Algebraic Topology M3P21 2015
Homework 3

AC
Imperial College London
a.corti@imperial.ac.uk

16$^{\text{th}}$ February 2015

N.B.

Turn in 5 questions by **Monday, 9$^{\text{th}}$ March**, at 12:00 either in class or in my pigeon-hole in the mail-room on the 6$^{\text{th}}$ floor.

(1) Let $0 \xrightarrow{\phi_0} V_1 \xrightarrow{\phi_1} V_2 \xrightarrow{\phi_2} ... \xrightarrow{\phi_{n-1}} V_n \xrightarrow{\phi_n} 0$ be a *complex of vector spaces*, meaning that the V_i are vector spaces and the ϕ_i are linear maps with $\phi_i \circ \phi_{i-1} = 0$ for $i = 1, ..., n$. In particular, $\ker \phi_i \supset \operatorname{im} \phi_{i-1}$, so it makes sense to define the quotient spaces $H_i = (\ker \phi_i)/(\operatorname{im} \phi_{i-1})$. Show that if all the V_i are finite-dimensional, then $\sum_{i=1}^n (-1)^i \dim H_i = \sum_{i=1}^n (-1)^i \dim V_i$.

(2)

(a) We say that a sequence $A \xrightarrow{\phi} B \xrightarrow{\psi} C$ of abelian groups and group homomorphisms is *exact at* B if $\ker \psi = \operatorname{im} \phi$, and we say that a sequence

$$0 \longrightarrow A \xrightarrow{\phi} B \xrightarrow{\psi} C \longrightarrow 0$$

is *short exact* if it is exact at A, B, and C (so in particular every short exact sequence is a complex). Show that, for a short exact sequence as above, $A \cong \phi(A)$ and $C \cong B/\phi(A)$.

(b) A short exact sequence as above *splits* if there exists a homomorphism $\rho : C \to B$ with $\psi \circ \rho = \mathrm{id}$. Show that the homomorphism $A \oplus C \to B$, $(a, c) \mapsto \phi(a)+\rho(c)$, is an isomorphism. (If you want, you can think about why $0 \to \mathbb{Z}_n \to \mathbb{Z}_{n^2} \to \mathbb{Z}_n \to 0$ with $\phi(x) = nx$ doesn't split, and/or why short exact sequences of vector spaces and linear maps always split.)

(3) Let U and V be two path-connected open subsets of $\mathbb{R}^n$ such that $U \cup V = \mathbb{R}^n$. Show that $U \cap V$ is path-connected.

(4) Let $X = S^1 \times S^1$ be the torus and $Y = S^1 \times B^2$ the solid torus. Compute the induced homomorphisms on H_1 of the following two continuous maps:

(a) $f : X \to X$, $f(z, w) = (z^a w^b, z^c w^d)$ with $a, b, c, d \in \mathbb{Z}$.

(b) The inclusion map $i : X \to Y$ of X as the boundary of Y.

(5) Let X be a topological space. Let $f : X \to X$ be a homeomorphism. The *mapping torus* M_f of f is the quotient $M_f = (X \times [0, 1])/{\sim}$ where $\sim$ denotes the equivalence relation generated by $(x, 1) \sim (f(x), 0)$ for all $x \in X$. Introduce open sets $U = (X \times (0, 1))/{\sim}$ and $V = (X \times ([0, \frac{1}{2}) \cup (\frac{1}{2}, 1]))/{\sim}$. Argue carefully that that there exists a commutative diagram

$$\begin{array}{ccc} H_i(U \cap V) & \xrightarrow{i_* \oplus j_*} & H_i(U) \oplus H_i(V) \\ \cong \downarrow & & \downarrow \cong \\ H_i(X) \oplus H_i(X) & \xrightarrow{\phi} & H_i(X) \oplus H_i(X) \end{array}$$

such that $\phi(a, b) = (a + b, a + f_*(b))$.

(6) Let X be the space formed by inserting n vertical "bars" in the sphere S^2. Compute the fundamental group and all the homology groups of X.

(7) For $i = 1, 2, 3$ let $L_i \subset \mathbb{R}^3$ be three general lines, and write $X = \mathbb{R}^3 \setminus (L_1 \cup L_2 \cup L_3)$. Compute the fundamental group and all the homology groups of X.

(8) Compute the fundamental group and all the homology groups of the complement X of a (complex) line and a point not on it in $\mathbb{P}^2(\mathbb{C})$.

(9) Let X be the complement of a small disk in a torus $T^2 = S^1 \times S^1$, and let $A = \partial X \cong S^1$ be the boundary of X. Compute all the relative homology groups $H_j(X, A)$.

(10) Construct a surjective map $f : S^n \to S^n$ of degree zero, for each $n \geq 1$.