Algebraic Topology – Comments on Problem Sheet 2

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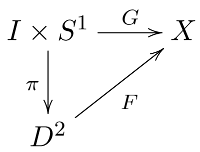
Exercise 1. To maintain mental sanity, I will denote points of I (resp. S^1, D^2) by the letters t, s (resp. z, w) and I will use the letters H, G, F for maps from $I \times I, I \times S^1, D^2$ to X, respectively.

(i) $\Rightarrow$ (ii): let $f\colon S^1 \to X$ be continuous map and we want to show that it can be extended to a continuous map on D^2. By (i), f is nullhomotopic, so there exists a continuous map $G\colon I \times S^1 \to X$ such that $G(0, \cdot)$ is constant and $G(1, \cdot) = f$. Let $x_0 \in X$ be the value of $G(0, \cdot)$. We define $F\colon D^2 \to X$ by

$$F(w) = \begin{cases} G\left(|w|, \frac{w}{|w|}\right) & \text{if } w \neq 0, \\ x_0 & \text{if } w = 0. \end{cases}$$

If F is continuous we are done because F is an extension of f to D^2. We have to show that F is continuous. This was missing in many of your solutions!

The situation is summarised by the following commutative diagram



where $\pi\colon I \times S^1 \to D^2$ is defined by $\pi(t, z) = tz$ for all $t \in I$ and $z \in S^1$. Since $I \times S^1$ is compact and D^2 is Hausdorff, π is closed. Since π is closed and surjective, π is a quotient map. Since π is a quotient map, F is continuous by the universal property of the quotient topology.

More explicitly, if C is a closed subset of X, then $G^{-1}(C)$ is closed in $I \times S^1$. Since π is closed and surjective, $F^{-1}(C) = \pi(G^{-1}(C))$ is closed in D^2.

Another method. I will show an explicit proof of the continuity of F. It is clear that it is enough to show that F is continuous at the point 0. So let U be a neighbourhood of x_0 in X. For every $z \in S^1$, the map G is continuous at $(0, z)$ and the products $[0, \varepsilon) \times V$ constitute a local basis of neighbourhoods of $(0, z)$ in $I \times S^1$, as $\varepsilon \in (0, 1]$ and V is a neighbourhood of z in S^1. Therefore, for every $z \in S^1$, there exist a real number $0 < \varepsilon_z \leq 1$ and a neighbourhood V_z of z in S^1 such that $G([0, \varepsilon_z) \times V_z) \subseteq U$.

Since S^1 is compact, we can extract from the open cover $\{V_z\}_{z \in S^1}$ a finite subcover $\{V_{z_1}, \dots, V_{z_n}\}$. This implies that if $\varepsilon = \min\{\varepsilon_{z_1}, \dots \varepsilon_{z_n}\}$ then each point of $[0, \varepsilon) \times S^1$ is contained in $[0, \varepsilon_{z_i}) \times V_{z_i}$ for some $i \in \{1, \dots, n\}$. Hence $G([0, \varepsilon) \times S^1) \subseteq U$. Therefore $F(w) \in U$ for all $w \in D^2$ such that $|w| < \varepsilon$. We have found a neighbourhood of 0 in D^2 which is mapped into U by F.

This is the only proof, known by me, that shows that $\lim_{w \to 0} F(w) = x_0$. You can see that both proofs of the continuity of F rely on a compactness argument.

(ii) $\Rightarrow$ (i): let $f\colon S^1 \to X$ be a continuous map. By (ii) it extends to D^2, so there exists a continuous map $F\colon D^2 \to X$ such that $F|_{S^1} = f$. Let $\pi\colon I \times S^1 \to D^2$ be defined by $\pi(t,z) = tz$ for $t \in I$ and $z \in S^1$. Consider the map $G\colon I \times S^1 \to X$ defined by $G = F \circ \pi$.

$$\begin{array}{ccc} I \times S^1 & \xrightarrow{G} & X \\ \pi\downarrow & \nearrow_{F} & \\ D^2 & & \end{array}$$

G is continuous, because it is the composite of two continuous maps, and such that $G(0,\cdot) = F(0)$ is constant and $G(1,\cdot) = f$. So G is a homotopy between f and a constant map.

(ii) $\Rightarrow$ (iii): this is the most difficult part of the exercise. A lot of you have not understood that in (i) and (ii) we consider free homotopies, whereas in (iii) we are considering homotopies relatively to $\{0,1\}$.

Let $\gamma\colon I \to X$ be a loop based at $x_0 \in X$. Consider $\phi\colon I \to S^1$ defined by $\phi(s) = \mathrm{e}^{2\pi \mathrm{i} s}$ for $s \in I$. In other words, ϕ glues 0 and 1 together. Since $\gamma(0) = \gamma(1)$, there exists a map $f\colon S^1 \to X$ such that $\gamma = f \circ \phi$.

$$\begin{array}{ccc} I & \xrightarrow{\gamma} & X \\ \phi\downarrow & \nearrow_{f} & \\ S^1 & & \end{array}$$

Is f continuous? Since I is compact and S^1, ϕ is closed. Since ϕ is closed and surjective, ϕ is a quotient map. Therefore f is continuous because of the universal property of the quotient topology.

By (i) (since we have already shown that (i) $\Leftrightarrow$ (ii)) we know that f is nullhomotopic, so there exists $G\colon I \times S^1 \to X$ such that $G(0,\cdot)\colon S^1 \to X$ is constant and $G(1,\cdot) = f$. Let $x_1 \in X$ be the value of $G(0,\cdot)$. The point x_1 can be different from x_0. Consider the map $p\colon I \times I \to I \times S^1$ defined by $p(t,s) = (t, \mathrm{e}^{2\pi \mathrm{i} s})$ for $t,s \in I$.

$$\begin{array}{ccc} I \times I & \xrightarrow{G \circ p} & X \\ p\downarrow & \nearrow_{G} & \\ I \times S^1 & & \end{array}$$

In general, the composite $G \circ p\colon I \times I \to X$ is *not* a homotopy between γ and the constant loop relatively to $\{0,1\}$. So we need to find another way.

By (ii) there exists a continuous map $F\colon D^2 \to X$ such that $F|_{S^1} = f$. Let $\psi\colon I \times D^2 \to D^2$ be a deformation retraction of D^2 onto the point $1 \in S^1 = \partial D^2$. For instance we can take

$$\psi(t,w) = (1-t) + tw \qquad \text{for } t \in I, w \in D^2$$

because D^2 is convex. Now, consider $H\colon I \times I \to X$ defined by

$$H(t,s) = F\left(\psi(t, (\pi \circ p)(1,s))\right) = F(\psi(t, \mathrm{e}^{2\pi \mathrm{i} s})) = F(1 - t + t\mathrm{e}^{2\pi \mathrm{i} s})$$

for all $t,s \in I$. Here $\pi\colon I \times S^1 \to D^2$ and $p\colon I \times I \to I \times S^1$ are defined above. We can see that $H(1,s) = \gamma(s)$ and $H(t,0) = H(t,1) = H(0,s) = x_0$

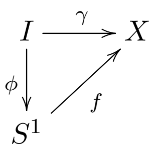
for all $t, s \in I$. So H is a homotopy between γ and the constant loop based at x_0 relatively to $\{0, 1\}$.

Another proof of (ii) ⇒ (iii). Let me sketch another proof that I learnt from the solution of some of you. Let $\gamma\colon I \to X$ be a loop based at x_0 and let $F\colon D^2 \to X$ be as above, i.e. $F(\mathrm{e}^{2\pi \mathrm{i}t}) = \gamma(t)$ for all $t \in I$. In particular, $F(1) = x_0$. Now, consider the loop $\beta\colon I \to D^2$ based at $1 \in D^2$ defined by $\beta(t) = \mathrm{e}^{2\pi\mathrm{i}t}$ for all $t \in I$. We have $\gamma = F \circ \beta$. Let $F_\star\colon \pi_1(D^2, 1) \to \pi_1(X, x_0)$ be the group homomorphism induced by F. So in $\pi_1(X, x_0)$ we have $[\gamma] = [F \circ \beta] = F_\star[\beta]$. Since D^2 is contractible, $[\beta] = e$ and hence $[\gamma] = e$.

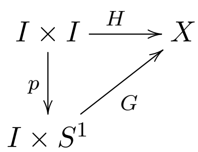
Remark. Some of you have used the presentation of $\pi_1(X, x_0)$ as homotopy classes of maps $S^1 \to X$. Firstly, here you have to be very careful with based points. Secondly, proving that this presentation is true is equivalent to solve this exercise. So, in my opinion, you cannot assume this. I will be more precise below.

By $[(S^1, 1), (X, x_0)]$ I denote the set of equivalence classes of continuous maps of pointed topological spaces $(S^1, 1) \to (X, x_0)$ with respect to the equivalence relation given by the homotopy rel $\{1\}$. Let $\phi\colon I \to S^1$ be the loop based at $1 \in S^1$ defined by $\phi(s) = \mathrm{e}^{2\pi\mathrm{i}s}$ for $s \in I$. It is well known that $[\phi]$ is a generator of the group $\pi_1(S^1, 1) \simeq \mathbb{Z}$. Consider the function $\Psi\colon [(S^1, 1), (X, x_0)] \to \pi_1(X, x_0)$ defined by $\Psi([f]) = f_\star([\phi])$ for all $f\colon (S^1, 1) \to (X, x_0)$. I think that proving that Ψ is well defined and bijective is more or less equivalent to solve this exercise.

(iii) ⇒ (i): let $f\colon S^1 \to X$ be continuous map. So $\gamma = f \circ \phi\colon I \to X$ is a loop based at $x_0 = f(1)$, where $\phi\colon I \to S^1$ is defined by $\phi(s) = \mathrm{e}^{2\pi\mathrm{i}s}$ for $s \in I$.



By (iii) there exists a homotopy between γ and the constant path based at x_0 relatively to $\{0, 1\}$, i.e. a continuous map $H\colon I \times I \to X$ such that $H(0, \cdot)$ is constant with value x_0, $H(1, \cdot) = \gamma$, $H(\cdot, 0) = H(\cdot, 1)$ is constant with value x_0. Consider $p = \mathrm{id}_I \times \phi\colon I \times I \to I \times S^1$. It is easy to see that there exists a map $G\colon I \times S^1 \to X$ such that $H = G \circ p$.



The map p is a quotient map because $I \times I$ is compact, $I \times S^1$ is Hausdorff, and p is surjective. Therefore G is continuous by the universal property of the quotient topology. G is a homotopy between $G(0, \cdot) = x_0$ and $G(1, \cdot) = f$.

Finally we prove that a space X is simply connected if and only if all maps $S^1 \to X$ are homotopic.

(⇒) assume that X is simply connected and let $f\colon S^1 \to X$ and $f'\colon S^1 \to X$ be two maps. From what we have proved above, we know that f and f'

are nullhomotopic, i.e. there exist two points $x, x' \in X$ such that $f \sim c_x$ and $f' \sim c_{x'}$. The symbol $\sim$ denotes the homotopy relation and $c_x \colon S^1 \to X$ denotes the constant map with value x. Since X is path connected, a path between x and x' provides a homotopy between c_x and $c_{x'}$. Since homotopy is an equivalence relation, $f \sim c_x \sim c_{x'} \sim f'$.

($\Leftarrow$) assume that all maps from S^1 to X are homotopic. For any $x, x' \in X$, a homotopy between the constant maps c_x and $c_{x'}$ provide a path between x and x'. Therefore X is path connected. We need to show that $\pi_1(X, x_0) = 0$ for all $x_0 \in X$. From what we have proven above, it is enough to show that every map $S^1 \to X$ is homotopic to a constant map. But this is the case.

Exercise 2. As you see in Exercise 4, the assumption that both U and V are open is necessary for Van Kampen's theorem to hold. So, here, you have to choose two open Möbius strips that overlap. In this case $U \cap V$ is homeomorphic to $(0, 1) \times S^1$.

Exercise 3. (a) Many of you have written that there is a natural inclusion $S^1 \hookrightarrow S^1 \vee S^1$. In my opinion there are *two* natural embeddings of S^1 in $S^1 \vee S^1$!

(b) For $n \in \mathbb{Z}$, let $r_n \colon S^1 \vee S^1 \to S^1$ such that the first circle of $S^1 \vee S^1$ is mapped identically on S^1 and the second circle of $S^1 \vee S^1$ is mapped onto S^1 via $z \mapsto z^n$. By a, b we denote the two standard generators of $\pi_1(S^1 \vee S^1) \simeq \mathbb{Z} * \mathbb{Z}$ and by x we denote the standard generator of $\pi_1(S^1) \simeq \mathbb{Z}$. It is clear that $(r_n)_\star(a) = x$ and $(r_n)_\star(b) = nx$. This shows that r_n and r_m are not homotopic if $n \neq m$.

Exercise 5. (a) I will adapt the proof of Proposition 1.26 in Hatcher's *Algebraic topology.* Let X be a path-connected topological space and let $n \geq 3$ be an integer. Let A be a set and, for $\alpha \in A$, let $\varphi_\alpha \colon S^{n-1} \to X$ be a continuous map. Let Y be obtained by gluing X and $|A|$ copies of D^n via φ_α's.

Let s_0 be a base point of S^{n-1}. Choose a basepoint $x_0 \in X$ and a path γ_α in X from x_0 to $\varphi_\alpha(s_0)$ for each α.

Let us expand Y to a slightly larger space Z that deformation retracts onto Y. The space Z is obtained from Y by attaching rectangular strips $S_\alpha = I \times I$, with the lower edge $I \times \{0\}$ attaced along γ_α, the right edge $\{1\} \times I$ attached along an arc in e^n_α, and all the left edges $\{0\} \times I$ of the different strips identified together. The top edges of the strips are not attached to anything, and this allows us to deformation retract Z onto Y.

In each cell e^n_α choose a point y_α not in the arc along which S_α is attached. Let $U = Z \setminus \cup_{\alpha \in A}\{y_\alpha\}$ and let $V = Z \setminus X$. Notice that U, V and $U \cap V$ are path-connected and $X = U \cup V$. It is clear that U deformation retracts onto X, and V is contractible. Since $\pi_1(V) = 0$, van Kampen's theorem applied to the cover $\{U, V\}$ says that $\pi_1(Z)$ is isomorphic to the quotient of $\pi_1(U)$ by the normal subgroup generated by image of the map $\pi_1(U \cap V) \to \pi_1(U)$.

But $U \cap V$ deformation retracts to a space made up of spheres S^{n-1} linked together by paths. One can see that $U \cap V$ is homotopically equivalent to

$\bigvee_\alpha S^{n-1}_\alpha$. Since $n \geq 3$, S^{n-1} is simply connected. Therefore $U \cap V$ is simply connected. This shows that $\pi_1(Z)$ is isomorphic to $\pi_1(U)$. Since Z deformation retracts to Y and U deformation retracts to X, we have $\pi_1(X) \simeq \pi_1(Y)$.