MI P1 ANALYSIS I

JAN 2011

§0 Logistics

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Office Hrs: Mon 17:00 – 18:00

Lectures: Mon 10:00 & 16:00
Thu 10:00

Problem Class: Mon 11:00

Assessment

- Exam 90%
- 2x progress tests 10%

Test 1 : Th 10th Feb
Test 2 : Th 17th March

(wks 4 & 9)

Course materials

http://www3.imperial.ac.uk/mathematics/students/undergraduate/coursematerials

Course Aims

- Rigorous treatment of:

  limits (sequences & series)

  continuity

- Basis for future study of analysis

Syllabus

1. Sequences & limits of sequences
2. Basic results on limits
3. The general principle of convergence
4. Series & convergence tests.
5. Rules for manipulating convergent series
6. Continuous functions

§1 Sequences & limits of sequences

Defn 1.1 A sequence of real/complex numbers is an infinite list of real/complex numbers:

$$a_1, a_2, a_3, \dots, a_n, \dots$$

one for each positive integer $n \in \mathbb{N}^\times$.

Rmk 1.2 A sequence is a function

$$a : \mathbb{N}^\times \longrightarrow \mathbb{R} \quad (\mathbb{C})$$
$$(\mathbb{R}^n, \mathbb{C}^n)$$

$$a(n) = a_n$$

Examples 1.3

(i) $1, \frac{1}{2}, \frac{1}{3}, \dots \qquad a_n = \frac{1}{n}$

(ii) $1, 0, 1, 0 \dots$

$$a_n = \frac{1-(-1)^n}{2} = \begin{cases} 1 & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$$

(iii) $1, 1, 2, 3, 5, 8 \dots$

a_n = Fibonacci sequence:

$$\begin{cases} a_1 = a_2 = 1 \\ a_n = a_{n-1} + a_{n-2} & n \geq 3 \end{cases}$$

(iv) $1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{3}, \dots$

$$a_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$$

Example 1.4

$a_n = \frac{n}{n+1}$ is the seq:

$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots$

Defn 1.5 The sequence (a_n) converges to ℓ if:

for all $\varepsilon > 0$, there exists $N > 0$ such that:

$n \geq N \Rightarrow |a_n - \ell| < \varepsilon$

Intuitively:

If n is large, then a_n is close to ℓ

Notation:

$a_n \to \ell$ or: $\lim_{n\to\infty} a_n = \ell$

Rmk 1.6 the definition works for complex numbers:

$a_n = b_n + i c_n$ where: $b_n, c_n \in \mathbb{R}$

$i : \sqrt{-1}$

$\ell = \ell_1 + i\ell_2$

$$|a_n - \ell| := \sqrt{(b_n - \ell_1)^2 + (c_n - \ell_2)^2}$$

The definition works for vector-valued sequences, e.g. in $\mathbb{R}^3$

$a_n = (b_n, c_n, d_n)$

$\ell = (\ell_1, \ell_2, \ell_3)$

$$|a_n - \ell| := \sqrt{(b_n - \ell_1)^2 + (c_n - \ell_2)^2 + (d_n - \ell_3)^2}$$

Example 1.7

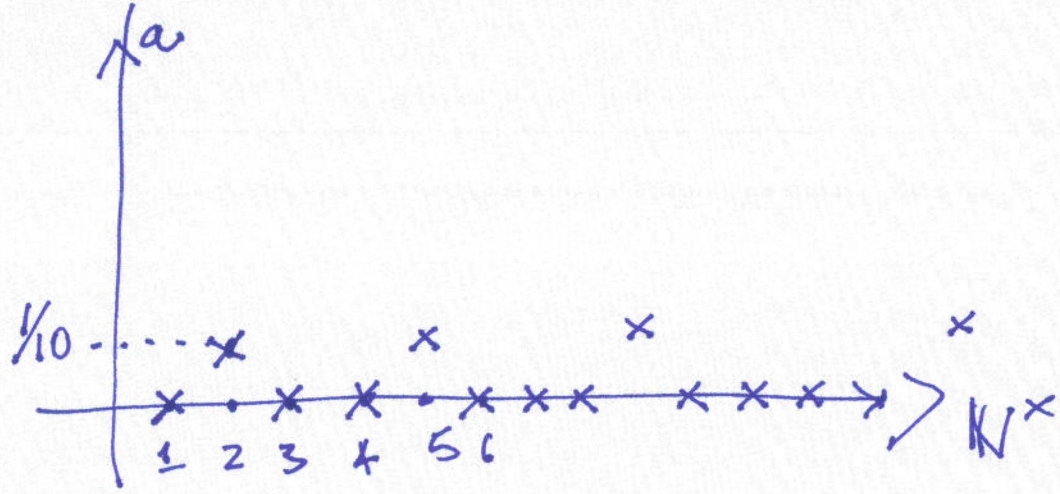
(i) $1, \frac{1}{2}, \frac{1}{3}, \dots \longrightarrow 0$

(ii) $\frac{1}{2}, 1, \frac{1}{4}, \frac{1}{3}, \frac{1}{6}, \frac{1}{5}, \dots \longrightarrow 0$

(iii) $0, \frac{1}{10}, 0, 0, \frac{1}{10}, 0, 0, 0, \frac{1}{10}, \dots$

has no limit
(does not converge)

for (iii) a graph of the sequence looks like:



(END L1)

Translation of Defn 1.5

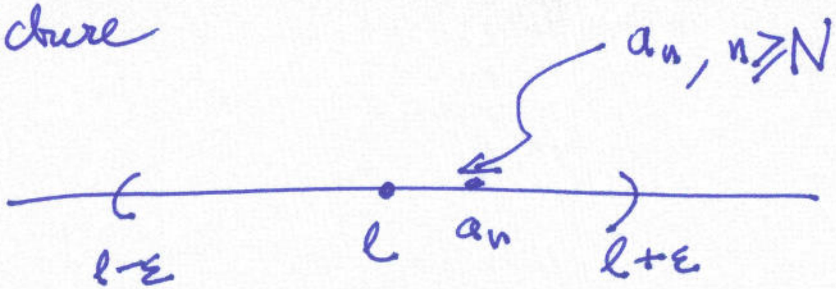
$\forall \varepsilon > 0,\ S(\varepsilon)$ "statement $S(\varepsilon)$ holds"

$S(\varepsilon)$ $\quad \exists\, N > 0$ s.t. $S(\varepsilon, N)$

where:

$S(\varepsilon, N)$ $\quad$ for all $n \geq N$, $|a_n - \ell| < \varepsilon$

picture



$(\ell-\varepsilon, \ell+\varepsilon) = \{a \in \mathbb{R} \mid |a - \ell| < \varepsilon\}$

$S(\varepsilon, N)$: for all $n \geq N$,

a_n is "within ε of ℓ"

Ex 1.8

(i) $a_n = 2$ $\qquad a_n \to 2$

(ii) $a_n = 1/n$ $\qquad \to 0$

(iii) $a_n = \dfrac{\cos n}{n} + i\,\dfrac{\sin n}{n}$ $\qquad \to 0$

(iv) $a_n = \dfrac{n+5}{n+1}$ $\qquad \to 1$

(v) $a_n = \dfrac{\cos n}{n}$ $\qquad \to 0$

(vi) $a_n = \dfrac{n^2 - 5}{n^3 - 50}$ $\qquad \to 0$

Strategy to show $a_n \to \ell$

1. Fix $\varepsilon > 0$
2. Calculate $|a_n - \ell|$
3. Solve inequality $|a_n - \ell| < \varepsilon$
4. Choose N s.t.
   $n \geq N \implies |a_n - \ell| < \varepsilon$
5. Put everything together into your proof.

Ex 1.8 (iv): $a_n = \dfrac{n+5}{n+1}$

take $\ell = 1$

1. Fix $\varepsilon > 0$
2. $|a_n - \ell| = \left|\dfrac{n+5}{n+1} - 1\right| = \dfrac{4}{n+1}$
3. $\dfrac{4}{n+1} < \varepsilon \iff \dfrac{4}{\varepsilon} < n+1$
4. choose $N = \dfrac{4}{\varepsilon}$
5. Fix $\varepsilon > 0$, choose $N = \dfrac{4}{\varepsilon}$, then

$$n \geq \frac{4}{\varepsilon} \Rightarrow |a_n - 1| = \frac{4}{n+1} < \frac{4}{4/\varepsilon} = \varepsilon$$

//

Modified strategy

1. Fix $\varepsilon > 0$

2. calculate $|a_n - l|$

$2\frac{1}{2}$ Find b_n s.t. $|a_n - l| \leq b_n$
for $n \geq n_0$

3 Solve $b_n < \varepsilon$

4 Choose $N \geq n_0$ s.t.:
$n \geq N \implies b_n < \varepsilon$

5. put everything together into your proof.

Ex. 1.8 (vi) $a_n = \dfrac{n^2 - 5}{n^3 - 50}$

1. Choose $\varepsilon > 0$, $l = 0$

2. $|a_n - l| = \left|\dfrac{n^2-5}{n^3-50}\right|$

$2\frac{1}{2}$ choose $b_n = \dfrac{2}{n}$

claim : if ~~##~~ $n > n_0 = 5$

then $\dfrac{n^2-5}{n^3-50} < \dfrac{2}{n}$

$\iff n^3 - 5n < 2n^3 - \cancel{}\, 100$

$\iff 100 - 5n < n^3$

so OK if $n^3 > 100$ e.g. $n \geq 5$

3. Solve $\frac{2}{n} < \varepsilon$

OK for $n > \frac{2}{\varepsilon}$

4 for $N = \frac{2}{\varepsilon}+1$ it is true

$n \geq N \Rightarrow b_n < \varepsilon$

5. put everything together:

Fix $\varepsilon > 0$. Choose $N = \max\{\frac{2}{\varepsilon}, 5\}$

Then I verify:

if $n \geq N$, then:

$$|a_n - 0| = \left|\frac{n^2-5}{n^3-50}\right| \leq \frac{2}{n} < \frac{2}{2/\varepsilon} = \varepsilon$$

by claim below, 'cause $N \geq 5$

Claim: If $n \geq 5$ then

$$\left|\frac{n^2-5}{n^3-50}\right| < \frac{2}{n}$$

pf. if $n \geq 5$, then $n^2-5>0$, $n^3-50>0$ and:

$$\left|\frac{n^2-5}{n^3-50}\right| = \frac{n^2-5}{n^3-50} < \frac{2}{n}$$

because, indeed:

$$n^3 - 5n < n^3 < 2n^3 - 100$$

(this last inequality is equivalent to $100 < n^3$)

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(END L2)

Ex 1.8 (iii)

$a_n = \frac{\cos n}{n} + i\frac{\sin n}{n} \longrightarrow 0$

Fix $\varepsilon > 0$, calculate:

$|a_n - \ell| = \left| \frac{\cos n}{n} + i \frac{\sin n}{n} \right|$

$= \sqrt{\left(\frac{\cos n}{n}\right)^2 + \left(\frac{\sin n}{n}\right)^2}$

$= \frac{1}{n} \sqrt{\cos^2 n + \sin^2 n} = \frac{1}{n}$

Choose N s.t. $n \geq N \Rightarrow |a_n - \ell| < \varepsilon$

i.e. if $n \geq N$ then $\frac{1}{n} < \varepsilon$

e.g. $N = \frac{1}{\varepsilon} + 1 > \frac{1}{\varepsilon}$:

$\frac{1}{n} \leq \frac{1}{N} < \varepsilon$.

put everything together:

fix $\varepsilon > 0$, choose $N = \frac{1}{\varepsilon} + 1$,

then if $n \geq N$:

$|a_n - \ell| = \left| \frac{\cos n}{n} + i \frac{\sin n}{n} \right|$

$= \frac{1}{n} \leq \frac{1}{N} < \varepsilon$

//

this shows that $a_n \to 0$ from the definition of limit.

Ex 1.8 (v) : $a_n = \frac{\cos n}{n} \to 0$

follow strategy:

fix $\varepsilon > 0$; calculate:

$$|a_n - \ell| = \frac{|\cos n|}{n}$$

Next we are supposed to solve:

$$\frac{|\cos n|}{n} < \varepsilon .$$

It is impossible to solve this inequality exactly.

Instead we follow the modified strategy: find b_n s.t. (for some $n \geq n_0$) $|a_n - \ell| \leq b_n$

$$|a_n - \ell| = \frac{|\cos n|}{n} \underset{(\text{all } n)}{\leq} \frac{1}{n} = b_n$$

We solve $b_n = \frac{1}{n} < \varepsilon$

i.e. $n > \frac{1}{\varepsilon}$.

then $N = \frac{1}{\varepsilon} + 1$

Put everything together:

fix $\varepsilon > 0$; choose $N = \frac{1}{\varepsilon} + 1$;

then: if $n \geq N$:

$$|a_n - \ell| = \frac{|\cos n|}{n} \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

//

This shows that

$$\lim_{n \to \infty} a_n = 0 .$$

On the choice of b_n in "modified strategy"

$|a_n - \ell| \leq b_n$

- b_n has to be simple enough

that we can confidently solve $b_n < \varepsilon$.

- For instance, suppose:

$$|a_n - \ell| = \frac{\alpha_n}{\beta_n}$$

we try to find simple $\begin{cases} \alpha_n \leq \gamma_n \\ \beta_n \geq \delta_n \end{cases}$

and then try $b_n = \dfrac{\gamma_n}{\delta_n}$:

indeed obviously:

$$\frac{\alpha_n}{\beta_n} \leq \frac{\gamma_n}{\delta_n}$$

I go back to Ex 1.8 (vi)

$$a_n = \frac{n^2-5}{n^3-50} \longrightarrow 0$$

I used the "modified strategy"

I apply this to 1.8 (vi)

$$|a_n - \ell| = \frac{|n^2-5|}{|n^3-50|} = \frac{\alpha_n}{\beta_n}$$

choose:

$$\alpha_n = |n^2-5| \leq n^2 := \gamma_n$$

$$\beta_n = |n^3-50| \geq \frac{1}{2}n^3 := \delta_n$$

If $n \geq 5$ then indeed $\beta_n \geq \delta_n$: $n^3 > 100$:

$$|n^3-50| = n^3-50 > \frac{1}{2}n^3$$

$$\iff \frac{1}{2}n^3 > 50$$

$$\iff n^3 > 100.$$

Example 1.9 $\quad a_n = \dfrac{n}{n+1}$.

Show that a_n does not tend to 0.8.

Observe that $a_n \to 1$ in this case.

We will show that limits are unique (if they exist).

Now I want to do this from the definition of limit.

How do I say that a_n does not tend to l?

Defn of limit:

$\forall \varepsilon > 0 \ \exists N$ s.t.
$n \geq N \Rightarrow$ for all $n \geq N$ a_n is within ε of l.

I "negate" this statement:

$\exists \varepsilon > 0$ s.t. $S(\varepsilon)$ not true.

$\exists \varepsilon > 0$ s.t. $\forall N$ $S(\varepsilon, N)$ not true.

$\exists \varepsilon > 0$ s.t. $\forall N \ \exists n \geq N$ s.t.
(a_n is within ε of l) is not true.

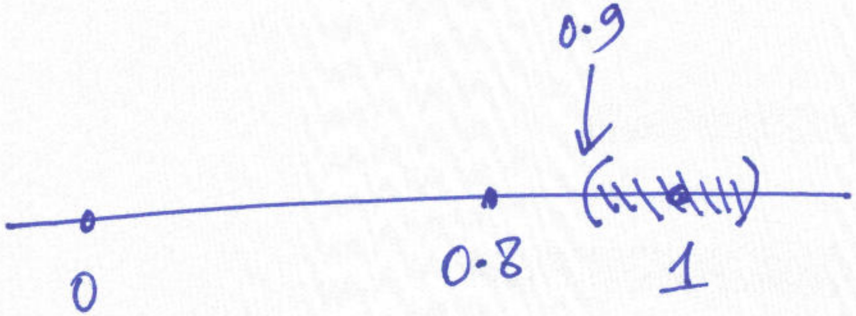
a_n does <u>not</u> tend to ℓ:

$\exists\, \varepsilon > 0$ s.t. $\forall N > 0\ \exists n \geq N$ s.t.

$$|a_n - \ell| > \varepsilon.$$

Back to Ex 1.9.

<u>Idea</u>: we know that $a_n \to \underline{1}$



We want to <u>choose</u> $\varepsilon > 0$ s.t.

$|a_n - 0.8| > \varepsilon$ "a lot of time"

(e.g. for infinitely many n)

It is clear from the picture that I want to choose:

$$\varepsilon = 0.1$$

Note $|a_n - 1| = \left|\frac{n}{n+1} - 1\right| = \frac{1}{n+1} < 0.1$

for all $n > 10$.

Note: if $n > 10$ then:

I want to show that a_n is "far" from 0.8:

$$0.2 = |1 - 0.8| = |1 - a_n + a_n - 0.8|$$

$$\leq |1 - a_n| + |a_n - 0.8| < 0.1 + |a_n - 0.8|$$

that is $|a_n - 0.8| > 0.2 - 0.1 = 0.1$

I summarise:

If $n > 10$, then

$$|a_n - 0.8| > 0.1$$

This finishes the problem:

choose $\varepsilon = 0.1$; fix N.

choose now <u>any</u> $n \geq \max\{N, 10\}$

then:

$$|a_n - 0.8| > \varepsilon = 0.1$$

this shows a_n does <u>not</u> tend to 0.8.

(END L3)

§2 Theorems and rules for calculating with convergent sequences

Philosophy:

It is awkward to always go back to the definition of limit.

Most sequences are built of simpler ones using elementary operations & functions

($+, -, \times, \div, \sqrt{\ }$, sin, cos, log ....)

Reminder (general properties of the absolute value — or more generally the norm)

1. $|a| \geq 0$, $|a| = 0 \iff a = 0$

2. $\lambda \in \mathbb{R}$, $|\lambda a| = |\lambda|\,|a|$

3. triangle inequality:

$$|a+b| \leq |a| + |b|$$

4. $|a-b| \geq \big|\,|a| - |b|\,\big|$

Pf of 4. :

recall that for $A \in \mathbb{R}$:

$$|A| \le B \iff -B \le A \le B$$

4. is equivalent to:

$$-|a-b| \le |a|-|b| \le |a-b|$$

similarly, this ineq. too is an instance of triangle ineq

$|b| \le |a-b| + |a|$

is an instance of 3:

$$|a| \le |a-b| + |b|$$

$$|a| = |(a-b) + b|$$

$$\le |a-b| + |b|$$

//

Thm 2.1 Limits are unique (when they exist)

In other words:

$$\left.\begin{array}{l} a_n \to l, \\ a_n \to m \end{array}\right\} \implies l = m.$$

proof.

Idea:

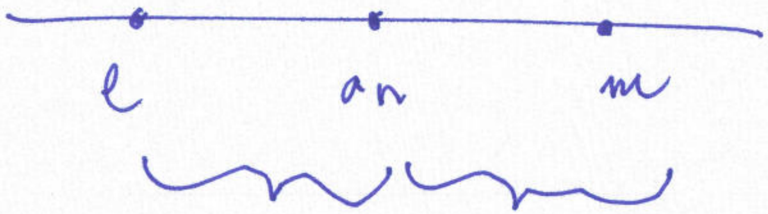
fix $\varepsilon > 0$,

$\exists N > 0$ s.t. $n \ge N$ a_n is within ε of l.

$\exists M > 0$ s.t. $n \ge M$ a_n is within ε of m.

choose $n \ge \max\{N, M\}$:

then:



$|a_n - l| < \varepsilon$ $\quad$ $|a_n - m| < \varepsilon$

$$|l - m| = |l - a_n + a_n - m|$$
$$\le |l - a_n| + |m - a_n|$$
$$< 2\varepsilon$$

this clearly implies that

$|l - m| = 0 \implies l - m = 0$

ie $l = m$.

this was the idea.

"formal" proof

Claim $\forall \varepsilon > 0$, $|l - m| < \varepsilon$.

(this clearly implies that $|l - m| = 0$ i.e. $l = m$)

Fix $\varepsilon > 0$.

(i) $a_n \to l$, therefore

$\exists N_1 > 0$ s.t. $n \geq N_1 \implies |a_n - l| < \frac{\varepsilon}{2}$

(ii) $a_n \to m$, therefore

$\exists N_2 > 0$ s.t. $n \geq N_2 \implies |a_n - m| < \frac{\varepsilon}{2}$

Now pick $n \geq \max\{N_1, N_2\}$:

$$|l - m| = |l - a_n + a_n - m| \le |a_n - l| + |a_n - m| < \underbrace{\frac{\varepsilon}{2} + \frac{\varepsilon}{2}}_{= \varepsilon} \quad //$$

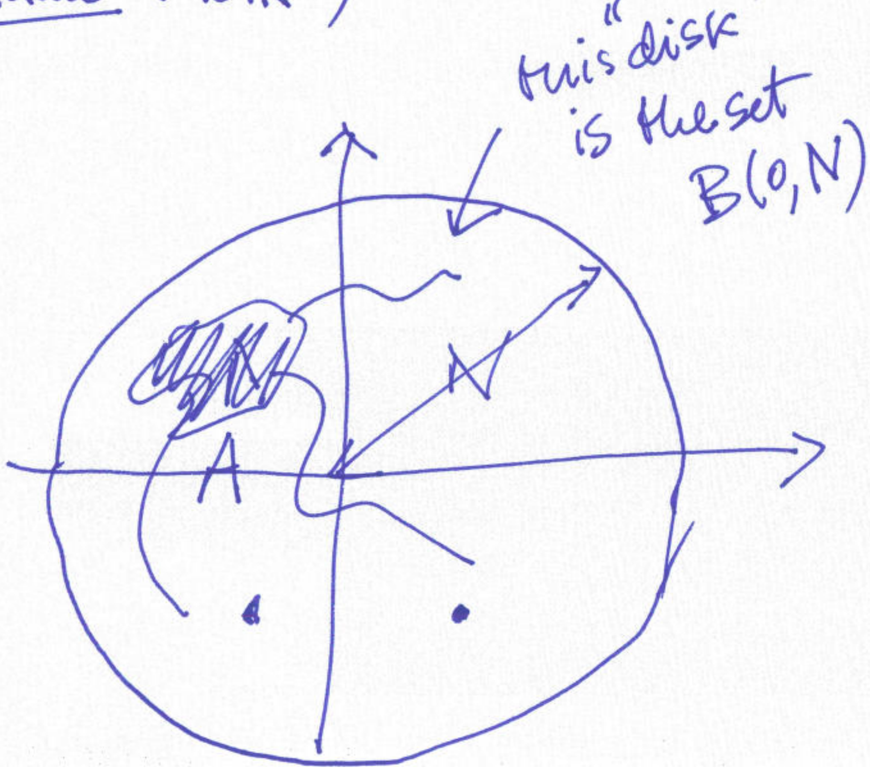
Bounded sets

A set $A \subset \mathbb{R}^n$ is <u>bounded</u>

if $\exists N > 0$ s.t.

$$A \subset B(0,N) = \{ x \in \mathbb{R}^n \mid |x| \leq N \}$$

<u>picture</u> (in $\mathbb{R}^2$)



<u>for example</u>:

$a_n = (-1)^n$ then:

$$\{a_1, a_2, a_3, \dots\} = \{+1, -1\}$$

is bounded.

$a_n = (-1)^n n$ then:

$$A = \{a_1, a_2, a_3, \dots\} = \{-1, 2, -3, 4, -5, \dots\}$$

is <u>not</u> bounded, that is:

$$\forall N > 0, \ \exists a \in A \text{ s.t. } |a| > N.$$

Now pick $N > 0$, pick $n > N$ then

$$a = a_n : \quad |a| = |a_n| = |(-1)^n n| = n > N.$$

Theorem 2.2

Let a_n be a convergent sequence (i.e. $a_n \to \ell$ for some ℓ)

Then the set:

$\{a_1, a_2, a_3, \dots\} \subset \mathbb{R}^n$

is bounded.

~~idea of~~ proof let $\ell = \lim_{n\to\infty} a_n$

fix $\varepsilon = 1$, then $\exists N$

s.t. $n \geq N \implies |a_n - \ell| < 1$

$$\implies |a_n| = |a_n - \ell + \ell| \leq |\ell| + |a_n - \ell| < |\ell| + 1$$

Choose $M = \max\{|a_1|, |a_2|, \dots |a_N|, |\ell|+1\}$

claim $\forall n$, $|a_n| \leq M$.

$\boxed{\text{END L4}}$

indeed $n \leq N$, $|a_n| \leq M$

$n > N$, $|a_n| \leq |\ell| + 1 \leq M$ //

Propos 2.3

(i) $a_n \to \ell \Rightarrow |a_n| \to |\ell|$

(ii) $a_n \to 0 \Leftrightarrow |a_n| \to 0$

pf part (i)

idea:

$$\big||a_n| - |\ell|\big| \leq |a_n - \ell|$$

pf: fix $\varepsilon > 0$.

I know $\exists N > 0$ s.t.

$$n \geq N \Rightarrow |a_n - \ell| < \varepsilon.$$

But then:

$$n \geq N \Rightarrow \big||a_n| - |\ell|\big| \leq |a_n - \ell| < \varepsilon$$

this shows that $|a_n| \to |\ell|$ qed (i)

part (ii): exercise //

Theorem 2.4 Let a_n, b_n be sequences, $a_n \to A$, $b_n \to B$. Then

(i) $a_n + b_n \to A + B$

(ii) $a_n b_n \to A \cdot B$

(iii) assume all $a_n \neq 0$, and $A \neq 0$

then $\frac{1}{a_n} \to \frac{1}{A}$.

(iv) assume all $a_n \neq 0$, and $A \neq 0$

then $\frac{b_n}{a_n} \to \frac{B}{A}$

//

pf (i) <u>idea:</u>

$$|a_n + b_n - A - B|$$
$$= |(a_n - A) + (b_n - B)|$$
$$\leq |a_n - A| + |b_n - B|$$

"formal" pf of (i)

fix $\varepsilon > 0$.

$\exists N_1 > 0$ s.t $n \geq N_1 \Rightarrow |a_n - A| < \frac{\varepsilon}{2}$

$\exists N_2 > 0$ s.t. $n \geq N_2 \Rightarrow |b_n - B| < \frac{\varepsilon}{2}$

Take $N = \max\{N_1, N_2\}$

then

$$n \geq N \Rightarrow |a_n + b_n - A - B| \leq |a_n - A| + |b_n - B| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

qed (i)

pf (ii) $a_n b_n \to AB$. Idea:

$$|a_n b_n - AB|$$
$$= |a_n b_n - A b_n + A b_n - AB|$$
$$\leq |a_n b_n - A b_n| + |A b_n - AB|$$
$$= |b_n| |a_n - A| + |A| |b_n - B|$$

formal proof of (ii).

We know by Thm 2.2 that $\{b_n\}$ is bounded; so choose M s.t. $|b_n| \leq M$ (for all n).

fix $\varepsilon > 0$. We know:

$\exists N_1$ s.t. $n \geq N_1$, $|a_n - A| < \dfrac{\varepsilon}{2M}$

$\exists N_2$ s.t. $n \geq N_2$, $|b_n - B| < \dfrac{\varepsilon}{2|A|}$

choose $N = \max\{N_1, N_2\}$

then

$n \geq N$:

$$|a_n b_n - AB| = |a_n b_n - Ab_n + Ab_n - AB|$$

$$\leq |b_n||a_n - A| + |A||b_n - B|$$

$$\leq M|a_n - A| + |A||b_n - B|$$

$$< M\frac{\varepsilon}{2M} + |A|\frac{\varepsilon}{2|A|} = \varepsilon$$

qed (ii)

pf (iii) Idea:

$$\left|\frac{1}{a_n} - \frac{1}{A}\right| = \left|\frac{A - a_n}{a_n A}\right|$$

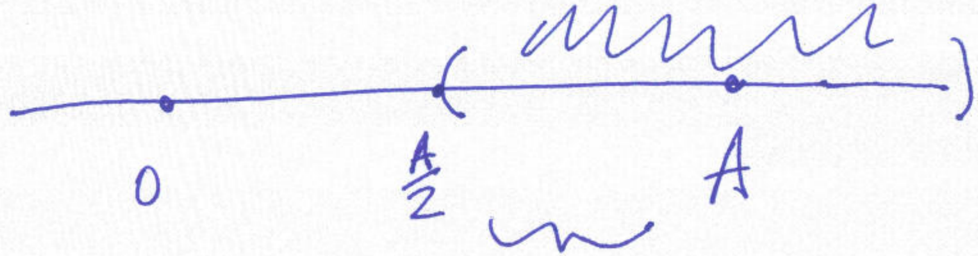
$$= \frac{1}{|A||a_n|}|a_n - A|$$

we want $\left\{\dfrac{1}{|a_n|} \,\middle|\, n = 1, 2, \ldots\right\}$

~~to~~ be a <u>bounded set</u>.

i.e. we want

$$\frac{1}{|a_n|} \leq (\text{something})$$



$\varepsilon = \dfrac{|A|}{2}$

$a_n \to A$

<u>Claim</u> the set

$$\left\{ \frac{1}{|a_n|},\ n=1,2,\dots \right\}$$

is <u>bounded</u>.

fix $\varepsilon = \frac{|A|}{2}$.

$\exists N$ s.t. $n \geq N$, $|a_n - A| < \frac{|A|}{2}$

then

$n \geq N$, $|A| = |A - a_n| + |a_n|$

$$< \frac{|A|}{2} + |a_n|$$

$$\Rightarrow |a_n| > |A| - \frac{|A|}{2} = \frac{|A|}{2}$$

$$\Rightarrow \frac{1}{|a_n|} < \frac{2}{|A|}$$

let $M = \max\left\{ \frac{1}{|a_1|}, \frac{1}{|a_2|}, \dots \frac{1}{|a_{N-1}|}, \frac{2}{|A|} \right\}$

manifestly $\frac{1}{|a_n|} \leq M$ all n. qed Claim.

formal pf of (iii)

fix $\varepsilon > 0$

$\exists N$ s.t. $n \geq N$

$$|a_n - A| < \frac{|A|\varepsilon}{M}$$

then if $n \geq N$, also:

$$\left| \frac{1}{a_n} - \frac{1}{A} \right| = \left| \frac{A - a_n}{a_n A} \right|$$

$$= \frac{1}{|A||a_n|} |a_n - A|$$

$$\leq \frac{M}{|A|} |a_n - A| < \frac{M}{|A|} \frac{|A|}{M} \varepsilon$$

$$= \varepsilon$$

qed (iii)

part(iv) is an immediate
consequence of part(ii) + part(iii)

qed Thm 2·4.

[END L5]

Corollary 2.5

Let $a_n \to A$, let B be a constant

(i) $a_n + B \longrightarrow A + B$

(ii) $a_n B \longrightarrow A \cdot B$

(iii) $B \neq 0$ then $\frac{a_n}{B} \to \frac{A}{B}$

Example (on how to use the Theorem)

$$\frac{n^2-5}{n^3-50} = \frac{n^2\left(1-\frac{5}{n^2}\right)}{n^3\left(1-\frac{50}{n^3}\right)}$$

$$= \frac{1}{n} \times \frac{1-\frac{5}{n^2}}{1-\frac{50}{n^3}}$$

$$= \frac{1}{n} \times \frac{1 - 5 \times \frac{1}{n} \times \frac{1}{n}}{1 - 50 \times \frac{1}{n} \times \frac{1}{n} \times \frac{1}{n}}$$

All I use is Thm 2.4, Cor 2.5, and the fact:

$$\frac{1}{n} \longrightarrow 0.$$

numerator:

$$\frac{1}{n} \to 0 \underset{2.4(ii)}{\Longrightarrow} \frac{1}{n} \times \frac{1}{n} \to 0$$

$$\underset{2.5(ii)}{\Longrightarrow} (-5) \times \frac{1}{n} \times \frac{1}{n} \to 0$$

$$\underset{2.5(i)}{\Longrightarrow} 1 - 5 \times \frac{1}{n} \times \frac{1}{n} \to 1$$

denominator

$$\frac{1}{n} \to 0 \underset{2.4(ii)}{\Longrightarrow} \frac{1}{n} \times \frac{1}{n} \to 0$$

$$\underset{2.4(ii)}{\Longrightarrow} \frac{1}{n} \times \frac{1}{n} \times \frac{1}{n} \to 0$$

$$\underset{2.5(ii)}{\Longrightarrow} (-50) \times \frac{1}{n} \times \frac{1}{n} \times \frac{1}{n} \to 0$$

$$\underset{2.5(i)}{\Longrightarrow} 1 - 50 \times \frac{1}{n} \times \frac{1}{n} \times \frac{1}{n} \to 1$$

By Thm 2.4 (iv) (Note: $1 - \frac{50}{n^3}$ is always $\neq 0$ prove it as an exercise)

$$\frac{1 - 5 \times \frac{1}{n} \times \frac{1}{n}}{1 - 50 \times \frac{1}{n} \times \frac{1}{n} \times \frac{1}{n}} \to \frac{1}{1} = 1$$

hence finally by 2.4 (ii)

$$\frac{n^2 - 5}{n^3 - 50} = \frac{1}{n} \times \frac{1 - 5 \times \frac{1}{n} \times \frac{1}{n}}{1 - 50 \times \frac{1}{n} \times \frac{1}{n} \times \frac{1}{n}}$$

$$\longrightarrow 0 \times 1 = 0$$

<u>Thm 2.6</u> (Ratio test for sequences)

Assume $\exists\ 0 < r < 1$ and:

$$n \geq n_0 \Rightarrow \frac{|a_{n+1}|}{|a_n|} < r$$

Then $a_n \to 0$.

<u>pf.</u> idea:

if $n \gg 0$

$$|a_n| < r\,|a_{n-1}| < r^2\,|a_{n-2}| < r^3\,|a_{n-3}|$$

$$< \cdots < r^{n-n_0}\,|a_{n_0}|$$

next we want to show:

$0 < r < 1$, then $r^n \to 0$.

<u>Lemma</u> If $0 < r < 1$

then $r^n \to 0$.

before doing this I'd like to do:

<u>Lemma'</u> If $1 < q$, then

$$q^n \to \infty.$$

"infinity"

where:

<u>dfn</u> A sequence $a_n \to \infty$

(tends to infinity)

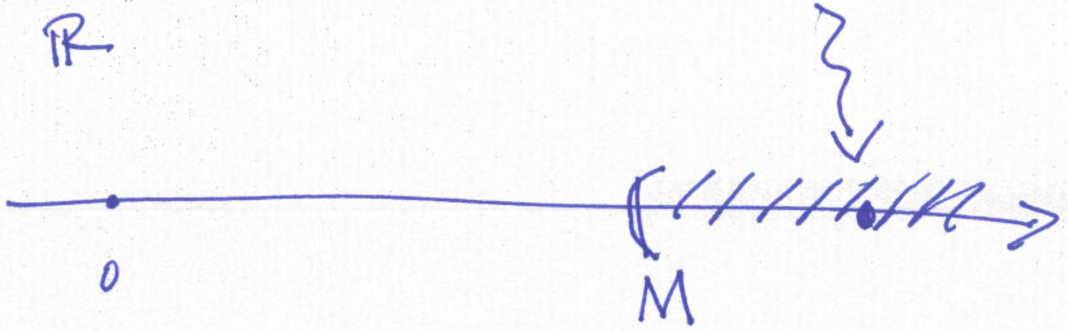
if $\forall\ M > 0\ \exists\ N > 0$

s.t.

$$n \geq N \Rightarrow a_n > M$$

picture:

for $n \geq N$, a_n



the interval (M, ∞) is a window around ∞.

write $q = 1+a$ with $a > 0$:

$$q^n = (1+a)^n$$
$$= 1 + na + \binom{n}{2}a^2 + \binom{n}{3}a^3 + \cdots + a^n$$
$$> 1 + na.$$

pf of Lemma!

we all know that exponential with base >1 tends to ∞. (rather dramatically) but, in this course, we want to go back to 1st principles.

fix M choose $N = \dfrac{M-1}{a}$

$$n \geq N, \implies q^n > 1 + na \geq 1 + a\,\frac{M-1}{a} = 1 + M - 1 = M$$

this shows $q^n \to \infty$

<u>general fact</u>

If $a_n > 0$, if $a_n \to \infty$
then $\frac{1}{a_n} \to 0$.

Pf: fix $\varepsilon > 0$.

choosing $M = \frac{1}{\varepsilon}$: there is N
s.t.
$$n \geq N \Rightarrow \quad a_n > \frac{1}{\varepsilon}$$

then:
$$n \geq N \Rightarrow \quad 0 < \frac{1}{a_n} < \varepsilon$$

i.e. also $\left| \frac{1}{a_n} - 0 \right| < \varepsilon$

that is, $\frac{1}{a_n} \to 0$ //

<u>pf of Lemma</u>

use Lemma' and the general fact:

$0 < r < 1$

take $q = \frac{1}{r} > 1$

by Lemma' $q^n \to \infty$

by general fact $\frac{1}{q^n} \to 0$

but $\frac{1}{q} = r$ so $\frac{1}{q^n} = r^n \to 0$.

//

pf of 2·6

Note:

$n \geq n_0 \implies |a_n| < \underbrace{r^n \frac{|a_{n_0}|}{r^{n_0}}}_{b_n}$

solve $b_n < \varepsilon$:

$$\frac{r^n |a_{n_0}|}{r^{n_0}} < \varepsilon$$

$$\iff r^n < \frac{\varepsilon r^{n_0}}{|a_{n_0}|}$$

formal pf :

fix $\varepsilon > 0$.

we know (Lemma) $r^n \to 0$.

so choose N' such that :

$n \geq N'$, $|r^n| = r^n < \frac{\varepsilon r^{n_0}}{|a_{n_0}|}$

If $n \geq N \equiv \max\{N', n_0\}$

$$|a_n| \underset{(\text{Note})}{<} \frac{r^n |a_{n_0}|}{r^{n_0}} < \frac{\varepsilon r^{n_0}}{|a_{n_0}|} \frac{|a_{n_0}|}{r^{n_0}}$$

$$= \varepsilon$$

[END L6] //

Subsequence

Suppose a_k is a sequence $(k = 1, 2, 3, \cdots \in \mathbb{N}^\times)$

picture of b_n, a subsequence of a_k

$a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6 \quad a_7 \quad a_8 \quad a_9 \quad \cdots \circ$

$b_1 \quad \cdots \quad b_2 \quad \cdots \quad b_3 \quad \cdots \quad b_4$

to pick the b's I need a strictly increasing sequence: $k_1 < k_2 < k_3$:

$$\underset{=k_1}{2} < \underset{= k_2}{5} < \underset{= k_3}{7} < \cdots$$

of natural numbers

Defn

- A sequence $a_n \in \mathbb{R}$ is increasing if

$n \geq m \Rightarrow a_n \geq a_m$

(equivalently $a_{n+1} \geq a_n$ all n)

- $a_n \in \mathbb{R}$ is strictly increasing if

$n > m \Rightarrow a_n > a_m$

(equivalently $a_{n+1} > a_n$ all n)

formal definition

A <u>subsequence</u> of a sequence a_k is a sequence b_n of the form

$$b_n = a_{k(n)} = a_{k_n}$$

where $k : \mathbb{N}^\times \to \mathbb{N}^\times$ is a strictly increasing sequence of natural numbers:

$$k(1) < k(2) < k(3) \cdots .$$

<u>Example</u> $k(n) = 2n$

$$b_1, b_2, b_3, \ldots$$

$$= a_2, a_4, a_6, a_8 \cdots$$

$$k(n) = 2n - 1$$

$$b_1, b_2, b_3, b_4, \ldots$$

$$= a_1, a_3, a_5, a_7, \cdots$$

<u>Defn</u> I say that a_n is <u>divergent</u> if a_n is not convergent.

<u>Propos 2.7</u> Assume $a_k \to l$

Let b_n be a subsequence of a_k.

Then $b_n \to l$.

pf. Fix $\varepsilon > 0$ We know

$\exists N > 0 \quad k \geq N \Rightarrow |a_k - l| < \varepsilon$

Just note: 'cause k increasing.

$n \geq N \Rightarrow \underbrace{k(n) \geq n}_{\text{'cause k increasing}} \geq N$

$|b_n - l| = |a_{k(n)} - l| < \varepsilon$ //

<u>Cor 2.8</u> Let a_k be a sequence. Assume we have 2 convergent subsequences:

$b_n = a_{k(n)} \to l$

$b'_n = a_{k'(n)} \to l'$

If $l \neq l'$, then a_n is divergent

<u>Example</u> $a_k = (-1)^k$

$= -1, 1, -1, 1, -1, 1, \ldots$

$k(n) = 2n \qquad b_n = 1$

$k'(n) = 2n-1 \qquad b'_n = -1$

Conclusion !

$a_n = (-1)^n$ does not converge.

To appreciate how cool this is, try to show this directly from the definition of limit.

Th 2.9 $a_n, b_n \in \mathbb{R}$

Suppose $a_n \le b_n$ (all $n \geq n_0$)

Suppose $a_n \to A$, $b_n \to B$

$\Longrightarrow A \le B$.

pf

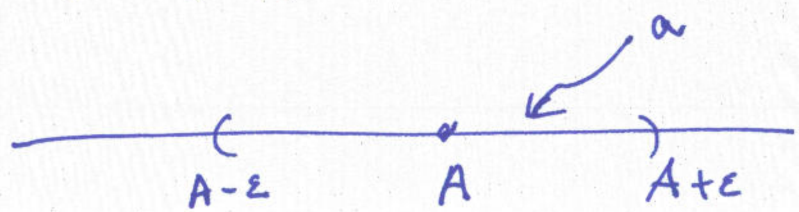
N.B. equivalent statements :

$$|A - a| < \varepsilon$$

$$\Updownarrow$$

$$A - \varepsilon < a < A + \varepsilon$$

picture



I claim

$\forall \varepsilon > 0 \quad A - \varepsilon \leq B$

(the claim manifestly implies the theorem.)

fix $\varepsilon > 0$.

$\exists N_1$ s.t. $n \geq N_1$ $|a_n - A| < \frac{\varepsilon}{2}$

$\exists N_2$ st $n \geq N_2$ $|b_n - B| < \frac{\varepsilon}{2}$

pick $N = \max\{N_1, N_2, n_0\}$

If $n \geq N$:

$$A - \frac{\varepsilon}{2} < a_n \leq b_n < B + \frac{\varepsilon}{2}$$

that is $A - \frac{\varepsilon}{2} < B + \frac{\varepsilon}{2}$

$\iff A - \varepsilon < B$ //

<u>Thm 2.10</u> (Sandwich)

$a_n, b_n, c_n \in \mathbb{R}$.

Suppose

$a_n \leq b_n \leq c_n \quad (n \geq n_0)$.

$a_n, c_n \longrightarrow A$

then $b_n \longrightarrow A$.

Pf fix $\varepsilon > 0$ [END L7]

we know:

$\exists N_1$ s.t. $n \geq N_1 \Rightarrow |a_n - A| < \varepsilon$

$\exists N_2$ s.t. $n \geq N_2 \Rightarrow |c_n - A| < \varepsilon$

pick $N = \max\{N_1, N_2, n_0\}$

$n \geq N$:

$$A - \varepsilon < a_n \leq b_n \leq c_n < A + \varepsilon$$

$$A - \varepsilon < b_n < A + \varepsilon$$

$$\Updownarrow$$

$$|b_n - A| < \varepsilon$$

That is, exactly $b_n \to A$ //