S = \text{Spec } k$, which was carried out in [CH].

(1) $\text{CH}^r(X) = F_S^{-\dim S} \text{CH}^r(X)$ and $F_S^\nu \text{CH}^r(X) = 0$ for ν large enough.

(2) If $f : X \rightarrow Y$ be a projective map over S , $f_* : \text{CH}_s(X) \rightarrow \text{CH}_s(Y)$ respects the filtrations F_S^\bullet . If in addition f is surjective, f_* is strictly compatible with the filtrations.

(3) If $j : U \hookrightarrow X$ is an open immersion, then $j^* : \text{CH}_s(X) \rightarrow \text{CH}_s(U)$ is strictly compatible with F_S^\bullet .

(4) If $f : X \rightarrow Y$ be an lci map of relative dimension d and f is over S , and

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & & \downarrow g \\ X & \xrightarrow{f} & Y \end{array}$$

a Cartesian square where $g : Y' \rightarrow Y$ is a quasi-projective map, then the refined Gysin map $f^! : \text{CH}_s(Y') \rightarrow \text{CH}_{s+d}(X')$ respects F_S^\bullet .

References

- [BBFGK] G. Barthel, J.-P. Brasselet, K.-H. Fieseler, O. Gabber and L. Kaup : Relèvement de cycles algébriques et homomorphismes associés en homologie d'intersection, Ann. Math. 141 (1995), 147–179.
- [BBD] A. Beilinson, J. Bernstein, and P. Deligne : *Faisceaux Pervers*. Astérisque 100, Société Mathématique de France, 1982.
- [CH] A. Corti and M. Hanamura: Motivic decomposition and intersection Chow groups I, Duke Math. J. 103 (2000), 459–522.
- [De] P. Deligne: Theorie de Hodge II, Publ. Math. IHES 40 (1972), 5-57; III, Publ. Math. IHES 44 (1975), 6-77.
- [Fu] W. Fulton: *Intersection Theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete, 3rd series, 2. Springer-Verlag, Berlin, 1998

- [GM-1] M. Goresky and R. MacPherson: Intersection homology. II, *Invent. Math.* 72 (1983), no. 1, 77-129.
- [GM-2] M. Goresky and R. MacPherson: *Stratified Morse Theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete, 14. Springer-Verlag, Berlin, 1988.
- [Ha-1] M. Hanamura: Mixed motives and algebraic cycles I, II, and III, *Math. Res. Letters* 2(1995), 811-821, *Invent. Math.* 158 (2004), 105-179, and *Math. Res. Letters* 6(1999), 61-82.
- [Ha-2] M. Hanamura: Motivic sheaves and intersection cohomology, to appear in: Proceedings of Franco-Japanese symposium in Singularity Theory.
- [Kl] S. Kleiman: Algebraic cycles and Weil conjectures, in *Dix Exposés sur la Cohomologie des Schémas*, pp. 359 – 386, North Holland, Amsterdam.
- [Mu] J. P. Murre: On a conjectural filtration on the Chow groups of an algebraic variety I and II, *Indag. Mathem.* 2(1993), 177–188 and 189–201.
- [SaM] Morihiko Saito: Modules de Hodge polarisables, *Publ. Res. Inst. Math. Sci.* 24 (1988), no. 6, 849-995 (1989).
- [SaS 1] S. Saito: Motives and filtrations on Chow groups, *Invent. Math.* 125(1996), 149-196,
- [SaS 2] S. Saito: Motives and filtrations on Chow groups II, in *The arithmetic and geometry of algebraic cycles* (Banff, AB, 1998), 321-346, NATO Sci. Ser. C Math. Phys. Sci., 548, Kluwer Acad. Publ., Dordrecht, 2000.

Department of Mathematics, Imperial College London,
 South Kensington Campus, 180 Queen's Gate, LONDON SW7 2AZ , Great Britain.

Mathematical Institute, Tohoku University
 Aramaki, Aoba-ku, Sendai 980-8587, Japan.