The Moduli Space of Curves

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This is a write up of my lecture in the Cambridge "Geometry seminar", an introduction to the construction and proof that the compactified moduli space $\overline{M}_g$ of curves of genus g is a projective variety. The modern technology is based on first constructing the stack $\overline{\mathcal{M}}_g$ [DM], showing that it is coarsely represented by an algebraic space [KM] $\overline{M}_g$, then finally endowing this space with a line bundle and showing that it is an ample line bundle [K], thus making $\overline{M}_g$ into a projective variety. The beauty of all this is not only that it is easier than the traditional method via [GIT], but every step of the way involves proving some fundamental geometric property of stable curves which is of general interest in its own right. There is also an approach to prove that (not necessarily complete) moduli spaces are quasiprojective [V]: this is all very nice but harder and I will not discuss it here. Stacks are mind boggling, but they are being used [BF] in interesting ways, and the lack of a decent foundational reference will soon be filled by [F] (I had the good fortune of sitting in during the lectures). [K] takes the point of view of algebraic spaces, but these are not really any easier than stacks. By [KM] a stack (with mild assumptions) is an algebraic space with the additional structure of a distinguished atlas (much like a Q-variety or orbifold discussed below). I would like to give the sense that there is some general machinery available which is easy and fun to use.

In the end I discuss an approach to the main result of [PD], different from the one they use, more within the spirit of this talk: the boundary points of a compactified moduli space better have a geometric interpretation.

This is true even for moduli spaces traditionally under the umbrella of [GIT] like vector bundles [Fa]. One should nevertheless remark that [GIT] still gives stronger results about what line bundles on the moduli space are ample.

Doing algebraic geometry in my generation, it is amazing to find oneself continually trying to climb on top of the very large shoulders of Deligne, Mumford and so on, having the uncomfortable feeling that one will never get there.

1 stable curves

The official reference for this section is [DM].

1. Definition. A *stable curve* of genus g is a proper curve C such that

(a) C has only nodes as singularities (by definition, a node is a singularity analytically equivalent to $\{xy = 0\} \subset \mathbb{C}^2$) and,

(b) ω_C is ample and $h^0(\omega_C) = g$.

If S is any scheme, a stable curve of genus g *over* S is a morphism

$$\begin{array}{c} X \\ \big\downarrow \\ S \end{array}$$

such that X_s is a stable curve of genus g for all closed points $s \in S$. We make these into a category $\overline{\mathcal{M}}_g$ by declaring that a morphism is a fibre square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow \\ S & \longrightarrow & T \end{array}$$

(The category just made *is* the moduli *stack* $\overline{\mathcal{M}}_g$.)

The important things about stable curves are

(1) ω_C is ample. If $\mathbb{P}^1 \cong D \subset C$ is a component and the rest of C intersects D in $\sum P_i$, $\omega_C|D = \omega_D(\sum P_i)$ is the sheaf of meromorphic differentials with *poles* along $\sum P_i$. For ω_C to be ample we then need at least 3 P_is and thus recover the original definition of a stable curve.

In fact one easily shows that $\omega_C^{\otimes 3}$ is very ample. Choosing a basis $\sigma_0, ...\sigma_N$ of $H^0(C, \omega_C^3)$ defines an embedding

$$C \hookrightarrow \mathbb{P}^N$$

into a fixed coordinatised $\mathbb{P}^N$.

(2) $\operatorname{Aut} C$ is finite. Indeed by (1) $\operatorname{Aut} C \subset \operatorname{Aut} \mathbb{P}^N$ is a *linear algebraic group*. It is also 0-dimensional, because if it were to contain a G_a or G_m, then a G_a or G_m orbit would be a component D of C with $D \setminus C \cong \mathbb{P}^1 \setminus \{0\}$ or $\mathbb{P}^1 \setminus \{0, \infty\}$.

(3) If $X \to \Delta^\times$ is a stable curve defined over a punctured disk (or the spectrum of a DVR), there is a unique limit

$$X_0 = \lim_{t \to 0} X_t$$

which is a stable curve. As I will explain below, this is a traditional manoeuvre involving semistable reduction after a base change [KKMS] (but much simpler in this case), followed by (relative) minimal and then canonical model, and is the way stable curves were first discovered.

It would be nice if there existed a space $\overline{M}_g$ representing $\overline{\mathcal{M}}_g$. By definition this means that we have a stable curve $U_g \to \overline{M}_g$ and every other stable curve $X \to S$ arises from a unique morphism

$$\begin{array}{ccc} X & \longrightarrow & U_g \\ \downarrow & & \downarrow \\ S & \longrightarrow & \overline{M}_g \end{array}$$

We now explain the only reason why such a space can never exist. Let C be a fixed stable curve with nontrivial automorphism group G, and $T \to S$ an étale covering on which G acts as the group of deck transformations. We can make a stable curve $X \to S$ by taking the quotient of $C \times T \to T$ by the equivariant G-action $(c, t) \xrightarrow{g} (g \cdot c, g \cdot t)$. Because we are assuming that there are no stabilisers for the action of G on T, the resulting $X \to S$ is a stable curve. The fibre of $X \to S$ is constant in moduli (it is C), but at the same time $X \to S$ it is not isomorphic to the product family: it is not so even topologically over a loop $\gamma \in \pi_1(S)$ which is the image of an element of G. Therefore $X \to S$ can never arise from a morphism from S to the putative $\overline{M}_g$.

2 Q-varieties

The definition of Q-variety is essentially taken from [M2]

We need an object, a variety with additional structure, capable of recording the information of all stable curves of genus g and their automorphisms. The simplest such object, which is sufficient for most but not all purposes, is a Q-variety:

2. Definition. An *analytic* Q-variety is an analytic space X, together with a distinguished atlas of open charts $\{X_\alpha/G_\alpha \subset X\}$ as follows

(a) X_α is a *smooth* analytic space and G_α is a finite group acting *faithfully* on X_α,

(b) Denote $p_\alpha : X_\alpha \to X$ the projection. Assume that two points $x \in X_\alpha$, $y \in X_\beta$ are identified, that is $p_\alpha(x) = p_\beta(y)$. Let $I_x \subset G_\alpha$ and $I_y \subset G_\beta$ be the stabilisers of the points x, y. Then we require that there be an identification $I \cong I_x \cong I_y$, open neighbourhoods $x \in V_x \subset X_\alpha$, $y \in V_y \subset X_\beta$, stable under I_x, I_y, and an I-equivariant isomorphism $V_x \cong V_y$ (we simply require that these identifications and isomorphisms exist, we do not choose any particular ones).

We are free to add a new chart provided is satisfies condition (b) when compared with any old chart.

The analytic space X is called the *coarse moduli space* of the Q-variety.

We say that the Q-variety is *algebraic* if its moduli space X is an algebraic space, in other words the field $\mathbb{C}(X)$ of meromorphic functions on X has transcendence degree $= \dim X$ over $\mathbb{C}$.

The goal of this lecture is to give a very informal discussion of the modern technology involved in proving the following important result due to too many people to list

3. Main Theorem. *For $g \geq 3$, there is a Q-variety $\overline{\mathcal{M}}_g$ recording all stable curves over all S. The coarse moduli space $\overline{M}_g$ is a projective variety.*

The problem with $g = 2$ is that every curve of genus g has an automorphism. The correct statement, valid for all $g \geq 2$ (and even $g = 1$ with suitable modifications) is that the category $\overline{\mathcal{M}}_g$ defined in the previous section *is* a Deligne-Mumford stack which is separated, proper, smooth and coarsely represented by a projective variety $\overline{M}_g$. The thing is, a Deligne-Mumford stack is only very slightly more general than an algebraic Q-variety.

3 proof of the main theorem

The official reference for stacks and 3.1, 3.2 is [DM]. [KM] proves that a Deligne-Mumford stack is coarsely represented by an algebraic space. If you want to learn about stacks you should start from [M1] (at least from looking in the literature, this is the place where it all started), then go to [F] before attempting [DM].

3.3 is in every textbook (semistable reduction is of course in [KKMS] but much easier in our case) on surfaces and 3.4 is [K].

3.1 local charts

Fix a stable curve C of genus g.

4. Definition. Let $0 \in S$ be an analytic germ. A deformation of C, parametrised by S, is a stable curve $X \to S$ with a fixed isomorphism $\varphi_X : X_0 \xrightarrow{\cong} C$. We define the *deformation functor* $\underline{\mathrm{Def}}_C$ from germs to sets by

$$\underline{\mathrm{Def}}_C(0 \in S) = \{\text{deformations of } C, \text{ parametrised by } S\}$$

($T \to S$ induces $\underline{\mathrm{Def}}_C\, S \to \underline{\mathrm{Def}}_C\, T$ by pull back).

5. Theorem. $\underline{\mathrm{Def}}_C$ *is represented by a universal deformation* $U_C \to \mathrm{Def}_C$*, parametrised by a smooth germ* $0 \in \mathrm{Def}_C$*.*

Note that $\mathrm{Aut}\, C$ naturally acts on Def_C, and we then impose that $\mathrm{Def}_C / \mathrm{Aut}\, C$ be a chart of $\overline{M}_g$.

To prove the theorem we observe that *infinitesimal* deformations of C are in 1-to-1 correspondence with elements $\theta \in H^1(C, T_C)$ (draw a picture; the correct space for stable curves is $\mathrm{Ext}^1(\mathcal{O}_C, \Omega^1_C)$). Now if we start off deforming C in the infinitesimal direction θ, we meet successive obstructions to extend the deformation to each higher order, each obstruction lying in $H^2(C, T_C) = (0)$ (more correctly if C is a stable curve $\mathrm{Ext}^2(\mathcal{O}_C, \Omega^1_C)$).

3.2 gluing

For my own edification, this subsection is written in technically precise language. There are various issues.

Axiom 1

The first requirement for an *algebraic stack* is that the diagonal $\overline{\mathcal{M}}_g \to \overline{\mathcal{M}}_g \times \overline{\mathcal{M}}_g$ be representable, separated and quasicompact.

This is equivalent to the following. Given stable curves $X \to S$, $Y \to S$, we make the functor $\underline{\mathrm{Isom}}(X, Y) : Sch/S \to Sets$ as follows

$$\underline{\mathrm{Isom}}(X, Y)(T) = \{T\text{-isomorphisms } X_T \stackrel{\cong}{\to} Y_T\}$$

Now we are thinking of $\overline{\mathcal{M}}_g$ to be a stack over Sch with its étale topology, which means that we have already checked that $\underline{\mathrm{Isom}}$ is a sheaf in the étale topology (i.e. étale descent data are effective, no big deal in this case at all).

The requirement is equivalent to $\underline{\mathrm{Isom}}(X, Y)$ to be represented by a scheme $\mathrm{Isom}(X, Y) \to S$ which is separated and quasicompact over S. Properly understood, this is just saying that the stack is made of local pieces which are of the form $[U/R]$ where $R \rightrightarrows U$ is a groupoid scheme. This is not saying that these groupoids are any nice or how many we need (that is taken care of by axiom 2). Indeed the data are equivalent to a pair of morphisms $X : S \to \overline{\mathcal{M}}_g$, $Y : S \to \overline{\mathcal{M}}_g$ and the functor $\underline{\mathrm{Isom}}(X, Y)$ *is* the stack $S \times_{\overline{\mathcal{M}}_g} S$ (which is also the pull back of $X \times Y : S \times S \to \overline{\mathcal{M}}_g \times \overline{\mathcal{M}}_g$ by the diagonal $\overline{\mathcal{M}}_g \to \overline{\mathcal{M}}_g \times \overline{\mathcal{M}}_g$).

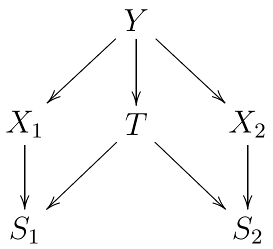
In down to earth terms, there is a scheme $I = \mathrm{Isom}(X, Y) \to S$, together with a universal isomorphism $\varphi_I : X_I \stackrel{\cong}{\to} Y_I$

$$\begin{array}{ccc} X_I & \xrightarrow{\ \varphi_I\ } & Y_I \\ & \searrow \quad \swarrow & \\ & I & \end{array}$$

such that, for any scheme $T \to S$, any isomorphism $\varphi_T : X_T \stackrel{\cong}{\to} Y_T$ arises as the pull back of φ_I from a unique S-morphism $T \to I$. Moreover $I \to S$ is separated and quasicompact. The existence of I is an application of the method of Hilbert schemes. More interesting is to show that $I \to S$ is *proper*, i.e., the stack $\overline{\mathcal{M}}_g$ is *separated*: this is done in §3.

Finally, all this also means that *absolute* products exist in the stack $\overline{\mathcal{M}}_g$. Let $X_1 \to S_1$, $X_2 \to S_2$ be 2 stable curves (objects of $\overline{\mathcal{M}}_g$). The product family which we are about to describe has a universal property w.r.t.

diagrams



The product family is the pullback by $\mathrm{Isom}_{S_1\times S_2}(X_1\times S_2, S_1\times X_2)\to S_1\times S_2$ of any of the 2 isomorphic families $X_1\times S_2$, $S_1\times X_2$. This is of course nothing more that what has already been said. It means that if we have now 2 morphisms $X_i : S_i \to \overline{\mathcal{M}}_g$, we know how to make the product $S_1\times_{\overline{\mathcal{M}}_g} S_2$ and it too is representable.

Note as a consequence of this that we can make a (possibly hugely infinite) cover of $\overline{\mathcal{M}}_g$ by taking all stable curves C of genus g, for each C we let $V_C\to D_C$ be an algebraisation of the universal family $U_C\to \mathrm{Def}_C$ (by openness of versality this gives the universal deformation of fibres over points close to $0\in\mathrm{Def}_C$). Then

$$\coprod D_C\to\overline{\mathcal{M}}_g$$

is an surjective cover (as we do below for the cover H_g, it is basically tautological that this is an *étale* cover, in other words we are here showing that $\overline{\mathcal{M}}_g$ has the *local* structure of a Deligne-Mumford stack). The absolute product construction just explained is specifying how to patch the D_Cs for different Cs according to the groupoid law

$$\coprod D_C\times_{\overline{\mathcal{M}}_g} D_{C'}\rightrightarrows\coprod D_C$$

Next we basically see that a *finite* number of D_Cs suffices.

Axiom 2

The second requirement for an algebraic stack is that it be covered by a smooth and surjective morphism $H_g\to\overline{\mathcal{M}}_g$. To construct this we rigidify stable curves further to get a representable functor.

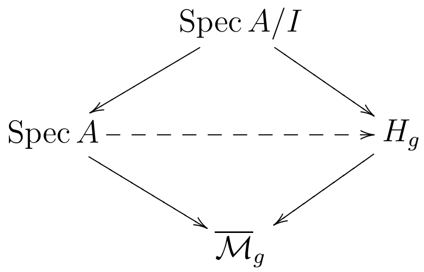
6. Definition. A *rigidified* stable curve of genus g is a closed subscheme $i : X\hookrightarrow S\times\mathbb{P}^N$ such that $p_1\circ i : X\to S$ is a stable curve of genus g and, for all closed points $s\in S$, $\mathcal{O}(1)|X_s\cong\omega_{X_s}^{\otimes 3}$

By standard Hilbert scheme theory, the functor

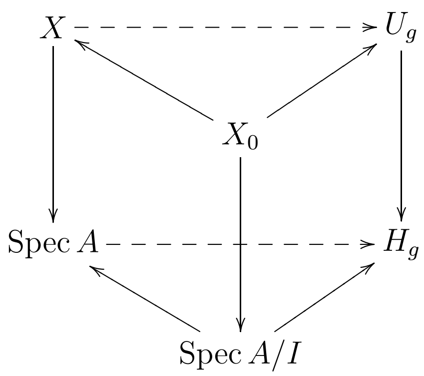
$$S \to \{\text{rigidified stable curves } X \subset S \times \mathbb{P}^N\}$$

is represented by a universal rigidified stable curve $U_g \subset H_g \times \mathbb{P}^N$.

Because every stable curve can be rigidified, the ensuing $H_g \to \overline{\mathcal{M}}_g$ is surjective, and we need to check that this morphism is (formally) smooth. Given an Artin algebra A with ideal $I \subset A$, of square zero $I^2 = (0)$, a stable curve over A and a rigidification over A/I, we need to be able to extend the rigidification to all of A (fill in the dashed arrow):



in other words, fill in the dashed arrows in the following diagram of morphisms of stable curves:



This is not a big deal.

Clearly here $H_g \times_{\overline{\mathcal{M}}_g} H_g = \underline{\mathrm{Isom}}_{H_g}(U_g, U_g) = PGL(N) \times H_g$, which by (1) identifies $\overline{\mathcal{M}}_g$ with the quotient stack $[H_g/PGL(N)]$

Deligne-Mumford stacks

The requirement for a Deligne-Mumford stack is that there be an étale cover by a scheme. This is done by using H_g to extract a finite subcover from the cover described at the end of Axiom 1.

A theorem of Keel and Mori guarantees that a Deligne-Mumford stack is always coarsely represented by an algebraic space. Thus we get our coarse moduli space $\overline{M}_g$. It is easy to convince oneself that $\overline{M}_g$ must be an algebraic space: according to A. Weil's construction of the "birational" variety of orbits under a group action, the function field $\mathbb{C}(\overline{M}_g)$ is the field of $PGL(N)$ invariants $\mathbb{C}(H_g)^{PGL(N)}$ and therefor has transcendence degree $\dim H_g - \dim PGL(N) = \dim \overline{M}_g$ over $\mathbb{C}$.

3.3 separated and proper

7. Theorem. *Let $U \to \Delta^\times$ be a stable curve of genus g, defined over a punctured disk (or the spectrum of a DVR). There is a unique limit*

$$U_0 = \lim_{t \to 0} U_t$$

which is a stable curve of genus g.

The proof of this is standard semistable reduction and (relative) minimal model program. Let $\overline{Z} \to \Delta$ be any proper morphism "closing" $U \to \Delta$. Let Z be a resolution of singularities. After a suitable base change by $\Delta' \ni s \to s^k = t \in \Delta$, the minimal resolution Z'' of the pull back Z' is *semistable*:

8. Definition. $X \to \Delta$ is semistable if X is smooth and the fibre X_0 over $0 \in \Delta$ is a semistable curve, i.e. a proper curve with only ordinary nodes as singularities.

The (relative) canonical model X' of Z'' is a stable curve over Δ' extending the pull back curve U'. We define

$$X'_0 = \lim_{t \to 0} U_t$$

It is easy to see that this does not depend on its construction.

3.4 projectivity

9. Theorem. *The algebraic space $\overline{M}_g$ is a projective variety.*

The proof consists of 2 parts, one of which is well known, the other is a neat piece of modern technology replacing the mind boggling stability estimates for stable curves.

10. Definition. A vector bundle V on an *algebraic space* X is *semipositive* if one of the following equivalent conditions

(a) $\mathcal{O}_{\mathbb{P}}(1)$ is *nef* on $\mathbb{P}V$,

(b) for every map $C \to X$ from a proper and smooth algebraic curve C to X, every quotient of $V|C$ has nonnegative degree,

(c) for every map $C \to X$ from a proper and smooth algebraic curve C to X and for every ample line bundle L on C, the bundle $L \otimes V|C$ is ample.

As a word of warning, semipositive really works only if you are on an algebraic space, a general analytic space won't do here.

Now if $p : X \to S$ is a stable curve of genus g, $T \to S$ any morphism, $\omega_{X/S}|T = \omega_{X_T/T}$ (canonical isomorphism) Therefore $V_k = p_*\omega^k_{X/S}$ is a vector bundle on the moduli stack $\overline{\mathcal{M}}_g$. This is a Deligne-Mumford stack, so this defines a Q-bundle on the space $\overline{M}_g$ which after tensoring a bit we may assume to be an honest vector bundle. The first point is:

11. Theorem. *V_k is semipositive.*

Proof. If S is a proper and smooth curve, and $p : X \to S$ a stable curve, we want to show that $V_k = p_*\omega^k_{X/S}$ is semipositive. There are 2 methods to do this:

METHOD 1: HODGE THEORY. This applies to $k = 1$. The PVHS for $p^0 : X^0 \to S$ over $S^0 = S \setminus \operatorname{Sing} f$ is $\mathcal{F}^1 = V_1^0 = Rp^0_*\omega_{X^0/S^0} \subset \mathcal{H} = R^1p^0_*\mathbb{Z}_{X^0} \otimes \mathcal{O}_S$. From Hodge theory then V_1^0 inherits a positive definite Herimitian metric from the Hodge bundle on the classifying space $D = h_g/Sp(2g, \mathbb{Z})$. One still has to check that things go ok over $\operatorname{Sing} f$ through the boundary of the (Satake?) compactification $\overline{D} = h_g \cup \text{cusps}/Sp(2g, \mathbb{Z})$, but they do so V_1 is positive. Note that this is a bit mind boggling: $\mathcal{H}$ possesses the Gauss-Manin (flat) connection, so it looks a bit like a trivial bundle and therefore $\mathcal{F}^1 \subset \mathcal{H}$ a bit like a universal *subbundle* on a Grassmannian, and yet it is *positive*. The bundle $\mathcal{H}$ is nice and flat but very unstable.

METHOD 2: REDUCTION TO CHARACTERISTIC P. By normalising, base change, resolution etc, we may assume that X is smooth and S has genus $q \geq 2$. Then X is a surface of general type. Assume that $V_k \twoheadrightarrow L^{-1}$ has a quotient line bundle L^{-1} of negative degree. Reducing to characteristic p and pulling back by a high power $F^n : S \to S$ of Frobenius, we may assume that L has arbitrarily large degree, namely $L = \omega_S^{k-1}(D)$ for some very ample divisor D on S, in other words we have a morphism

$$\omega_S^k \otimes L \otimes V_k \to \omega_S$$

which implies that

$$H^1(S, \omega_S^k(D) \otimes V_k) \neq (0)$$

and

$$H^1\big(X, \omega_X^k \otimes f^*(D)\big) \neq 0$$

Indeed, by the projection formula, $\omega_S^k(D) \otimes V_k = f_*\big(f^*\omega_S^k(D) \otimes \omega_{X/S}^k\big) = f_*\big(\omega_X^k \otimes f^*(D)\big)$. Now this is a contradiction to Kodaira Vanishing on X (by Ekedahl this is ok in characteristic p except possibly in characteristic 2, but there too because we can make the H^1 arbitrarily large if we want to). □

12. Remark. One may justifiably ask what *are* the bundles V_k on $\overline{\mathcal{M}}_g$. For instance, the tangent space $T_C\overline{\mathcal{M}}_g = H^1(C, T_C) = H^0(\omega_C^{\otimes 2})^\vee$ so a first (uneducated) guess would be that $V_2 = \omega_{\overline{\mathcal{M}}_g}$. Since the singularities of Def_C / G are *canonical* [HM], the considerations here would then immediately imply that $\omega_{\overline{\mathcal{M}}_g}$ is ample, i.e. $\overline{\mathcal{M}}_g$ is the minimal model of a variety of *general type*. Now it is clearly wrong that $\overline{\mathcal{M}}_g$ is a minimal model and (for g small) of general type (it is so for g large), but this philosophy is basically correct. It is true that $V_2|\mathcal{M}_g = \omega_{\mathcal{M}_g}$ but there is a correction on the boundary. My guess here would be that $V_2 = \omega_{\overline{\mathcal{M}}_g}(B)$ and we'd be getting at the statement that the pair (M_g, B) is a log minimal model of a variety of log general type.

The second point, which is done properly in [K], is to show that $\det V_k$ is an ample line bundle on $\overline{M}_g$ for k large. The idea is as follows. Let $X \to S$ be a stable curve over a proper base S. Assuming that the induced morphism $S \to \overline{\mathcal{M}}_g$ is generically finite, we'd like to show that $\det V_k$ is big, i.e. (since we already know, by semipositivity, that it is *nef*) that $\det V_k^{\dim S} > 0$. If by any chance $X \to S$ was rigidified, then V_3 would be the trivial bundle on S and

$$S^2 V_3 \twoheadrightarrow V_6$$

(which is, fibre by fibre $S^2H^0(C, \omega_C^{\otimes 3}) \twoheadrightarrow H^0(C, \omega_C^{\otimes 6})$) gives a morphism $S \to \mathit{Grass}$ to a suitable Grassmannian Grass, making $\det V_6$ the pull back of the (ample) determinant of the universal quotient bundle on Grass. Now the homogeneous ideal of a stable curve C, embedded in $\mathbb{P}^N$ by the sections of $\omega_C^{\otimes 3}$, is generated by quadrics, in other words we can reconstruct C from the map $S^2H^0(C, \omega_C^{\otimes 3}) \twoheadrightarrow H^0(C, \omega_C^{\otimes 6})$, which means that $S \to \mathit{Grass}$ is generically finite and $\det V_6$ is big. In particular this teaches that the V_k become more positive as k grows. Unfortunately, we can not assume that $X \to S$ is rigidified. On the other hand, we know that V_k is semipositive, which should increase our chances to get ampleness. The actual technicalities [K] are a bit messy, but things work.

4 Pikaart-De Jong

I will discuss a different approach to [PD]. Let G be a group, C a smooth and proper curve. A G-level structure on C is a surjective homomorphism $\pi_1(C) \twoheadrightarrow G$ modulo inner automorphisms of $\pi_1(C)$ (this is needed to get rid of the base point). If G is all of π_1, a G-level structure is the same as a Teichmüller structure. Because we are doing algebraic geometry, we will restrict to finite groups G from now on.

We denote $\mathcal{M}_g^G$ the moduli stack for curves of genus g with a G-level structure.

13. Proposition. *If $n \geq 3$ and $G = \mathbb{Z}^{2g}/n\mathbb{Z}^{2g}$, $\mathcal{M}_g^G$ is the coarse moduli space M_g^G (in particular, it is smooth and has a universal curve on top of it).*

Proof. Easy: to endow C with a G-level structure is equivalent to fix an isomorphism

$$\mathbb{Z}^{2g}/n\mathbb{Z}^{2g} \xrightarrow{\cong} H^1(C, \mathbb{Z}/n\mathbb{Z})$$

and it is an exercise to see that $\operatorname{Aut} C$ acts faithfully on $H^1(C, \mathbb{Z}/n\mathbb{Z})$, in other words, a curve with level structure has no automorphisms. □

Now there is a natural finite-to-one morphism $\mathcal{M}_g^G \to \overline{\mathcal{M}}_g$ which is forgetting the level structure. Pikaart-de Jong define the *compactification* $\overline{\mathcal{M}}_g^G$ of $\mathcal{M}_g^G$ to be the normalisation of $\overline{\mathcal{M}}_g$ in the function field of $\mathcal{M}_g^G$ and go on to prove

14. Theorem. *Let π_g be the fundamental group of a smooth proper curve of genus g, $\pi_g^k = [...[\pi_g, \pi_g], \pi_g]...]$ the k-th derived subgroup, $\pi_g^{k,n}$ the subgroup generated by n-th powers of elements of π_g^k, and $G = \pi_g/\pi_g^{k+1,n}$. If $n \geq 3$, then*

(a) if $k = 1$, $\overline{\mathcal{M}}_g^G$ is smooth iff $g = 2$,
(b) if $k = 2$, $\overline{\mathcal{M}}_g^G$ is smooth iff n is odd,
(c) if $k = 3$, $\overline{\mathcal{M}}_g^G$ is smooth iff n is odd or divisible by 4,
(d) if $k \geq 4$, $\overline{\mathcal{M}}_g^G$ is smooth.

The proof consists in tough monodromy calculations at the boundary of $\overline{\mathcal{M}}_g$ to determine precisely the local structure of the morphism $\mathcal{M}_g^G \to \overline{\mathcal{M}}_g$ there. I want to describe a more natural approach. The boundary points of a natural compactification of a moduli space should have some geometric significance.

15. Definition. Let C be a stable curve and G a finite group acting faithfully on C. The action is *admissible* if the only stabilisers occur (possibly) at singular points $x \in C$ and there the stabiliser $I_x \cong \mathbb{Z}/n_x\mathbb{Z}$ is a cyclic group acting with opposite weights on the 2 branches near x.

If G acts admissibly on C, it is easy to see that the morphism $f : C \to C/G$ to the quotient is *unramified*, that is $\omega_C = f^*\omega_{C/G}$. I propose to study the following stack:

16. Definition. An admissible G-stable curve over S is a stable curve $X \to S$, with an action $G \times X \to X$, covering the identity of S, which is admissible on each fibre. We denote $\overline{\mathcal{M}}^G$ the stack of admissible G-stable curves.

17. Proposition. *$\overline{\mathcal{M}}^G$ is proper.*

Proof. If $X \to \Delta$ is a stable curve with a G-action which is admissible on the generic fibre, then the G-action is admissible. This is an easy local calculation on X. $\square$

It is easy to see that the deformation theory of $\overline{\mathcal{M}}^G$ is unobstructed, just as in the case of stable curves without G-action. Therefore, the natural approach to show that $\overline{\mathcal{M}}^G$ is smooth is to show that an admissible G-stable curve has no automorphisms.

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