Algebraic Topology M3P21 2015
solutions 1

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(1)

(a) Quotient maps are continuous, so preimages of closed sets are closed (preimages of open sets are open, and $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$ for all $A \subset Y$). Since Y is Hausdorff, $\{y\}$ is closed (given any $x \in Y \setminus \{y\}$, the definition of Hausdorff provides an open neighborhood U_x of x which is disjoint from some open neighborhood of y, so that $Y \setminus \{y\} = \bigcup_{x \in Y \setminus \{y\}} U_x$ is open).

(b) No. Here are two counterexamples:

1. As we have seen in lecture, the quotient space $(\mathbb{R} \times \{0,1\})/\sim$ with $(x,0) \sim (x,1)$ for $x \neq 0$ is not Hausdorff, but the equivalence classes (i.e. the fibers of the quotient map) are $\{(x,0),(x,1)\}$ for $x \neq 0$, $\{(0,0)\}$, and $\{(0,1)\}$, all of which are closed in $\mathbb{R} \times \{0,1\}$.
2. Let Y be any topological space which is not Hausdorff but in which all singletons are closed, and take $f = \mathrm{id}_Y$. (For example, let Y be an infinite set together with the cofinite topology, i.e. the topology whose open sets are precisely the complements of finite subsets of Y.)

(2) We can say that $X \sqcup Y = (X \times \{0\}) \cup (Y \times \{1\})$ as a set, and that $U \sqcup V$ is open in $X \sqcup Y$ if and only if U is open in X and V is open in Y.

Let us say that $X \cup_f Y = (X \sqcup Y)/\equiv$, where $\equiv$ is the equivalence relation generated by $(x, 0) \equiv (f(x), 1)$ for all $x \in A$ (informally: $x \equiv f(x)$).

Given this, we now define a map of sets $F : X/\sim \to X \cup_f Y$ that will turn out to be bijective and bicontinuous. The only definition that comes to mind is $F([x]) := [(x, 0)]$, square brackets denoting equivalence classes. As usual, we first need to check whether this really *is* a definition, i.e. whether $[x_1] = [x_2]$ implies $[(x_1, 0)] = [(x_2, 0)]$. But $[x_1] = [x_2]$ if and only if either $x_1 = x_2$, or $x_1, x_2 \in A$ and $f(x_1) = f(x_2)$. In the latter case, $(x_1, 0) \sim (f(x_1), 1) = (f(x_2), 1) \sim (x_2, 0)$, as desired.

Next, F is surjective because every element of $X \cup_f Y$ can be written as either $[(x, 0)]$ or $[(y, 1)]$; elements of the form $[(x, 0)]$ are in the image of F by definition, while given any $y \in Y$ we can write $y = f(x)$ for some x since f is surjective, so that $[(y, 1)] = [(f(x), 1)] = [(x, 0)] = F([x])$. To see that F is injective, notice that $[(x_1, 0)] = [(x_2, 0)]$ if and only if $x_1 = x_2$ or $f(x_1) = f(x_2)$.

To show bicontinuity, let us write $p : X \to X/\sim$ and $q : X \sqcup Y \to X \cup_f Y$ for the quotient maps. Given a set $U \subset X \cup_f Y$, we need to prove that $q^{-1}(U)$ is open if and only if $(F \circ p)^{-1}(U)$ is open. Let us abbreviate $(F \circ p)^{-1}(U) = V$. Then $V = \{x \in X : [(x, 0)] \in U\}$, and hence

$$q^{-1}(U) = (V \times \{0\}) \cup (f(V \cap A) \times \{1\})$$

Thus, it suffices to show that V open in X implies $f(V \cap A)$ open in Y. But $V \cap A$ is open in A by definition of the subspace topology, and is a union of equivalence classes because V is.

(3) Let $X = \mathbb{R} \times \{0, 1\}$ and let $p : X \to X/\sim$ denote the quotient map. We will write down a bijection of sets $f : X/\sim \to S^1$, and then show that this is a homeomorphism.

Fix a homeomorphism $F : \mathbb{R} \to S^1 \setminus \{i\}$ (we are thinking $S^1 \subset \mathbb{C}$) such that $\lim_{x\to\pm\infty} F(x) = i$. Just to convince ourselves that such maps actually exist, we can specify F to be the inverse of stereographic projection centered at $i \in S^1$. The explicit equations are $F(x) = \frac{2x}{x^2+1} + \frac{x^2-1}{x^2+1} i$ and $F^{-1}(z) = \frac{\mathrm{Re}(z)}{1-\mathrm{Im}(z)}$ for $z \neq i$.

Now we define $f : X/\sim \to S^1$ by setting $f([(x, 0)]) := F(x)$ for all $x \in \mathbb{R}$ and $f([(0, 1)]) := i$. This is a legitimate definition because every equivalence class either contains exactly one element of the form $(x, 0)$, or contains the element $(0, 1)$. Since F is bijective from $\mathbb{R}$ onto $S^1 \setminus \{i\}$, it is clear that f is bijective from $X/\sim$ onto S^1. It now suffices to show that f and f^{-1} are both continuous.

We will show that $U \subset S^1$ is open if and only if $f^{-1}(U)$ is open. By the definition of the quotient topology on $X/\sim$, it suffices to prove that U is open if and only if $(f \circ p)^{-1}(U)$ is open. Now

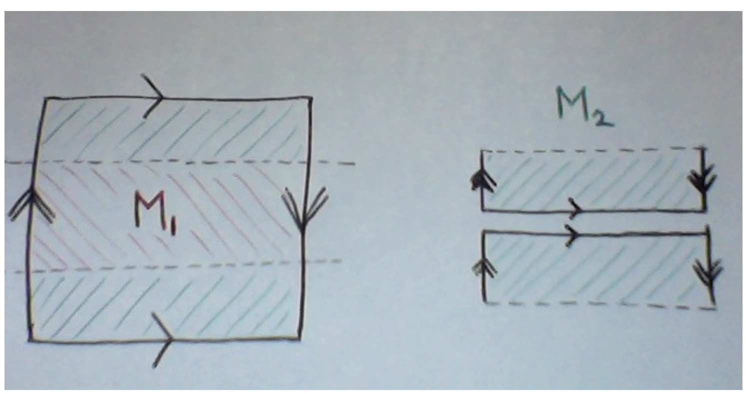
$$(f \circ p)^{-1}(U) = [F^{-1}(U) \times \{0\}] \cup [(I(F^{-1}(U) \setminus \{0\}) \cup Y) \times \{1\}], \qquad (*)$$

where $I : \mathbb{R}^* \to \mathbb{R}^*$ is defined by $I(x) = \frac{1}{x}$, and $Y = \{0\}$ if $i \in U$ whereas $Y = \emptyset$ if $i \notin U$.

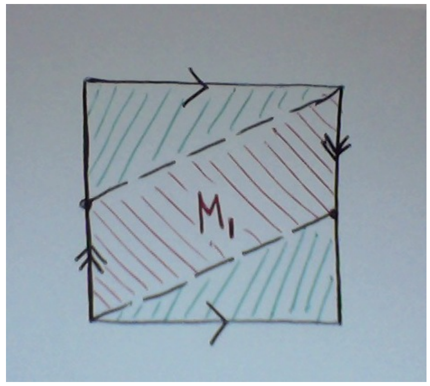
Case 1: $i \notin U$**.** In this case, U is open if and only if $F^{-1}(U)$ is open. Since $Y = \emptyset$, this is in turn obviously equivalent to the right-hand side of $(*)$ being open.

Case 2: $i \in U$**.** In this case, U is open if and only if $F^{-1}(U)$ is open *and* U in addition contains an open neighborhood of i. This is in turn equivalent to $F^{-1}(U)$ being open *and* containing a set of the form $|x| > N$ for some $N \gg 1$ (since $\lim_{x\to\pm\infty} F(x) = i$). But that is again equivalent to both of the disjoint parts on the right-hand side of $(*)$ being open since now $Y = \{0\}$.

(4) One possibility:



Another one:



The important point is that the points on the boundary of the square that are being identified *before* the cut are still being identified *the same way* after the cut. For instance, $(0, 0.9)$ on the left vertical edge must always be identified with $(1, 0.1)$ on the right vertical edge before *and after* cutting. Most of you identified $(0, 0.9)$ with $(1, 0.6)$, and $(0, 0.4)$ with $(1, 0.1)$, after cutting.

(5) I don't feel like writing this up. Indeed it is similar to (4). I will do it in class if you ask me. Perhaps I can give a hint. Use the model $\mathbb{P}^2(\mathbb{R}) = D^2 \cup_f S^1$ to give $\mathbb{P}^2(\mathbb{R})$ the structure of a CW complez: a polygon with certain identifications on the boundary. Then cut and paste, a bit like in question (4).

(6) (a) By definition, the restriction of p_n to $H^n_{\mathbb{C}}$ is surjective if and only if every equivalence class (in other words, every S^1-orbit) in S^{2n+1} contains an element of $H^n_{\mathbb{C}}$. But this is easy to see: Let $z \in S^{2n+1}$. If $z_{n+1} = 0$, then $z \in H^n_{\mathbb{C}}$ anyway. If $z_{n+1} \neq 0$, then $t := |z_{n+1}|/z_{n+1} \in S^1$, and the $(n+1)$-st coordinate of tz is equal to $|z_{n+1}| \in [0, \infty)$, so that $tz \in H^n_{\mathbb{C}}$.

(b) If $z \sim w$, then $w = tz$ for some $t \in S^1$, hence $w_{n+1} = tz_{n+1}$. Since $z, w \in H^n_{\mathbb{C}}$, both w_{n+1} and z_{n+1} are nonnegative real numbers. If $z_{n+1} = 0$, then $w_{n+1} = 0$, and there is nothing to show. But if $z_{n+1} > 0$, then $t = w_{n+1}/z_{n+1} \in [0, \infty)$, and so $t = 1$ because $t \in S^1$; thus, $z = w$.

(c) Let us denote this map by f. Then f is clearly continuous as a map from $H^n_{\mathbb{C}}$ to $\mathbb{C}^n = \mathbb{R}^{2n}$. Moreover, the image of f is contained in the unit ball because $|z_1|^2 + ... + |z_n|^2 = 1 - |z_{n+1}|^2 \leq 1$. We now claim that the map

$g : B^{2n} \to \mathbb{C}^{n+1}$ given by $g(z) = (z, \sqrt{1 - |z|^2})$ is a continuous inverse to f. Indeed, g is obviously continuous, takes values in $H^n_{\mathbb{C}}$, and satisfies $f \circ g = \mathrm{id}$, $g \circ f = \mathrm{id}$.

(7) f is nullhomotopic if and only if there exists a continuous map $h \colon S^1 \times [0, 1] \to X$ with $h(z, 0) = f(z)$ and $h(z, 1) = x_0$ for all $z \in S^1$ and some fixed $x_0 \in X$. Thus, given g, and viewing D^2 as a subspace of $\mathbb{C}$ as usual, we can simply define $h(z, s) := g((1 - s)z)$. Conversely, given h, we can define $g(z) := h(z/|z|, 1 - |z|)$ for $z \neq 0$ and then only need to observe that $\lim_{z\to 0} g(z) = x_0$, so that defining $g(0) := x_0$ yields a continuous extension of g from $B^2 \setminus \{0\}$ to B^2.

Remark 1: It is possible to phrase this argument more geometrically by showing that

$$\begin{aligned} (S^1 \times [0, 1])/(S^1 \times \{1\}) &\to B^2 \\ [(z, s)] &\mapsto (1 - s)z \end{aligned}$$

is well-defined and a homeomorphism; i.e. the cone over a circle is homeomorphic to a disk.

Remark 2: We may wish to require that h be a homotopy *relative endpoints*, i.e. that $h(1, s) = x_0$ for all $s \in [0, 1]$ with $x_0 = f(1)$. In this case $h(z, s) := g((1 - s)z + s)$ would work. In some sense we are using here that B^2 with the subspace $[0, 1]$ collapsed is homeomorphic to B^2.

(8)

$$F_s(t) = \begin{cases} f(\frac{4t}{1+s}) & \text{for } t \in [0, \frac{1+s}{4}], \\ g(4t - (1 + s)) & \text{for } t \in [\frac{1+s}{4}, \frac{2+s}{4}], \\ h(\frac{4t-(2+s)}{2-s}) & \text{for } t \in [\frac{2+s}{4}, 1]. \end{cases}$$

(9) Set $n = \phi(1)$. Then $\phi(x) = nx$ for all $x \in \mathbb{Z}$, and we can take $f(z) = z^n$ since it holds by definition that $f_*(\Phi(x)) = [t \mapsto f(\Phi(x)(t))] = [t \mapsto e^{2\pi i x n t}] = \Phi(\phi(x))$.

(10) Notice that $z, w \in S^1$ so we can just set $f(z, w) = z$ to get a counterexample. We could also realize T^2 as a surface of revolution in $\mathbb{R}^3$ as usual and note that the map given by projection onto the plane perpendicular to the axis of symmetry is a counterexample.