Algebraic Topology – Comments on Problem Sheet 1

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Exercise 2. Let me show a nice trick that I have learned from the solution of some of you. The following diagram sums up the situation.

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "p"'] & X \coprod Y \arrow[d, "q"] \\ X/\sim \arrow[r, "F"] & X \cup_f Y \end{tikzcd}$$

It is quite easy to show that F is bijective and continuous. We want to show that the inverse F^{-1} is continuous.

Consider the function $\alpha\colon Y \to X/\sim$ which is defined by $\alpha\colon y \mapsto [a]_\sim$, for $a \in A$ such that $y = f(a)$. It is easy to show that α is well defined. Is α continuous? Since $f\colon A \to Y$ is a quotient map, by the universal property of the quotient topology, it is enough to check that $\alpha \circ f\colon A \to X/\sim$ is continuous. But $\alpha \circ f$ coincides with the restriction of p to A, so it is continuous. This shows that α is continuous.

Since $\{X, Y\}$ is an open cover of $X \coprod Y$, the function $\Phi\colon X \coprod Y \to X/\sim$ defined by $\Phi|_X = p$ and $\Phi|_Y = \alpha$ is continuous. It is clear that $\Phi = F^{-1} \circ q$. Since q is a quotient map, F^{-1} is continuous.

Exercise 3. A lot of you were able to construct a continuous bijective map from X to S^1, but the proof that it was a homeomorphism was usually wrong, incomplete or not very rigorous. Let me sketch a proof that uses an argument slightly different from Prof. Corti's one.

Let $X = (\mathbb{R} \times \{0,1\})/\sim$ be the quotient space considered in this exercise and let $p\colon \mathbb{R} \times \{0,1\} \to X$ be the quotient map. Let us consider the function $\varphi\colon \mathbb{R} \times \{0,1\} \to S^1$ defined by

$$\varphi(x,k) = \begin{cases} \dfrac{2x}{x^2+1} + \dfrac{x^2-1}{x^2+1}\mathrm{i} & \text{if } x \in \mathbb{R}, k = 0; \\ \dfrac{2x}{x^2+1} + \dfrac{1-x^2}{x^2+1}\mathrm{i} & \text{if } x \in \mathbb{R}, k = 1. \end{cases}$$

Compare φ with Prof. Corti's F. It is obvious that φ is continuous. One can easily check that if $z_1, z_2 \in \mathbb{R} \times \{0,1\}$ then $z_1 \sim z_2$ if and only if $\varphi(z_1) = \varphi(z_2)$. This implies that φ induces an injective function $f\colon X \to S^1$, which is defined by $f([z]) = \varphi(z)$ for all $z \in \mathbb{R} \times \{0,1\}$. In other words we have a commutative diagram of functions:

$$\begin{tikzcd} \mathbb{R} \times \{0,1\} \arrow[r, "\varphi"] \arrow[d, "p"'] & S^1 \\ X \arrow[ur, "f"'] & \end{tikzcd}$$

Since X has the quotient topology, f is continuous because φ is continuous.

To prove that f is surjective, it is enough to prove that φ is surjective. Check that φ is surjective!

So f is a bijective continuous function. We have to check that f is a homeomorphism, or equivalently that f is open, or equivalently that f is closed. To show that f is closed I will use a trick that can be used in many occasions. Consider the subset $K = [-1,1] \times \{0,1\}$ of $\mathbb{R} \times \{0,1\}$. One can see that K is compact (via Tychonoff, or because it is closed and bounded in $\mathbb{R}^2$) and that the restriction $p|_K \colon K \to X$ is surjective. This implies that X is a compact[1] topological space. Since S^1 is a Hausdorff space, the continuous map f is closed: indeed, if C is a closed subset of X, then C is compact, so $f(C)$ is compact, hence $f(C)$ is closed in S^1 because S^1 is Hausdorff.

Another possibility to show that the inverse f^{-1} of f is continuous is to find two open subsets U_0 and U_1 of S^1 such that the restrictions $f^{-1}|_{U_0}$ and $f^{-1}|_{U_1}$ are continuous, showing that for $k \in \{0,1\}$ the restriction $f^{-1}|_{U_k}$ coincides with the composite $p \circ g_k$ for some continuous function $g_k \colon U_k \to \mathbb{R} \times \{k\} \subseteq \mathbb{R} \times \{0,1\}$ for which you can write down formulae.

Actually there are a lot of choices for φ, but essentially the scheme of solution is always the same: show that φ is continuous and constant on the fibres of p, then show that f is bijective, then show that f^{-1} is continuous. Here I write down another possibility for the choice of $\varphi \colon \mathbb{R} \times \{0,1\} \to S^1$ which was implicitly taken in the solution of some of you:

$$\varphi(x,k) = \begin{cases} x - \mathrm{i}\sqrt{1-x^2} & \text{if } |x| \leq 1, k = 0; \\ \frac{1}{x} + \mathrm{i}\sqrt{1 - \frac{1}{x^2}} & \text{if } |x| \geq 1, k = 0; \\ x + \mathrm{i}\sqrt{1-x^2} & \text{if } |x| \leq 1, k = 1; \\ \frac{1}{x} - \mathrm{i}\sqrt{1 - \frac{1}{x^2}} & \text{if } |x| \geq 1, k = 1. \end{cases}$$

The function φ is continuous because it is well-defined and its restriction to each of the four closed subsets of $\mathbb{R} \times \{0,1\}$ above is continuous.

Final remark toward algebraic geometry: the topological space X, which is obtained by glueing two copies of $\mathbb{R}$ along the open set $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ via $x \mapsto \frac{1}{x}$, is exactly the real projective line $\mathbb{P}^1(\mathbb{R})$. In this exercise you have shown that $\mathbb{P}^1(\mathbb{R})$ is homeomorphic to S^1. The images in X of the two copies of $\mathbb{R}$ are the standard affine charts of $\mathbb{P}^1(\mathbb{R})$. Indeed, you could prove that the restriction $p|_{\mathbb{R} \times \{k\}} \colon \mathbb{R} \times \{k\} \to X$ is an open embedding for $k = 0, 1$. What happens with $\mathbb{P}^1(\mathbb{C})$ and $\mathbb{P}^2(\mathbb{R})$?

Exercise 9. Let $\Phi \colon \mathbb{Z} \to \pi_1(S^1, 1)$ be the group isomorphism you have seen in lectures. Many of you have written $\Phi^{-1}(\mathrm{e}^{2\pi \mathrm{i} kt})$ or $\Phi^{-1}([\mathrm{e}^{2\pi \mathrm{i} kt}])$ or $f_*(\mathrm{e}^{2\pi \mathrm{i} kt})$ for some continuous map $f \colon S^1 \to S^1$, but this notation is wrong because $\mathrm{e}^{2\pi \mathrm{i} kt}$ is a complex number and the arguments of Φ^{-1} and f_* are

[1]For me a topological space is compact if every open cover admits a finite refinement. In the definition of compactness I don't require Hausdorffness, but I know that some people do.

in $\pi_1(S^1, 1)$, i.e. the set of homotopy classes of loops of S^1 based at 1. Therefore the right way to write down the things is, for example, $f_*([t \mapsto \mathrm{e}^{2\pi \mathrm{i} kt}]) = [t \mapsto f(\mathrm{e}^{2\pi \mathrm{i} kt})]$.

Warning: it is meaningless to evaluate a homotopy class of paths at a certain time $t \in [0, 1]$, because homotopic paths can be at different places at the same time.