Algebraic Topology M3P21 2015 solutions 3

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11th$^{\text{th}}$ March 2015

A small disclaimer

This document is a bit sketchy and it leaves some to be desired in several other respects too. I thought it is more useful to you if I show you this now than if I show you a much better document at a time infinitely far into the future.

(1)

$$\sum_{i=1}^{n}(-1)^i \dim H_i =$$
$$= \sum_{i=1}^{n}(-1)^i \dim\ker\phi_i + \sum_{i=1}^{n}(-1)^{i-1}\dim\operatorname{im}\phi_{i-1} =$$
$$= \sum_{i=1}^{n}(-1)^i(\dim\ker\phi_i + \dim\operatorname{im}\phi_i)$$

since $\dim\operatorname{im}\phi_0 = \dim\operatorname{im}\phi_n = 0$. But $\dim\ker\phi + \dim\operatorname{im}\phi = \dim V$ for all linear maps $\phi\colon V \to W$.

(2) (a) By exactness at A, $0 = \operatorname{im}(0) = \ker(\phi)$ so ϕ is an isomorphism onto its image as desired. Exactness at C gives $C = \ker(0) = \operatorname{im}(\psi)$, and

so $C = \operatorname{im}\psi \cong B/\ker(\phi) = B/(\operatorname{im}\phi)$ by the first isomorphism theorem of group theory and exactness at B.
(b) Injectivity: If $\phi(a) + \rho(c) = 0$, then applying ψ yelds $c = 0$ by exactness of the sequence and the defining property of ρ. Hence $\phi(a) = 0$ and so $a = 0$ by exactness. Surjectivity: For any $b \in B$ put $c = \psi(b)$ and write $b = (b - \rho(c)) + \rho(c)$. Notice that $b - \rho(c) \in \ker(\psi)$ since $\psi(b - \rho(c)) = c - (\psi \circ \rho)(c) = 0$, and hence $b - \rho(c) = \phi(a)$ for some a by exactness.

Woffle The maps in the exact sequence $0 \to \mathbb{Z}_n \to \mathbb{Z}_{n^2} \to \mathbb{Z}_n \to 0$ are given by $\phi(x \mod n) = (nx \mod n^2)$ (well-defined since if x changes by a multiple of n then nx changes by a multiple of n^2; and clearly a group homomorphism) and $\psi(x \mod n^2) = (x \mod n)$ (well-defined since n^2 is a multiple of n. The obvious guess $\rho(x \mod n) = x \mod n^2$ is not well defined since n is not a multiple of n^2. We can force well-definedness by insisting that $x \in \{0, 1, \ldots, n-1\}$ but the resulting ρ would not be a group homomorphism (consider $\rho(x + x)$ and $\rho(x) + \rho(x)$ for $x = n - 1$).

In fact $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is not isomorphic to $\mathbb{Z}_4$ (it has no elements of order 4) and, more generally, and for the same reason, $\mathbb{Z}_n \oplus \mathbb{Z}_n$ is not isomorphic to $\mathbb{Z}_{n^2}$ (all elements have order dividing n, so no element has order n^2).

A theorem of linear algebra states that linear subspaces of vector spaces over a field always have complements, and if $D \subset B$ complements $\ker(\psi)$ then the natural map $D \to B/\ker(\psi)$ is an isomorphism. Thus $D \cong B/\ker(\psi) \cong \operatorname{im}\psi = C$, and composing the inverse of the composition of these isomorphisms with the inclusion of D into B yelds the desired linear map ρ.

(3) This is a straight-up Mayer-Vietoris question. We have a long exact sequence

$$\cdots \to H_1(\mathbb{R}^n) \to H_0(U \cap V) \to H_0(U) \oplus H_0(V) \to H_0(\mathbb{R}^n) \to 0.$$

Now $H_1(\mathbb{R}^n) = 0$ since $\mathbb{R}^n$ is contractible (or even only since $\mathbb{R}^n$ is simply-connected) so $H_0(U \cap V)$ injects into $H_0(U) \oplus H_0(V)$ with image equal to the kernel of the surjection $\phi : H_0(U) \oplus H_0(V) \to H_0(\mathbb{R}^n)$. Now $H_0(U) \cong H_0(V) \cong H_0(\mathbb{R}^n) \cong \mathbb{Z}$ since all three spaces are path-connected; conversely $H_0(U \cap V) \cong \mathbb{Z}$ would *imply* that $U \cap V$ is path-connected by Hatcher Proposition 2.6.

It seems intuitively obvious that a surjection $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ must have a "1-dimensional" kernel. In fact, this would be obvious from the standard rank-nullity theorem if we were using homology with coefficients in the abelian group $G = (k, +)$, where k is a field—for example $k = \mathbb{Q}$ (so this is then one way of answering the question, assuming that Mayer-Vietoris works with G-coefficients).

Alternatively "rank-nullity for abelian groups" (Google!) shows that $H_0(U \cap V)$ has *rank* 1, and since $H_0(U \cap V)$ is torsion-free by Hatcher Proposition 2.6, $H_0(U \cap V) \cong \mathbb{Z}$.

Arguably the least pretentious way out is to note that $\phi(a, b) = a - b$ if we use augmentation to identify H_0 of each of $U, V, \mathbb{R}^n$ with $\mathbb{Z}$, and so then obviously $\ker(\phi) = \{(a, a) : a \in \mathbb{Z}\} \cong \mathbb{Z}$.

(4) It suffices to compute the induced maps on π_1^{ab}, or equivalently on π_1 since all fundamental groups in this question are abelian.

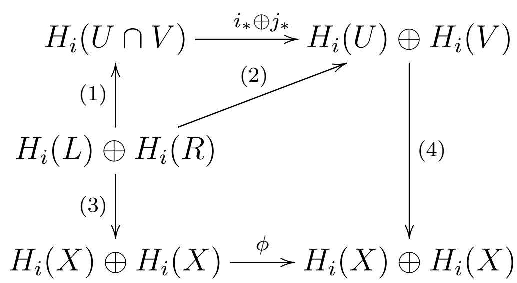
(a) We know that $\pi_1(S^1 \times S^1) \cong \mathbb{Z}^2$ via the isomorphism whose inverse sends $(m, n) \in \mathbb{Z}^2$ to the homotopy class of the loop $t \mapsto (e^{2\pi imt}, e^{2\pi int})$. Thus (exactly as in Homework 3) it follows straight from the definitions that $f_*(m, n) = (am + bn, cm + dn)$, or in other words $f_* = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

(b) Similarly, what we have proved about fundamental groups is enough to deduce that the map $\mathbb{Z} \to \pi_1(S^1 \times B^2)$, $m \mapsto [t \mapsto (e^{2\pi imt}, 1)]$, is a group isomorphism (using that B^2 is contractible). It is then once again immediate from the definitions that $i_*(m, n) = m$.

(5) Notice that $U \cap V = L \sqcup R$, where $L = (X \times (0, \frac{1}{2}))/{\sim}$ and $R = (X \times (\frac{1}{2}, 1))/{\sim}$. Hence the question comes down to understanding the relation between the homology groups of U, V, L, R and X. I claim that there exist four homotopy equivalences $\alpha : L \to X$, $\beta : R \to X$, $\gamma : U \to X$ and $\delta : V \to X$ such that the following four diagrams commute:

$$\begin{array}{ccc} L & \xrightarrow{i|_L} & U \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\gamma} \\ X & \xrightarrow{\text{id}} & X \end{array} \qquad \begin{array}{ccc} L & \xrightarrow{j|_L} & V \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\delta} \\ X & \xrightarrow{\text{id}} & X \end{array} \qquad \begin{array}{ccc} R & \xrightarrow{i|_R} & U \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ X & \xrightarrow{\text{id}} & X \end{array} \qquad \begin{array}{ccc} R & \xrightarrow{j|_R} & V \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\delta} \\ X & \xrightarrow{f} & X \end{array}$$

Let's assume for now that this is true. Then we have the following commutative diagram:



The arrows (1), (2), (3), (4) will be explained presently; (1), (3), (4) are isomorphisms. Hence there is one and only one way of defining ϕ such that the diagram commutes. Once (1), (2), (3), (4) have been defined, it will therefore be possible to compute ϕ, and the result will be as claimed.

(1) is the canonical isomorphism identifying the homology of a disjoint union with the direct sum of the homologies of its pieces (Hatcher Proposition 2.6). So (1) maps (l, r) to $\lambda_*(l) + \rho_*(r)$, where $\lambda : L \to U \cap V$ and $\rho : R \to U \cap V$ are the inclusions.

(2) is defined to be the composition of $i_* \oplus j_*$ with (1). In other words, (2) maps a pair (l, r) to $(i_*(\lambda_*(l) + \rho_*(r)), j_*(\lambda_*(l) + \rho_*(r))) = ((i|_L)_*(l) + (i|_R)_*(r), (j|_L)_*(l) + (j|_R)_*(r))$.

(3) maps (l, r) to $(\alpha_*(l), \beta_*(r))$. This is an isomorphism since α, β are homotopy equivalences.

(4) maps (u, v) to $(\gamma_*(u), \delta_*(v))$. This is an isomorphism since γ, δ are homotopy equivalences.

As indicated above, with these definitions in place we can compute $\phi(a, b)$. In order to do so, we write $(a, b) = (3)(l, r) = (\alpha_*(l), \beta_*(r))$. Then $\phi(a, b) = (4)(2)(l, r)$ by commutativity of the diagram. Thus, $\phi(a, b) = (4)(2)(l, r) = (\gamma_*(i|_L)_*(l) + \gamma_*(i|_R)_*(r), \delta_*(j|_L)_*(l) + \delta_*(j|_R)_*(r))$. Finally, using the four small diagrams, $\phi(a, b) = (\alpha_*(l) + \beta_*(r), \alpha_*(l) + f_*\beta_*(r)) = (a + b, a + f_*(b))$ as desired.

It remains to discuss the four small diagrams, i.e. to define $\alpha, \beta, \gamma, \delta$, check that they are indeed homotopy equivalences as claimed, and that the diagrams actually commute.

Defining α, β, γ is easy since L, R, U are obviously homeomorphic to $X \times$ *interval*. For example, $L = (X \times (0, \frac{1}{2}))/\sim$ and all equivalence classes are single points, so we can just put $\alpha([(x, t)]) = x$; likewise $\beta([(x, t)]) = x$ and $\gamma([(x, t)]) = x$. Clearly the first and the third diagram commute.

In order to define δ we produce a homeomorphism $\hat{\delta} : V \to X \times (0,1)$ and define $\delta = \mathrm{pr}_X \circ \hat{\delta}$. In order to define $\hat{\delta}$ we recall that $V = (X \times ([0, \frac{1}{2}) \cup (\frac{1}{2}, 1]))/\!\sim$ and employ a piecewise definition: For $t \in [0, \frac{1}{2})$ put $\hat{\delta}([(x,t)]) = (x, t+\frac{1}{2})$. For $t \in (\frac{1}{2}, 1]$ put $\hat{\delta}([(x,t)]) = (f(x), t-\frac{1}{2})$. This is well-defined since the only nontrivial equivalence relations are $(x,1) \sim (f(x), 0)$, and indeed $(x,1)$ gets mapped to $(f(x), \frac{1}{2})$ by the second part of the definition of $\hat{\delta}$, as does $(f(x), 0)$ by the first part.

Given that $\hat{\delta}$ is a homeomorphism (which is the only step of the proof I'm skipping—this follows easily from f being a homeomorphism), it is clear that δ is a homotopy equivalence. Commutativity of the second and fourth diagrams is clear from the definitions.

(6) X is homotopic to $(\vee^n S^1) \vee S^2$ so by repeated applications of Seifert-van Kampen $\pi_1(X) = \mathbb{Z}^{\star n}$. There are various ways to compute homology. An efficient way is to use the exact sequence of the pair (X, A) where $A = \vee^n S^1$. First check that $H_1\, A = \mathbb{Z}^n$, $H_0\, A = \mathbb{Z}$ and all other $H_i\, A = (0)$. Then note that (X, A) is a good pair so $H_i(X, A) = H_i\, X/A = H_i\, S^2$. The exact sequence looks like

$$H_2\, A\big((= (0)\big) \to H_2\, X \to H_2\, X/A(= \mathbb{Z}) \to H_1\, A(= \mathbb{Z}^n) \to H_1\, X \to H_1\, X/A\big(= (0)\big)$$

Note that by excision $H_2(X, A) = H_2(S^2, \mathrm{pt})$ and the natural map $H_2(X, A) \to H_1\, A$ factors through $H_1\, \mathrm{pt} \to H_1\, A$ induced by the natural inclusion $\mathrm{pt} \to A$ hence it is the zero map. It follows that:

$$H_i\, X = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z}^n & \text{if } i = 1 \\ \mathbb{Z} & \text{if } i = 2 \\ (0) & \text{otherwise} \end{cases}$$

(7) This is dead easy: upto homeomorphism it doesn't matter where the lines are: $X = (\mathbb{R}^2 \setminus \{P_1, P_2, P_3\}) \times \mathbb{R}$ is homotopic to $\vee^3 S^1$, hence $\pi_1(X) = \mathbb{Z}^{\star 3}$ and $H_0\, X = \mathbb{Z}$, $H_1\, X = \mathbb{Z}^3$, and all other H_i are (0).

(8) $\mathbb{P}^2(\mathbb{C})$ minus a line is the affine plane $\mathbb{C}^2$ so $X = \mathbb{C}^2 \setminus \{0\} = \mathbb{R}^4 \setminus \{0\} \sim S^3$. So $\pi_1(X) = (0)$, $H_3\, X = H_0\, X = \mathbb{Z}$ and all other $H_i\, X = (0)$.

(9) Right. Well (X, A) is a good pair so $H_i(X, A) = H_i(X/A)$ and X/A is the torus again: $H_i(X, A) = H_i\, S^1 \times S^1$ and everybody knows what these groups are: $H_0 = H_2 = \mathbb{Z}$, $H_1 = \mathbb{Z}^2$ and all other $H_i = 0$.

(10) Oh, I don't know. If $A = S^{n-1} \subset S^n$ then $S^n/A = S^n \vee S^n$. So let $\pi\colon S^n \to S^n \vee S^n$ be the quotient map and $h = \mathrm{id} \vee \varphi\colon S^n \vee S^n \to S^n$ where $\varphi\colon S^n \to S^n$ is your favourite map of degree -1 fixing the point where the two sphere are joined. Now $f = h \circ \pi$ will do the trick.