If you are a non-Maths student and you plan to take this course for Examination, then you <u>must</u> tell:

Mr Chris Sisson
Huxley 647
c.sisson@imperial.ac.uk
020 7594 8482

§3 General theory of convergent sequences

$\mathbb{R}$ is a complete ordered field

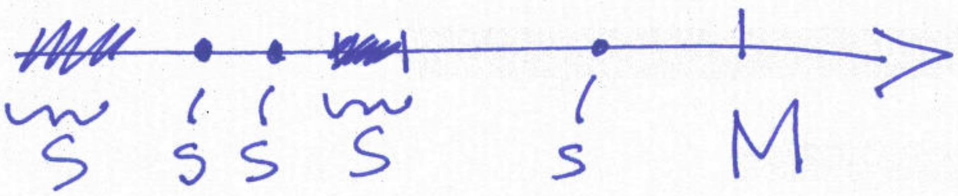
definition

$S \subset \mathbb{R}$ is bounded above if there is $M \in \mathbb{R}$ s.t.

$$s \leq M \quad \text{for all } s \in S$$

M is called an upper bound of S

picture : M is an upper bound:



Similarly, $S \subset \mathbb{R}$ is bounded below if there is $M \in \mathbb{R}$ s.t.

$$s \geq M \quad \text{for all } s \in S$$

M is called a lower bound of S

$\mathbb{R}$ complete :

If S is bounded above, then S has a least upper bound

(If S is bounded below, then S has a greatest lower bound)

Where: L is a least upper bd means:

(1) L is an upper bound

(2) If M is an upper bound then $L \leq M$.

(similar for greatest lower bounds)

defn

S has a <u>maximum</u> if there $\exists$ $s_M \in S$ called the maximum, such that

$$s \leq s_M \quad \text{for all } s \in S.$$

<u>Rmk</u> A set need not have a maximum.

<u>If</u> S <u>does</u> have a maximum, s_M then s_M is also a least upper bound

<u>Examples</u>

(i) $\{x \in \mathbb{Q} \mid x^2 < 2\} \subset \mathbb{Q}$

does not have a l.u.b. in $\mathbb{Q}$.

That is, $\mathbb{Q}$ is not <u>complete</u>

(ii) $\{x \in \mathbb{R} \mid x^2 < 2\}$

$= \{x \in \mathbb{R} \mid |x| < \sqrt{2}\}$

the l.u.b. is $\sqrt{2}$.

<u>Note</u>: $\sqrt{2}$ does not itself belong.

There is no maximum!

definition

A **field** is a set K together with operations

$$K \times K \xrightarrow{+} K$$

$$K \times K \xrightarrow{\times} K$$

$K, +$ is an abelian group w. identity $0 \in K$

$K^\times = K \setminus \{0\}, \times$ is an abelian group. with identity 1

$$a \times (b + c) = a \times b + a \times c.$$

ordered: there is a $<$ on K satisfying various more-or-less obvious properties.

Characterisation

$$S \subset \mathbb{R}$$

M = least upper bound of S

$$\Updownarrow$$

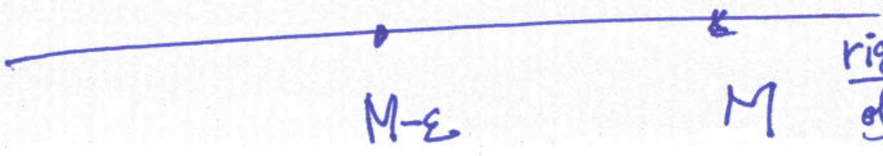
1) M is an upper bound of S

$$(s \in S \Rightarrow s \leq M)$$

2) $\forall \varepsilon > 0 \quad \exists s \in S$ s.t

$$M - \varepsilon < s$$

picture $\longrightarrow$ property 2) says there $\exists s \in S$ to the right of $M-\varepsilon$



<u>defn</u> A sequence $a_n \in \mathbb{R}$ is:

<u>increasing</u> if $n \leq m \Rightarrow a_n \leq a_m$

<u>decreasing</u> if $n \leq m \Rightarrow a_n \geq a_m$

(a_n is monotone if it is either decreasing or increasing)

A sequence $a_n \in \mathbb{R}$ is <u>bounded above</u> (resp. <u>below</u>)

if the set $\{a_n \mid n = 1, 2, 3 \ldots\} \subset \mathbb{R}$ is bounded above (resp. below)

<u>Thm 3.1</u> Let $a_n \in \mathbb{R}$ be <u>increasing</u> and <u>bounded above</u>.

Then a_n is convergent.

<u>Pf.</u> Let $l = \text{l.u.b.}\{a_n \mid n = 1, 2, 3 \ldots\}$

<u>claim</u> $a_n \longrightarrow l$.

Fix $\varepsilon > 0$. By 2) in the characterisation above, there $\exists\ a_N \in \{a_n \mid n = 1, 2, 3 \ldots\}$ s.t.

$$l - \varepsilon < a_N$$

but now take $m \geq N$:

$$L - \varepsilon < a_N \le a_n \le L$$

'cause seq. is increasing.

'cause L is an u.b.

To summarise

$$n \geq N \implies L - \varepsilon < a_n \le L$$

in part

$$|a_n - L| < \varepsilon$$

that is what was to be shown //

Rmk Similarly, if a_n is decreasing & bounded below, then it converges to the g.l.b.

I'm working towards:

Thm 3.3 (Bolzano-Weierstrass)

If $a_n \in \mathbb{R}$ is bounded (above and below) then a_n has a convergent subsequence

Before I can do the pf of 3.3

I do:

<u>Thm 3.2</u> Let $a_n \in \mathbb{R}$

then a_n has

<u>either</u> a decreasing subsequence

<u>or</u> an increasing subsequence.

<u>Note</u> $3.2 \Rightarrow 3.3$.

s'pose a_n has an increasing subseq:

$$a_{n_1} \leq a_{n_2} \leq a_{n_3} \leq \cdots$$

Of course

$$\{a_{n_k} \mid k=1,2,\ldots\} \subset \{a_n \mid n=1,2,\ldots\}$$

therefore a_{n_k} is also <u>bounded</u>.

by Thm 3.1 a_{n_k} tends to a limit.

Similarly if a_n had a <u>decreasing</u> subsequence a_{n_k},

then a_{n_k} is decreasing & bounded below so again by Thm 3.1 a_{n_k} has a limit.

this proves 3.3 assuming 3.2. //

[END L.8]

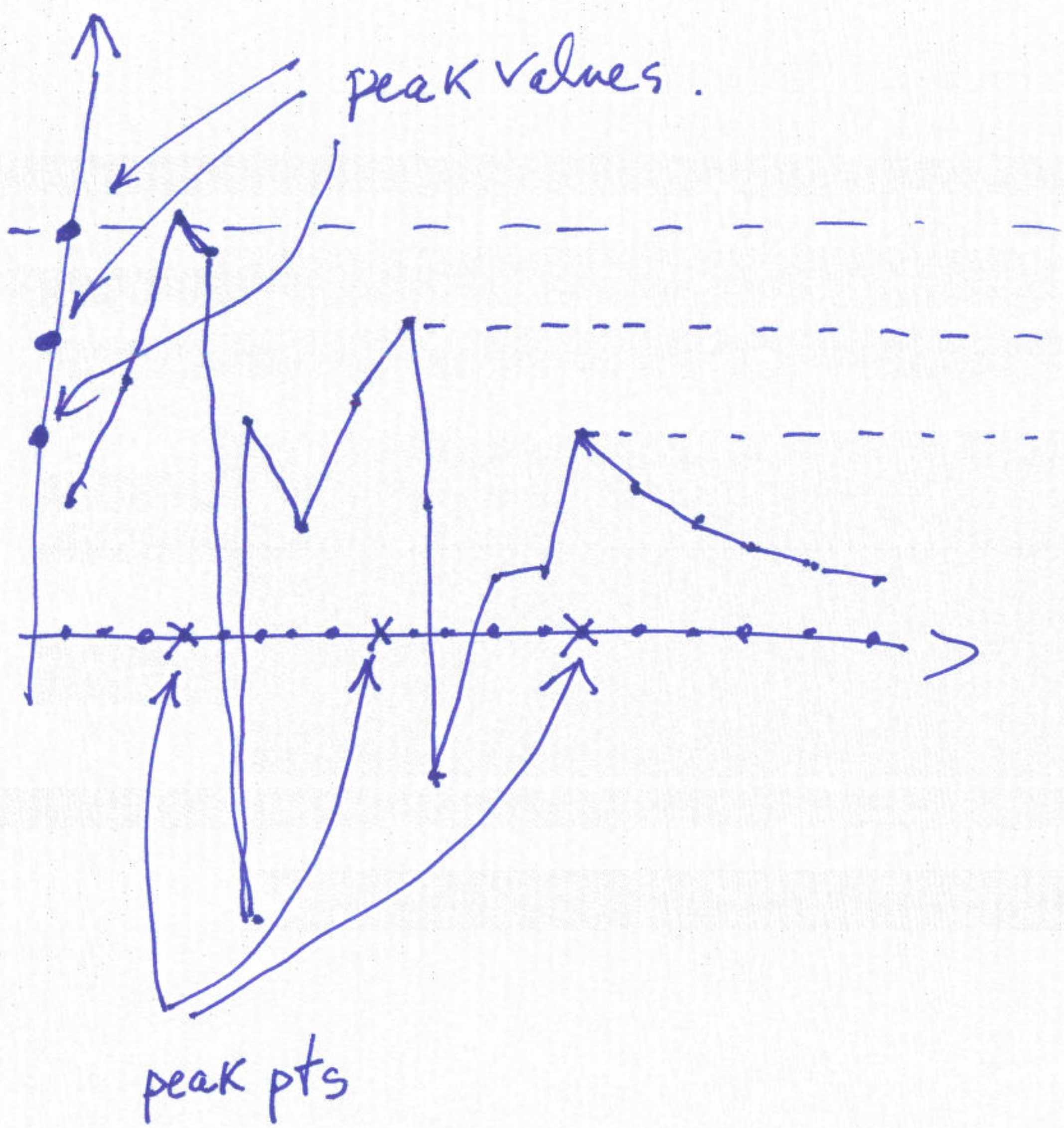
defn Let $a_n \in \mathbb{R}$

I say that $n \in \mathbb{N}^\times$ is a **peak pt** (a_n is a **peak value**) if

$$m \geq n \implies a_m \leq a_n .$$

Idea: if you look ahead, you see nothing higher.

Picture: peak points



~~Pf~~ of Thm 3.2

2 cases: case 1: a_n has infinitely many peak pts. Cas2: a_n has only finitely many peak pts (or none)

Case 1

Let $n_1 < n_2 < n_3 < n_4 < \cdots$

be the sequence of peak pts.

$$a_{n_1} \geq a_{n_2} \geq a_{n_3} \geq a_{n_4} \geq \cdots$$

$b_k = a_{n_k}$ is your decreasing subseq.

Case 2

~~Start~~

Let $m_1 < m_2 < \cdots < m_K$ be the peak pts of a_n.

Start with $n_1 > m_1, \dots, m_K$

(if there were no peak pts start with $n_1 = 1$)

n_1 is not a peak point!

This means:

$\exists\ n_2 > n_1$, s.t. $a_{n_2} > a_{n_1}$.

n_2 is not a peak point!

This means:

$\exists\, n_3 > n_2$, s.t. $a_{n_3} > a_{n_2}$

n_3 is not a peak point!

This means:

$\exists\, n_4 > n_3$, s.t. $a_{n_4} > a_{n_3}$

n_4 is not a peak point!

This means:

$\exists\, n_5 > n_4$, s.t. $a_{n_5} > a_{n_4}$

$\vdots$

We found $n_1 < n_2 < n_3 < n_4 < \cdots$

such that $a_{n_1} < a_{n_2} < a_{n_3} < a_{n_4} < \cdots$

$b_k = a_{n_k}$ is your increasing subseq //

Cauchy sequences

<u>defn</u> 3.4 a_n is <u>Cauchy</u> if $\forall \varepsilon > 0 \quad \exists N$ s.t.

$$n, m \geq N \implies |a_n - a_m| < \varepsilon.$$

<u>Idea</u>: as $n \gg 0$, values of the sequence become very close.

<u>Non-example</u>

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$

is <u>not</u> a Cauchy sequence.

how do I say that a_n is not a Cauchy sequence?

$\exists \varepsilon > 0 \quad \forall N$

$\exists\, n, m \geq N$ s.t. $|a_n - a_m| \geq \varepsilon$.

I bet you $\varepsilon = \frac{1}{2}$.

You give me N.

I bet you $n = N$, $m = 2N$

$$|a_n - a_m| = \underbrace{\frac{1}{N+1} + \frac{1}{N+2} + \cdots + \frac{1}{N+N}}_{N \text{ summands}}$$

$$\geq N \frac{1}{2N} = \frac{1}{2} \quad //$$

Lemma 3.5

Let a_n be a Cauchy sequence.

Then a_n is bounded.

pf.

fix $\varepsilon = 1$ $\exists N$ s.t.

$n, m \geq N \Longrightarrow |a_n - a_m| < 1$

Note:

$n \geq N$ then $|a_n| = |a_n - a_N + a_N|$

$\leq |a_n - a_N| + |a_N| < |a_N| + 1$

Therefore if $L = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, |a_N| + 1\}$

then $|a_n| \leq L$ for all n //

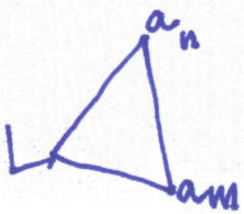
Theorem 3.6 ("general principle of convergence")

a_n is Cauchy $\iff$ a_n converges.

pf today I prove the easy part $\Leftarrow$

idea: if a_n is close to L
and a_m is close to L
then a_n is close to a_m

quantify with the triangle ineq.:



$$|a_n - a_m| \leq |a_n - L| + |a_m - L|$$

Let $L = \lim a_n$

fix $\varepsilon > 0$

$\exists$ N s.t. $n \geq N \Rightarrow |a_n - L| < \frac{\varepsilon}{2}$

but then:

$n, m \geq N \Rightarrow$

$$|a_n - a_m| < |a_n - L| + |a_m - L|$$
$$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

this proves $\Leftarrow$.

Next time I do $\Longrightarrow$.

[END L9]

Pf of Thm 3.6, $\Longrightarrow$:

a_k Cauchy $\Longrightarrow$ a_k converges.

Recall 3.5: a_k is bounded.

Recall thm 3.3 (Bolzano–Weierstrass)

says that a_k has a convergent subsequence

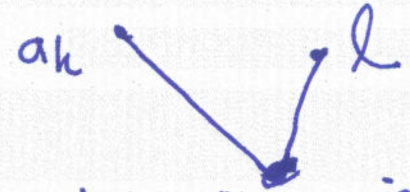
$$b_n = a_{k(n)} \to l$$

($k_1 < k_2 < k_3 < \dots$ increasing)

Plan: is to prove that $l = \lim a_k$.

Idea:

for $n \gg 0$, $k \gg 0$ $\rightarrow$ a_n is "close" to a_k



$b_n = a_{k(n)}$ is "close" to l for $n \gg 0$.

a_n must also be "close" to l:

$$|a_n - l| = |a_n - a_k + a_k - l|$$
$$\leq |a_n - a_k| + |a_k - l|$$

formal proof that $a_k \to \ell$.

Fix $\varepsilon > 0$.

$\exists\ K'$ s.t. $k, h \geq K' \Rightarrow |a_k - a_h| < \frac{\varepsilon}{2}$

$\exists\ N$ s.t. $n \geq N \Rightarrow |b_n - \ell|$

$= |a_{k(n)} - \ell| < \frac{\varepsilon}{2}$

Fix n_0 s.t. $\begin{cases} n_0 \geq N \\ k(n_0) \geq K' \end{cases}$

[I can accomplish this because:

$k(1) < k(2) < k(3) \ \ldots$

so e.g. $k(n) \geq n$ for all n

if take $n_0 \geq \max\{N, K'\}$

then $n_0 \geq N$

$k(n_0) \geq n_0 \geq K'$]

take $K = k(n_0)$.

$h \geq K \Rightarrow$

$|a_h - \ell| \leq |a_h - a_K| + |a_K - \ell|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

[note ① $h \geq K = k(n_0) \geq K'$

$K = k(n_0) \geq K'$

so $h, K \geq K' \Rightarrow |a_h - a_K| < \frac{\varepsilon}{2}$

② $K = k(n_0)$, $n_0 \geq N$

$\Rightarrow |b_{n_0} - \ell| = |a_{k(n_0)} - \ell| = |a_K - \ell|$

$< \frac{\varepsilon}{2}$]

this shows $a_k \to \ell$

//

<u>Exercise</u> $a_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix} \in \mathbb{R}^2$

a_n is Cauchy $\iff$ x_n, y_n are both Cauchy.

In $\mathbb{R}^2$ the standard "norm" of a vector (used to measure if a_n is Cauchy)

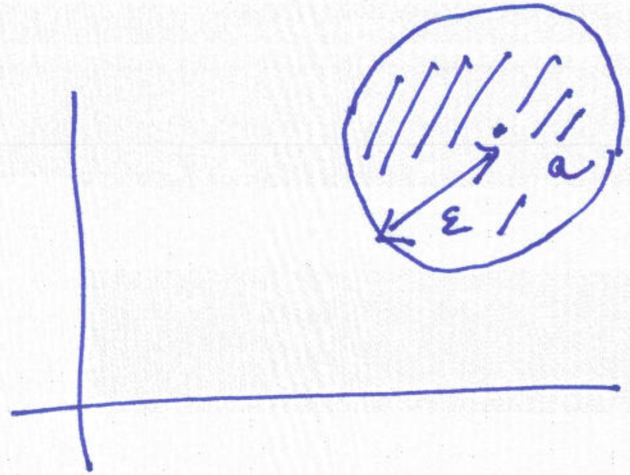
$$\| a_n \|_1 = \sqrt{x_n^2 + y_n^2}$$

It is interesting to use another "norm":

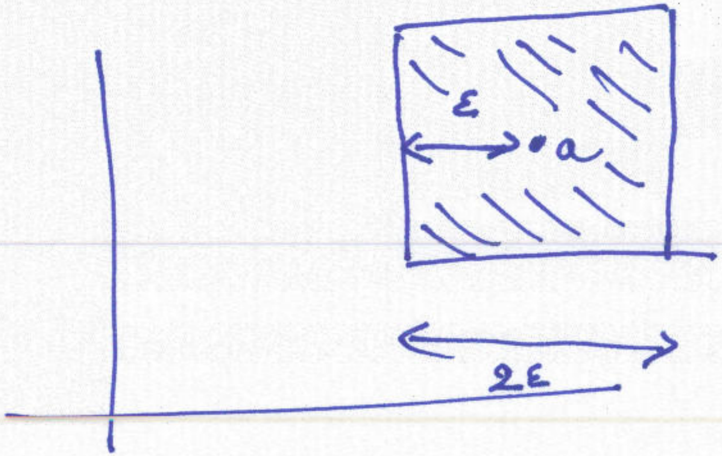
$$\| a_n \|_2 = \max \{ |x_n|, |y_n| \}$$

fix $a \in \mathbb{R}^2$, $a = \begin{pmatrix} x \\ y \end{pmatrix}$

$$B_\varepsilon (a) = \{ b \in \mathbb{R}^2 \mid \| b - a \|_1 < \varepsilon \}$$



$S_\varepsilon(a) = \{ b \in \mathbb{R}^2 \mid \|b-a\|_2 < \varepsilon \}$



A sequence $a_n \in \mathbb{R}^2$ is :

<u>Cauchy$_1$</u> if :

$\forall \varepsilon \; \exists N : n, m \geq N \Rightarrow \|a_n - a_m\|_1 < \varepsilon$

($\| \ \|_1$ = "standard norm")

$\Rightarrow$ Cauchy$_1$ $\Longleftrightarrow$ Cauchy in the usual, "normal" way

<u>Cauchy$_2$</u> if

$\forall \varepsilon \; \exists N : n, m \geq N \Rightarrow \|a_n - a_m\|_2 < \varepsilon$

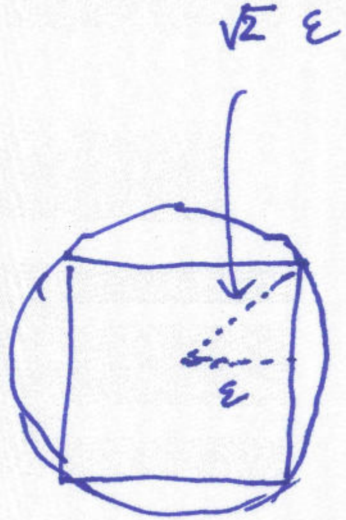
<u>Exercise!</u> prove :

a_n is Cauchy$_2$ $\Longleftrightarrow$ x_n, y_n are Cauchy.

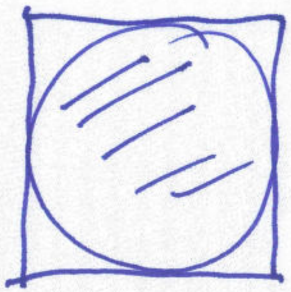
now we can re-state the original Exercise

as follows:

an $\text{Cauchy}_1 \iff$ an Cauchy_2.



the key thing for this is the foll picture:



$$B_\varepsilon(a) \subset S_\varepsilon(a)$$

$$\Longrightarrow$$

$$S_\varepsilon(a) \subset B_{\varepsilon\sqrt{2}}(a)$$

$$\Longleftarrow$$

fix $\varepsilon > 0$ $\exists N$ s.t.

$$n, m \geq N \Rightarrow \|a_n - a_m\|_2 < \frac{1}{\sqrt{2}}\varepsilon$$

but then:

$$\|a_n - a_m\|_1 = \sqrt{(x_n - x_m)^2 + (y_n - y_m)^2}$$

$$\leq \sqrt{2 \max\{|x_n - x_m|, |y_n - y_m|\}^2}$$

$$= \sqrt{2} \quad \| a_n - a_m \|_2 < \sqrt{2} \frac{\varepsilon}{\sqrt{2}}$$

$$= \varepsilon$$

//

[END L10]

Summary of §3

Th 3.1 $a_n \in \mathbb{R}$

a_n increasing, bdd above $\Rightarrow$ a_n converges

a_n decreasing, bdd below $\Rightarrow$ "

Th 3.2 $a_n \in \mathbb{R}$

$\Rightarrow$ either a_n has increasing subseq.
or decreasing
(or both)

Th 3.3 (Bolzano-Weierstrass)

$a_n \in \mathbb{R}$

a_n bounded $\Rightarrow$ a_n has a convergent subsequence.

Def 3.4 (Cauchy seq)

a_n Cauchy if $\forall \varepsilon > 0 \ \exists N$

$n, m \geq N \Rightarrow |a_n - a_m| < \varepsilon$

Thm 3.6

a_n Cauchy $\iff$ a_n converges.

Remark

3.1 and 3.2 can only be stated for $a_n \in \mathbb{R}$ (real numbers, not vectors)

On the other hand, 3.3 and 3.6 can be stated, and <u>they are true</u> for $a_n \in \mathbb{R}^k$ (though our proofs don't work for vectors)