§4 Series

An infinite **series** is an expression:

$$a_1+a_2+a_3+a_4+\cdots = \sum_{n=1}^{\infty} a_n = \sum a_n$$

where $a_n \in \mathbb{R}^k$ is a **sequence**

The sequence of **partial sums** of a series is the sequence:

$$S_n = a_1 + a_2 + \cdots a_n$$

$$= \sum_{k=1}^{n} a_k .$$

~~Thm~~ **Defn 4.1**

the series $\sum_{k=1}^{\infty} a_k$ converges if and only if the sequence of partial sums

$$S_n = \sum_{k=1}^{n} a_k$$

converges.

$$S = \lim_{n\to\infty} S_n = \text{the value of the series}$$

$$= \sum_{k=1}^{\infty} a_k .$$

The series **diverges** if it does not converge.

Examples

(i)
$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

this series diverges.

the sequence of partial sums

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$

We proved earlier this week that

S_n is <u>not a Cauchy sequence</u>

Therefore it doesn't converge.

Therefore by definition $\sum \frac{1}{n}$ does not converge. //

(ii) $a_n = \frac{1}{2^n}$

$$\sum_{n=0}^{\infty} a_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

$$S_n = 1 + \frac{1}{2} + \frac{1}{4} + \dots + \left(\frac{1}{2}\right)^n$$

$$1 + q + q^2 + \dots + q^n = \frac{1 - q^{n+1}}{1 - q}$$

$$S_n = \frac{1 - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}} = 2 - \frac{1}{2^n}$$

we do know that $\lim_{n\to\infty} \frac{1}{2^n} = 0$

hence $\lim_{n\to\infty} S_n = 2 = \sum_{n=0}^{\infty} \frac{1}{2^n}$ //

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<u>small generalization</u>

$0 < q < 1$

$$\Rightarrow \sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$$

(iii) $a_n = (-1)^n$

$$\sum_{n=1}^{\infty} a_n = -1+1-1+1-1+1-\cdots$$

$S_1 = -1$

$S_2 = -1+1 = 0$

$S_3 = -1+1-1 = -1$

$S_4 = -1+1-1+1 = 0$

$S_5 = \dots \quad -1$

$$S_n = \begin{cases} -1 & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even.} \end{cases}$$

we see two obvious subsequences:

$t_n = S_{2n} = 0 \longrightarrow 0$

$t'_n = S_{2n+1} = -1 \longrightarrow -1$

tending to two <u>different</u> limits.

This shows that S_n does not converge.

Hence, <u>by definition</u>, $\sum (-1)^n$ does not converge.

(iv) $a_n = n$.

$$S_n = 1+2+3+4+\cdots+n$$

$$= \frac{n(n+1)}{2}$$

does not converge. because it is not bounded

thence by definition $\sum n$ does not converge.

Thm 4.2 a_n a sequence.

$$\sum_{n=1}^{\infty} a_n \text{ converges} \implies a_n \to 0$$

BIG WARNING the converse is not true

for instance $a_n = \frac{1}{n} \to 0$

but $\sum \frac{1}{n}$ does not converge.

Nt. to say $\sum_{n=1}^{\infty} a_n$ converges

is to say $S_n = a_1 + \cdots + a_n$

converges.

S_n converges, so it is Cauchy.

$\forall \varepsilon > 0 \quad \exists N$ s.t.

$n, m \geq N \quad |s_n - s_m| < \varepsilon$.

in particular

$$n \geq N \Rightarrow |S_{n+1} - S_n|$$

$$= |a_1 + a_2 + \cdots + a_{n+1} - (a_1 + a_2 + \cdots + a_n)|$$

$$= |a_{n+1}| < \varepsilon.$$

this clearly means that $|a_n| \to 0$,

hence also $a_n \to 0$

//

Example $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

this thing does converge.

~~Note~~

I study instead the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

I show that this series converge.

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$\left(= \frac{n+1 - n}{n(n+1)} = \frac{1}{n(n+1)} \right)$$

$$S_n = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \dots + \frac{1}{n(n+1)}$$

$$= 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} - \frac{1}{5} + \dots \frac{1}{n} - \frac{1}{n+1}$$

$$= 1 - \frac{1}{n+1}$$

$S_n \longrightarrow 1$ and then so does the series:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

We were interested in $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$T_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2}$$

$$\leq 1 + S_n$$

$$= 1 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \dots \frac{1}{n(n-1)} + \frac{1}{n(n+1)}$$

T_n is increasing and bounded above; hence T_n converges; hence so does the series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

[END L11]

Divergence to ∞

A sequence $a_n \in \mathbb{R}$ "diverges to ∞" $a_n \to \infty$ ("tends to ∞") if $\forall K \in \mathbb{R}$ $\exists N$ s.t. $n \geq N \Rightarrow a_n > K$.

Warning We do not say that a_n converges !!!!

Similarly, $a_n \in \mathbb{R}$ diverges to $-\infty$ (tends to $-\infty$) $a_n \to -\infty$ if $\forall K \in \mathbb{R}$ $\exists N$ s.t. $n \geq N \Rightarrow a_n < K$.

Propos 4.3

Let $a_n \in \mathbb{R}$ be increasing. Then:

(1) a_n bounded above $\Longrightarrow$ a_n converges.

(2) a_n not bounded above $\Longrightarrow$ a_n diverges to ∞

pf. (1) we already know (Thm 3.1)

(2) fix $K \in \mathbb{R}$.

$\exists n_0$ s.t. $a_{n_0} > K$.

(this is exactly saying a_n not bounded above) pick any $n \geq n_0$:

$$a_n \geq a_{n_0} > K.$$

This is exactly saying that a_n diverges to ∞.

<u>Remark</u>: Similarly if $a_n \in \mathbb{R}$ is decreasing, then :

(1) a_n bdd below $\Rightarrow$ a_n converges

(2) a_n not bdd below $\Rightarrow$ a_n diverges to $-\infty$

<u>Question</u> suppose $a_n \geq 0$.

What can I say about $\sum_{n=1}^{\infty} a_n$?

<u>The key observation</u>:

If $a_n \geq 0$, then the <u>sequence of partial sums</u> :

$$S_n = a_1 + a_2 + \dots + a_n$$

is <u>increasing</u>.

Indeed $S_{n+1} = S_n + a_{n+1} \geq S_n$.

Thm 4.4 $a_n \geq 0$.

$\sum_{n=1}^{\infty} a_n$ converges $\Leftrightarrow$ sequence s_n (of partial sums) is bdd (above).

~~Pf this is obvious~~

$\sum_{n=1}^{\infty} a_n$ diverges to ∞ $\Leftrightarrow$ s_n is not bounded (above)

Pf: clear from 4.3

Thm 4.5 (Comparison test)

$a_n, b_n \in \mathbb{R}$ $0 \leq a_n \leq b_n$

$\sum b_n$ converges $\Rightarrow \sum a_n$ converges

$\sum a_n$ diverges (to ∞) $\Rightarrow \sum b_n$ diverges (to ∞)

Pf $s_n := a_1 + a_2 + \cdots + a_n$

(sequence of partial sums for $\sum a_n$)

$t_n := b_1 + b_2 + \cdots + b_n$

$\sum b_n$ converges $\Rightarrow t_n \leq M$ for some M.

$\Rightarrow s_n \leq t_n \leq M$ is also bounded

$\Rightarrow \sum a_n$ converges

$\sum a_n$ diverges $\Rightarrow$

s_n is unbounded.

$\forall K \ \exists n_0$ s.t $K < s_{n_0}$

$\Rightarrow \quad K < t_{n_0}$

$(K < s_{n_0} \leq t_{n_0})$

that is, t_n is unbounded

$\Rightarrow \sum b_n$ diverges to ∞.
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Example

$$\sum_{n=1}^{\infty} \frac{1}{2^n + 3^n} \quad \text{converges.}$$

Indeed $0 \leq \dfrac{1}{2^n + 3^n} \leq \dfrac{1}{2^n}$

& we know from last time that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges.

Example $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges.

Indeed $\dfrac{1}{\sqrt{n}} \geq \dfrac{1}{n} \geq 0$ &

we know that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

More generally

$$\sum \frac{1}{n^{\alpha}} \quad \text{diverges.}$$

where $\alpha \leq 1$

$$\left(\frac{1}{n^{\alpha}} \geq \frac{1}{n} \geq 0 \right)$$

Example

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{converges.}$$

Indeed $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$

and $\displaystyle 0 \leq \frac{1}{(n+1)^2} \leq \frac{1}{n(n+1)}$

& we know from last time $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges.

Example

More generally

$$\alpha > 1 \implies \sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} \quad \text{converges}$$

Key idea

$$S_n = 1 + \frac{1}{2^{\alpha}} + \cdots + \frac{1}{n^{\alpha}}$$

$$S_{2n} - S_n = \underbrace{\frac{1}{(n+1)^{\alpha}} + \frac{1}{(n+2)^{\alpha}} + \cdots + \frac{1}{(2n)^{\alpha}}}_{n}$$

$$\cancel{\leq \underbrace{\frac{1}{(n+1)^{\alpha}} + \frac{1}{(n+1)^{\alpha}} + \cdots + \frac{1}{(n+1)^{\alpha}}}_{n}}$$

$$\leq \underbrace{\left(\frac{1}{n}\right)^{\alpha} + \cdots + \left(\frac{1}{n}\right)^{\alpha}}_{n \text{ times}} = \frac{1}{n^{\alpha-1}}$$

so for instance:

$$S_4 = S_4 - S_2 + S_2 - S_1$$

$$S_8 = S_8 - S_4 + S_4 - S_2 + S_2 - S_1 + S_1$$

$$S_{16} = S_{16} - S_8 + S_8 - S_4 + S_4 - S_2 + S_2 - S_1 + S_1$$

$$\le \frac{1}{8^{\alpha-1}} + \frac{1}{4^{\alpha-1}} + \frac{1}{2^{\alpha-1}} + 1 + 1$$

$$= 1 + 1 + \frac{1}{2^{\alpha-1}} + \left(\frac{1}{2^{\alpha-1}}\right)^2 + \left(\frac{1}{2^{\alpha-1}}\right)^3$$

More generally:

$$S_{2^{k+1}} \le 1 + 1 + \left(\frac{1}{2^{\alpha-1}}\right) + \left(\frac{1}{2^{\alpha-1}}\right)^2 + \left(\frac{1}{2^{\alpha-1}}\right)^3 + \dots + \left(\frac{1}{2^{\alpha-1}}\right)^k$$

$$\le 1 + \sum_{k=0}^{\infty} \left(\frac{1}{2^{\alpha-1}}\right)^k$$

$$= 1 + \frac{1}{1 - \frac{1}{2^{\alpha-1}}}$$

$$= A$$

this shows that s_n is bounded

indeed : $s_n \leq A$ for all n.

pick k s.t. $n \leq 2^{k+1}$:

$$s_n \leq s_{2^{k+1}} \leq A \, .$$

s_n bounded, increasing $\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$

converges //

[END L12]

Def 4.6 $a_n \in \mathbb{R}^k$

I say that $\sum_{n=1}^{\infty} a_n$

is absolutely convergent

if $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Theorem 4.7

$\sum_{n=1}^{\infty} a_n$ absolutely convergent

$\Longrightarrow \sum_{n=1}^{\infty} a_n$ is convergent.

This important because $\sum |a_n|$ is a series of terms ≥ 0; and we understand this kind of series very well.

pf of the Theorem.

Let

$s_n = a_1 + a_2 + \dots + a_n$

be the sequence of partial sums.

I want to show that s_n is a convergent sequence.

I will show that s_n is Cauchy.

(then we know s_n converges, 'cause all Cauchy sequences converge) idea: say $n > m$

$$|S_n - S_m| = |a_1 + a_2 + \cdots + a_n - a_1 - a_2 - \cdots - a_m|$$

$$= |a_{m+1} + a_{m+2} + \cdots + a_n|$$

$$\le |a_{m+1}| + |a_{m+2}| + \cdots + |a_n|$$

we can handle this if the series is absolutely convergent.

The formal pf.

We know that $\sum |a_n|$ converges.

that is, the sequence of partial sums:

$$t_n = |a_1| + |a_2| + \cdots + |a_n|$$

converges. Then t_n is Cauchy.

~~That is~~

Fix $\varepsilon > 0$. Because t_n Cauchy, there $\exists N$ s.t.

$$n > m \ge N \Rightarrow |a_m| + |a_{m+1}| + \cdots + |a_n| <$$

But then if $n > m \ge N$, also:

$$|S_n - S_m| = |a_{m+1} + \cdots + a_n|$$

$$\le |a_{m+1}| + \cdots + |a_n| < \varepsilon.$$

This shows that s_n is Cauchy

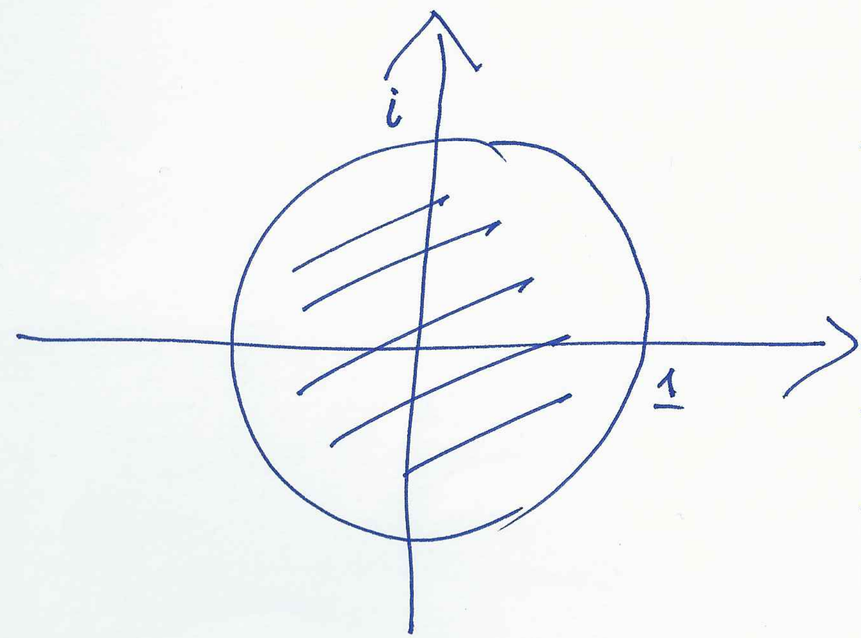
&, finally, by general principle of convergence, s_n converges, that is,

$$\sum_{n=1}^{\infty} a_n \quad \text{converges} \quad //$$

Examples for $z \in \mathbb{C}$, consider:

$$\sum_{n=1}^{\infty} z^n$$

the series converges $\Longleftrightarrow$ $|z| <$



$\Longleftrightarrow$ $z \in \mathbb{C}$ is contained in the (open) disk of radius 1 in Argand plane.

dead $\quad (|z^n| = |z|^n)$

$| < 1 \Rightarrow \sum |z|^n$ converges

$\left(\to \dfrac{1}{1-|z|} \right)$

$\Rightarrow \sum z^n$ converges.

by the Theorem

$| \geq 1 \Rightarrow |z^n| \geq 1$

$\Rightarrow |z^n| \not\to 0$

$\Rightarrow z^n \not\to 0$

$\Rightarrow \sum z^n$ does not converge

Th 4.

Example

The converse of the Theorem is **not** true.

I.E. there are convergent series $\sum a_n$ which are not absolutely convergent. For instance:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \dots$$

i.e. $a_n = \dfrac{(-1)^{n+1}}{n}$

we will show that $\sum a_n$ converges. but we do know $\sum |a_n|$ diverges.

Next .... I will prove a general theorem about series which alternate sign. That theorem will immediately imply that $\sum \frac{(-1)^{n+1}}{n}$ converges.

Theorem 4.8 (comparison test II)

$a_n \in \mathbb{R}$ a sequence.

$b_n \geq 0$ "

Assume $\sum b_n$ converges; if there exist $n_0 \geq 0$, $C \geq 0$ s.t. $n \geq n_0 \Rightarrow |a_n| \leq C\, b_n$

then $\sum a_n$ converges.

(in fact absolutely).

Pf. Denote:

$t_n = b_1 + \cdots + b_n$

$s_n = |a_1| + \cdots + |a_n|$.

I know t_n is Cauchy.

I prove s_n is Cauchy.

fix $\varepsilon > 0$. there $\exists N$ s.t:

$n > m \geq N$

$$\Rightarrow |t_n - t_m| = b_{m+1} + \dots + b_n < \frac{\varepsilon}{C}$$

pick $N' = \max\{N, n_0\}$:

$$n > m \geq N' \Rightarrow |s_n - s_m|$$

$$= |a_{m+1}| + \dots + |a_n|$$

$$\leq C b_{m+1} + C b_{m+2} + \dots C b_n$$

$$= C(b_{m+1} + b_{m+2} + \dots b_n)$$

$$< C \frac{\varepsilon}{C} = \varepsilon$$

that is s_n Cauchy $\Rightarrow$ s_n converges //

Examples

$$\sum_{n=1}^{\infty} \frac{n^2+5}{n^4-10}$$ converges.

we want: $\exists n_0$, $C \geq 0$ such that

$$\frac{n^2+5}{n^4-10} \leq C \frac{1}{n^2}$$

note:

$n^2 + 5 \leq 2n^2$ for $n \geq 3$

$n^4 - 10 \geq \frac{1}{2} n^4$

for $n^4 \geq 20$ or $n \geq 3$

take $n_0 = 3$, $C = 4$:

$n \geq 3$:

$$\frac{n^2+5}{n^4-10} \leq \frac{2n^2}{\frac{1}{2}n^4} = 4\,\frac{1}{n^2}$$

we know $\sum \frac{1}{n^2}$ converges;

we conclude <u>by</u> Comparison $\underline{\text{II}}$:

$$\sum \frac{n^2+5}{n^4-10} \quad \text{converges.}$$

[END L13]

~~Next week:~~

~~Test Fr. 19th Feb at 15:00~~

~~Chapters 1 – 4~~

~~(everything so far until today + possible "easy" Qs on material next week)~~

~~worksheets 1–6 (inclusive)~~

~~(sht 6 goes on the web today + some notes)~~

Theorem 4.8 $|a_n| \leq c\, b_n$ for $n \geq n_0$

$\Rightarrow$ if $\sum_{n=1}^{\infty} b_n$ converges

then $\sum_{n=1}^{\infty} a_n$ converges

(Comparison test II)

Example

$$\sum \frac{n^2+1}{2n^4-16}$$

everybody can see this converges:

$a_n \sim \cancel{as}\ \frac{1}{2n^2}$ & $\sum \frac{1}{n^2}$ converge we want to apply comparison test in a very precise way.

$n^2+1 < 2n^2$

(works $\iff 1 < n^2$
ie. $n \geq 2$)

$2n^4 - 16 > n^4$

(works $\iff n^4 > 16$
ie. $n \geq 3$)

Note: for $n \geq n_0 = 3$:

$$\frac{n^2+1}{2n^4-16} \leq \frac{2n^2}{n^4} = 2\,\frac{1}{n^2}$$

By the comparison test ($n_0 = 3$, $C = 2$)

series converges.

Example

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

everybody can see this diverges:

$$\frac{n}{n^2+1} \sim \frac{1}{n}$$

before I continue:

Exercise $\boxed{0 \le a_n}$

$\exists\, n_0,\ C > 0$ s.t. $n \geq n_0$ $Ca_n \leq b_n$

$$\sum_{n=1}^{\infty} a_n \text{ diverges} \Rightarrow \sum_{n=1}^{\infty} b_n \text{ diverges}$$

(a very small strengthening of 1st comparison test)

$$n \geq n$$

$$n^2 + 1 \leq 2n^2 \quad \text{for all } n:$$

$$\frac{n}{n^2+1} \geq \frac{n}{2n^2} = \frac{1}{2} \quad \frac{1}{n}$$

$$\Rightarrow \sum \frac{n}{n^2+1} \text{ diverges} //$$

Thm 4.9 (Ratio test aka "d'Alembert test")

$a_n \neq 0$ for all n.

Assume $\exists\ 0 < r < 1,\ n_0$ s.t.

$$n \geq n_0 \Rightarrow \frac{|a_{n+1}|}{|a_n|} \leq r$$

Then $\sum_{n=0}^{\infty} a_n$ converges absolutely.

pf. $A = |a_{n_0}|$

$n = n_0 + 1$ $\quad |a_n| \leq A \cdot r$

$n = n_0 + 2$ $\quad |a_n| = |a_{n_0+2}| \leq |a_{n_0+1}| r$

$$\leq |a_{n_0}| r^2 = A r^2$$

$n = n_0 + 3$ $\quad |a_n| \leq A r^3$

and so on:

$n \geq n_0$ $\quad |a_n| \leq A r^{n-n_0}$

$$= \frac{A}{r^{n_0}} \cdot r^n$$

In other words:

$n \geq n_0$, $\quad C = \dfrac{A}{r^{n_0}}$:

$$|a_n| \leq C r^n$$

& we know that $\sum_{n=1}^{\infty} r^n$

converges; so by comparison test (II),

$\sum_{n=1}^{\infty} a_n$ converges

//

Corollary $a_n \neq 0$

Assume $\frac{|a_{n+1}|}{|a_n|} \longrightarrow r$

If $r < 1$ then $\sum a_n$ converges absolutely

If $r > 1$ then $\sum a_n$ diverges.

If $r = 1$ ~~we don't know~~

(the test gives no information)

Pf. Assume $r < 1$,

choose $\varepsilon > 0$ such that $r + \varepsilon = q < 1$

$\exists\, n_0$ s.t.

$n \geq n_0 \implies \frac{|a_{n+1}|}{|a_n|} < r + \varepsilon = q$

then apply ratio test.

Assume $r > 1$ choose $\varepsilon > 0$ such that $r - \varepsilon = q > 1$

$\exists\, n_0$ s.t.

$n \geq n_0$ $\frac{|a_{n+1}|}{|a_n|} > r - \varepsilon = q$

As before, if $n \geq n_0$:

$$|a_n| \geq |a_{n_0}| q^{n-n_0}$$

$$= \left(\frac{|a_{n_0}|}{q^{n_0}} \right) q^n$$

But $q > 1$ so ~~[illegible]~~

diverges to ∞.

So $|a_n| \not\to 0 \Rightarrow a_n \not\to 0$

$\Rightarrow$ series does not converge.

What about the case $r = 1$?

Example

$\sum \frac{1}{n}$ diverges $\frac{|a_{n+1}|}{|a_n|} = \frac{n}{n+1} \to 1$

$\sum \frac{1}{n^2}$ converges $\left|\frac{a_{n+1}}{a_n}\right| = \frac{n^2}{n^2+1} \to 1$

When $\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$ the test is inconclusive. We (you) have to study the series by some other means —

Note $\lim \left| \frac{a_{n+1}}{a_n} \right|$ may not exist.

Test can be useful in that case too.

In all cases

$$\left| \frac{a_{n+1}}{a_n} \right| \leq \frac{2}{3} < 1$$

hence series converges.

Example

$$1 + \frac{2}{3} + \frac{1}{3^2} + \frac{2}{3^3} + \frac{1}{3^4} + \frac{2}{3^5} + \cdots$$

(you could directly compare with $\frac{1}{3^n}$: $a_n \leq 2 \cdot \left(\frac{1}{3}\right)^n$)

$$\frac{a_{n+1}}{a_n} = \begin{cases} \frac{2}{3} & n \text{ even} \\ \frac{1}{6} & n \text{ odd} \end{cases}$$

Example $\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{3^n}$

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)^3}{3^{n+1}} \frac{3^n}{n^3}$$

$$= \left(\frac{n+1}{n}\right)^3 \frac{1}{3} \to \frac{1}{3} < 1$$

hence series converges.

Ex 6 q 2 (ii)

$$a_n = \frac{1}{\sqrt{n} + \sqrt{n+1}}$$

does not converge

(e.g compare w. $\sum \frac{1}{n^{1/2}}$).

Q. wants a simple formula for

value $s_n = a_1 + a_2 + a_3 + \cdots + a_n$

$$a_n = \frac{1}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} \frac{(\sqrt{n+1} - \sqrt{n})}{(\sqrt{n+1} - \sqrt{n})}$$

$$= \sqrt{n+1} - \sqrt{n}$$

$$s_n = a_1 + a_2 + \dots + a_n$$

$$= \cancel{\sqrt{2}} - 1$$
$$+ \cancel{\sqrt{3}} - \cancel{\sqrt{2}}$$
$$+ \cancel{\sqrt{4}} - \cancel{\sqrt{3}}$$
$$\vdots$$
$$\sqrt{n+1} - \cancel{\sqrt{n}}$$

$$= \sqrt{n+1} - 1$$

manifestly s_n does not converge

(it's unbounded)

so $\sum_{n=1}^{\infty} a_n$ does not converge.

[END L. 14]

Alternating series

Let $b_n \geq 0$ be a sequece of positive numbers.

An alternating series is a series $\sum_{n=1}^{\infty} a_n$

where

either $a_n = (-1)^n b_n$

or $a_n = (-1)^{n+1} b_n$

Theorem (alternating series test)

Let $b_n \geq 0$ be a decreasing sequence. Assume $b_n \to 0$.

Let $a_n = (-1)^{n+1} b_n$ $\left(a_n = (-1)^n b_n \right)$

Then $\sum_{n=1}^{\infty} a_n$ converges.

Example the series:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{ converges.}$$

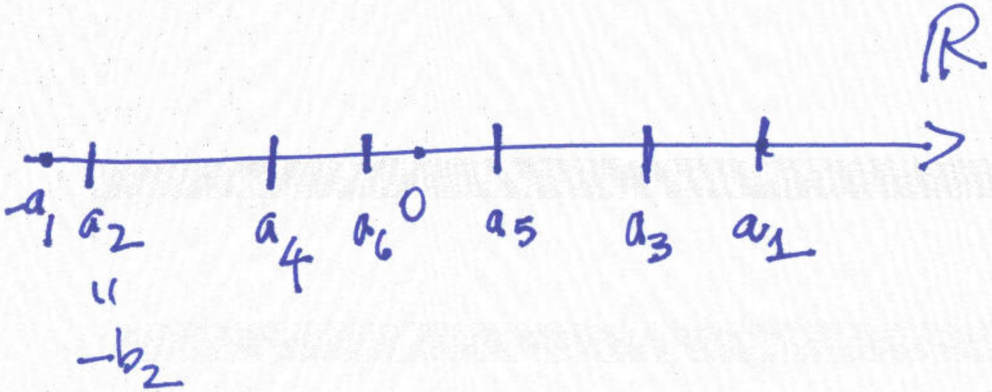
indeed $b_n = \frac{1}{n} \geq 0$

(1) is <u>decreasing</u>
$(n \leq m \Rightarrow \frac{1}{n} \geq \frac{1}{m})$

(2) tends to 0.

proof of the theorem.

<u>picture</u>: $a_n = (-1)^{n+1} b_n$



pf is in <u>several</u> <u>steps</u>:

$$S_n = a_1 + a_2 + \cdots + a_n$$

the steps will be as follows:

<u>Step 1</u>

S_{2n} increasing, $\leq a_1$

$\Rightarrow S_{2n} \to l$

<u>Step 2</u>

S_{2n+1} decreasing ≥ 0

$\Rightarrow S_{2n+1} \to m$

<u>Step 3</u> $l = m$.

<u>Step 4</u> $S_n \to l \; (= m)$

this shows the theorem.

pf Step 1

I study s_{2n}:

$$s_2 = a_1 + a_2 \leq a_1$$

$$s_4 = a_1 + a_2 + a_3 + a_4$$

$$= s_2 + \underbrace{(b_3 - b_4)}_{\geq s_2}$$

$$= a_1 + (a_2 + a_3) + a_4$$

$$= a_1 + \underbrace{(-b_2 + b_3)}_{\leq 0} - b_4 \leq a_1$$

$\leq$

- s_{2n} is increasing:

$$s_{2n} = s_{2(n-1)} + a_{2n-1} + a_{2n}$$

$$= s_{2(n-1)} + \underbrace{(b_{2n-1} - b_{2n})}_{\geq 0}$$

$$\geq s_{2(n-1)}$$

- $s_{2n} \leq a_1$:

$$s_{2n} = a_1 + (a_2 + a_3) + (a_4 + a_5) + \dots + (a_{2n-2} + a_{2n-1}) + a_{2n}$$

$$= a_1 + (-b_2 + b_3) + (-b_4 + b_5) + \dots + (-b_{2n-2} + b_{2n-1}) - b_{2n}$$

$$= a_1 + (\text{bunch of negative terms}) \leq a_1$$

pf Step 2

I study S_{2n-1}:

$$s_1 = a_1 = b_1 \geq 0 .$$

$$s_3 = a_1 + a_2 + a_3$$

$$= s_1 + \underbrace{(-b_2 + b_3)}_{\leq 0} \leq s_1$$

$$s_3 = (a_1 + a_2) + a_3$$

$$= \underbrace{(b_1 - b_2)}_{\geq 0} + \underbrace{b_3}_{\geq 0} \geq 0$$

• s_{2n-1} is decreasing

$$s_{2n-1} = s_{2n-3} + a_{2n-2} + a_{2n-1}$$

$$= s_{2n-3} + \underbrace{(-b_{2n-2} + b_{2n-1})}_{\leq 0} \leq s_{2n-3}$$

• $s_{2n-1} \geq 0$:

$$s_{2n-1} = (a_1 + a_2) + (a_3 + a_4) + \dots + (a_{2n-3} + a_{2n-2}) + a_{2n-1}$$

$$= \underbrace{(b_1 - b_2)}_{\geq 0} + \underbrace{(b_3 - b_4)}_{\geq 0} + \dots + \underbrace{(b_{2n-3} - b_{2n-2})}_{\geq 0} + \underbrace{b_{2n-1}}_{\geq 0}$$

$$\geq 0 .$$

this finishes Step 2.

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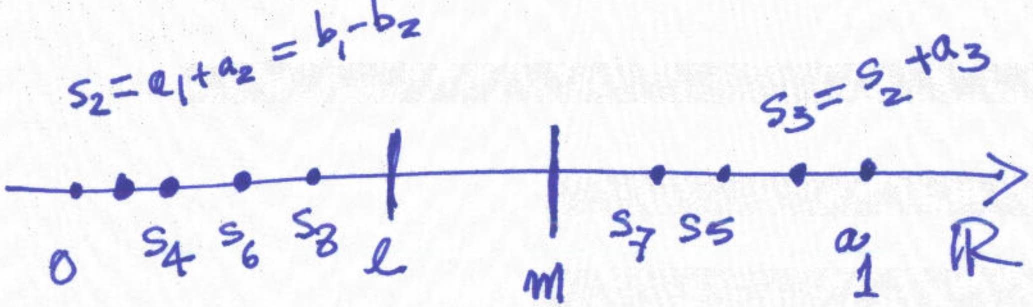
picture of where we are : Step 3 $\ell = m$.



$= S_1$
$= b_1$

$s_{2n} \uparrow$ $\quad s_{2n-1} \downarrow$

$\leq a_1$ $\quad \geq 0$

$\lim s_{2n} = \ell$ $\quad \lim s_{2n-1} \longrightarrow m$

$$s_{2n+1} = s_{2n} + a_{2n+1}$$
$$= s_{2n} + b_{2n+1}$$

$\Longrightarrow$

$$m = \lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} b_{2n+1}$$
$$= \ell + 0$$

so $\ell = m$.

Step 4 $s_n \to \ell$ (Fin)

[END L. 15]

fix $\varepsilon > 0$.

$\exists N_1$ s.t. $n \geq N_1$, $|s_{2n} - \ell| < \varepsilon$

$\exists N_2$ s.t. $n \geq N_2$, $|s_{2n+1} - \ell| < \varepsilon$

If $n \geq 2 \max\{N_1, N_2\} + 2$:

(a) if n even, then

$n = 2k \geq 2N_1 \Rightarrow k \geq N_1$

so $|s_n - \ell| = |s_{2k} - \ell| < \varepsilon$.

(b) if n odd, then

$n = 2k+1 \geq 2N_2 + 1 \Rightarrow k \geq N_2$

$|s_n - \ell| = |s_{2k+1} - \ell| < \varepsilon$

//

Revision

Thm 4.5 (Comparison I)

$0 \le a_n \le b_n$

$\sum b_n$ converges $\Rightarrow \sum a_n$ converges

$\sum a_n$ diverges $\Rightarrow \sum b_n$ diverges.

(OK also if $\exists\, n_0, C > 0$ s.t.

$0 \le a_n \le C\, b_n$ for $n \ge n_0$)

Def 4.6 (absolutely convergent)

$\sum a_n$ absolutely convergent if

$\sum |a_n|$ is convergent.

Thm 4.7 absolutely convergent

$\Longrightarrow$ convergent.

Key examples:

- $\displaystyle\sum_{n \ge 0} q^n$ $\begin{cases} \text{converges if } 0 < q < 1 \\ \text{diverges } q \ge 1 \end{cases}$

- $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ $\begin{cases} \text{converges } \alpha > 1 \\ \text{diverges } \alpha \le 1 \end{cases}$

- $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

  converges but not absolutely.

<u>Thm 4.8</u> (Comparison II)

$\exists\, n_0,\ C > 0$ st.

$n \geq n_0 \Rightarrow |a_n| \leq C\, b_n$

$\sum b_n$ converges $\Rightarrow \sum a_n$ absolutely converges.

<u>Example</u> $\displaystyle\sum_{n=1}^{\infty} \frac{n^4+2}{n^6-100}$

converges :

$n^4 + 2 \leq 2n^4$ if $n \geq 2$

$n^6 - 100 \geq \frac{1}{2} n^6$ if $n \geq 3$

$n_0 = 3 \quad C = 4$:

$$\frac{n^4+2}{n^6-100} \leq \frac{2n^4}{\frac{1}{2}n^6} = 4\frac{1}{n^2} \quad n \geq 3$$

<u>Th 4.9</u> (Ratio test)

$a_n \neq 0$ all n. <u>Assume</u>:

$\exists\, n_0,\ 0 < r < 1$ s.t.

$n \geq n_0 \Rightarrow \left|\frac{a_{n+1}}{a_n}\right| \leq r$

Then $\displaystyle\sum_{n=0}^{\infty} a_n$ converges. (absolutely)

<u>Philosophy</u> The ratio test is not <u>very</u> deep. It is <u>basically</u> comparison test with $\displaystyle\sum_{n=0}^{\infty} r^n$.

Corollary (user-friendly ratio test)

$a_n \neq 0$ all n. Assume

$\left|\frac{a_{n+1}}{a_n}\right| \longrightarrow r$.

If $r < 1 \Rightarrow \sum a_n$ absolutely convergent

If $r > 1 \Rightarrow \sum a_n$ divergent

If $r = 1 \Rightarrow$ TEST INCONCLUSIVE

Example test inconclusive for $\sum \frac{1}{n^\alpha}$ (any α)

Thm 4.10 (Alternating series test)

$b_n \downarrow 0$

(b_n decreasing, $\lim b_n = 0$)

$\Rightarrow \sum (-1)^n b_n$, $\sum (-1)^{n+1} b_n$ converge.

Example $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

You need to check that b_n is decreasing

(COMPLEX) POWER SERIES

A (complex) power series is a series like so:

$$\sum_{n=0}^{\infty} a_n z^n \quad (*)$$

here • $a_n \in \mathbb{C}$ is a fixed sequence.

• $z \in \mathbb{C}$

We study for what $z \in \mathbb{C}$ (*) converges; and then we study (*) as a (complex-valued) function of z

Examples:

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \cdots$$

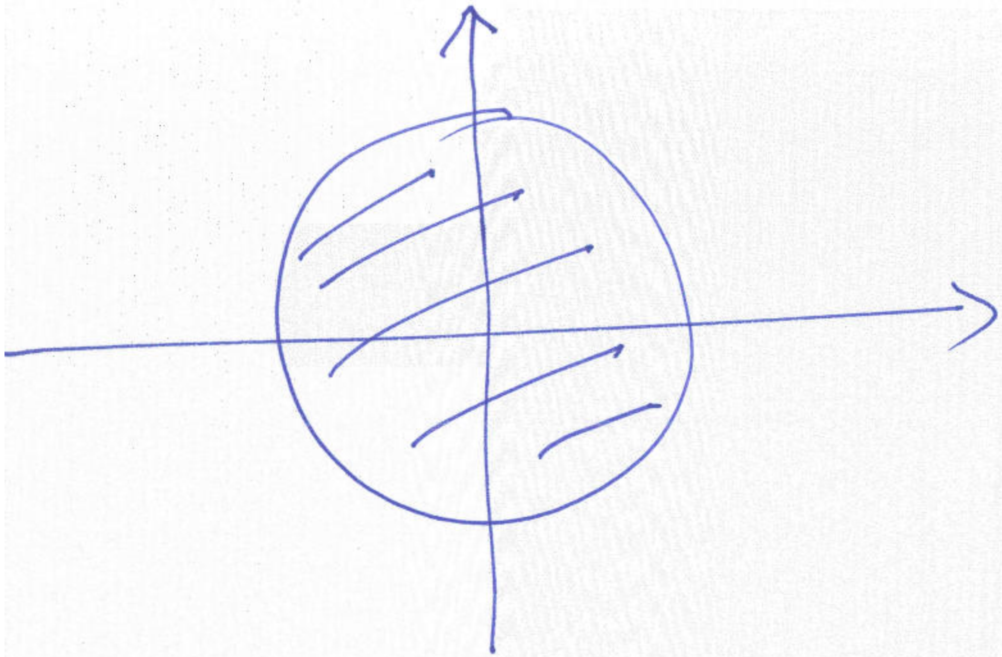
converges iff $|z| < 1$

diverges otherwise.

(If $|z| = r < 1$, then compare with $\sum r^n$;

If $|z| \geq 1$, then $|z|^n \geq 1$ hence $z^n \not\to 0$, hence $\sum$ diverges)

- the domain of convergence is a disk



- We know that:

$|z|<1$ then: $\displaystyle\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$

Indeed:

$$S_n = 1+z+z^2+\dots+z^n = \frac{1-z^{n+1}}{1-z}$$

$$\lim_{n\to\infty} S_n(z) = \frac{1}{1-z}$$

Note:

the complex valued function $f(z) = \dfrac{1}{1-z}$

is defined on

$$\{ z \in \mathbb{C} \mid z \neq 1 \}$$

On the other hand the power series

$\sum z^n$ only converges on $\{ |z|<1 \}$

Example:

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$= 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \cdots$$

converges for all $z \in \mathbb{C}$.

Pick $z \in \mathbb{C}$ (fixed)

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{z^{n+1}}{(n+1)!} \frac{n!}{z^n} = \frac{z}{n+1}$$

$$\longrightarrow 0$$

So OK by ratio test.

Example

$$\sum_{n=0}^{\infty} n!\, z^n$$

only converges if $z = 0$.

Ratio test again: fix $z \neq 0$:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!\, z^{n+1}}{n!\ z^n} = (n+1) z$$

$$\longrightarrow \infty.$$

Example $\sum_{n=1}^{\infty} \frac{z^n}{n}$

- converges if $|z| < 1$
- diverges if $|z| > 1$

The situation is somewhat complicated if $|z| = 1$

E.G.

We know:

$$z=1 \rightsquigarrow \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$

$$z=(-1) \rightsquigarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{ converges.}$$

In fact $\sum \frac{z^n}{n}$

converges when $|z|=1$, $z \neq 1$

but this is not so easy to see.

We will see for all $\sum a_n z^n$ $\exists$ $0 \leq R \in \mathbb{R}$

s.t. $\sum a_n z^n$ converges $|z|<R$
diverges $|z|>R$

[END L: 16]

Power series

$a_n \in \mathbb{C}$ sequence $(n \geqslant 0)$

$$\sum_{n=0}^{\infty} a_n z^n \qquad (*)$$

where $z \in \mathbb{C}$ variable.

defn A real no. $0 \leq R \in \mathbb{R} \cup \{\infty\}$ is the radius of convergence of the series $(*)$ if:

(1) $(*)$ converges (absolutely) if $|z| < R$

(2) $(*)$ diverges if $|z| > R$

N.B. no judgement on $|z| = R$.

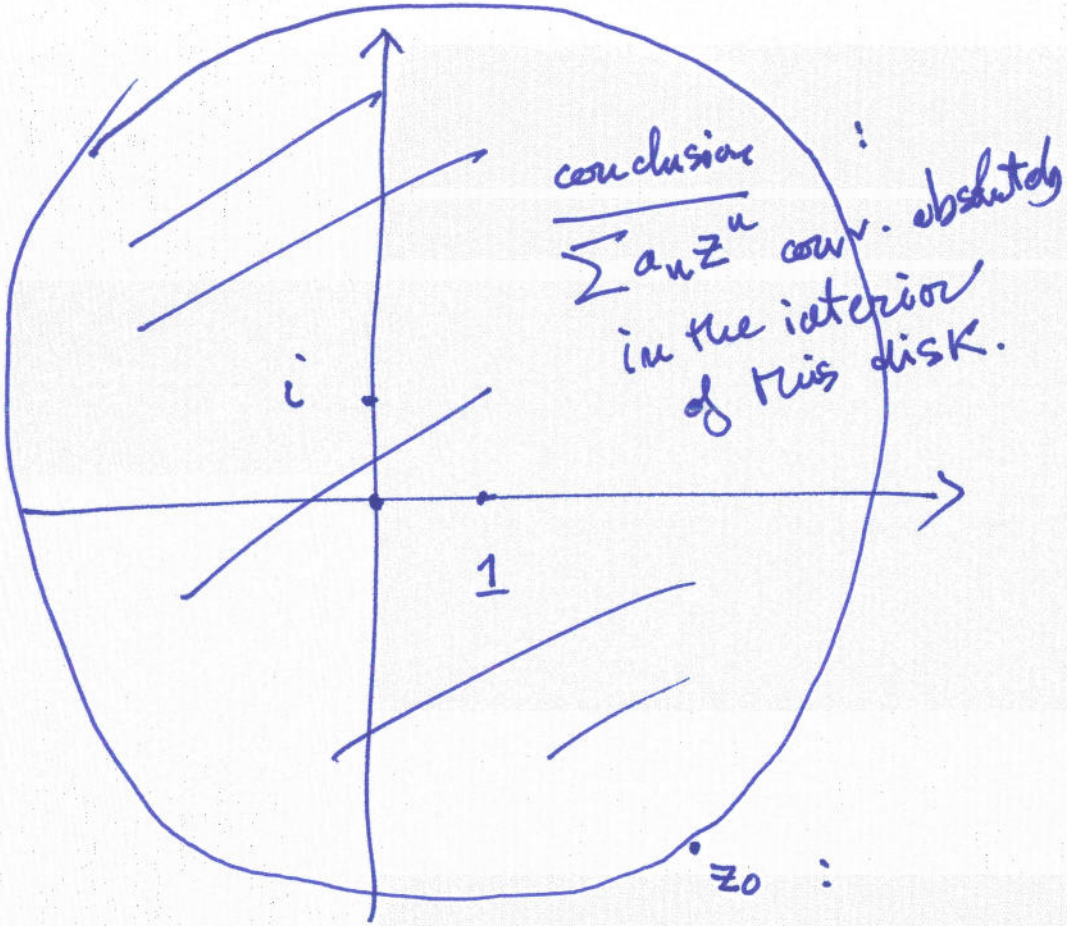
Theorem 4.11 Every complex power series has a radius of convergence.

Before I prove this, I prove the following:

Lemma Let $z_0 \in \mathbb{C}$, assume $\sum_{n=0}^{\infty} a_n z_0^n$ converges.

then $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely if $|z| < |z_0|$

picture of what the lemma says:



$\cdot\, z_0$:

We assume $\sum_{n=0}^{\infty} a_n z_0^n$ converges

pf of Lemma:

$$a_n z^n = a_n z_0^n \cdot \left(\frac{z}{z_0}\right)^n$$

Note $\sum_{n=0}^{\infty} a_n z_0^n$ converges

so $\lim_{n\to\infty} a_n z_0^n = 0$.

so $\{a_n z_0^n\}$ is bounded

so $\exists\ C>0$ s.t. $|a_n z_0^n| \leq C$ for all n.

So now if $|z| < |z_0|$,

then $q = \left|\frac{z}{z_0}\right| = \frac{|z|}{|z_0|} < 1$

(here I have used some standard property of absolute value of complex no.s :

$|wz| = |w|\,|z|$)

please go to next page

Now :

$|a_n z^n| = |a_n z_0^n| \left|\frac{z}{z_0}\right|^n$

$\leq C\, q^n$

by Comparison Test (II):

$$\sum_{n=0}^{\infty} q^n \text{ converges } (q<1)$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n z^n \text{ absolutely converges.}$$

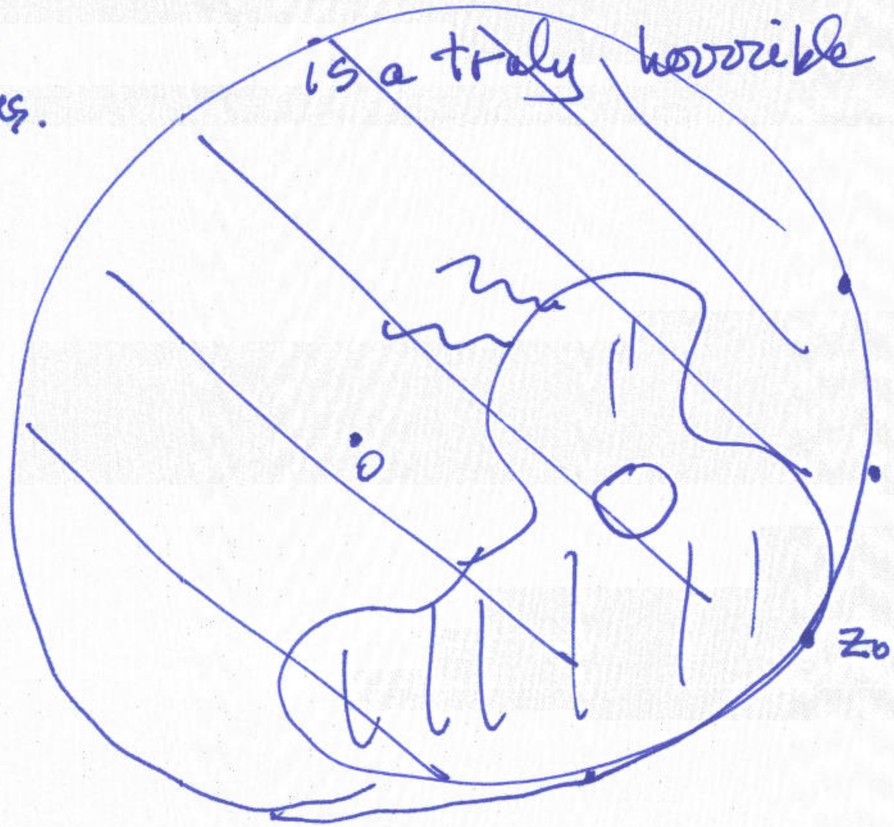
qed Lemma.

picture of why Lemma $\Rightarrow$ Thm 4.11

Assume that

$$S = \{ z \in \mathbb{C} \mid \sum_{n=0}^{\infty} a_n z^n \text{ converges} \}$$

is a truly horrible set.



choose $s \in S$ with largest norm.

Pf of Thm 4.11

$$R = \begin{cases} \sup\left\{ z_0 \in \mathbb{C} \;\middle|\; \sum_{n=0}^{\infty} a_n z_0^n \text{ converges} \right\} & \text{if set is bounded} \\ \infty & \text{otherwise} \end{cases}$$

Claim R = radius of converges.

Pf; say $R < \infty$.

If $R = 0$ nothing to prove,

so say $0 < R$.

(1) If $|z| < R$,

then $\exists\, z_0$ s.t. $|z| < |z_0| < R$

& $\sum_{n=0}^{\infty} a_n z_0^n$ converges.

By Lemma, $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely.

(2) Take $|z| > R$.

Then clearly $\sum_{n=0}^{\infty} a_n z^n$ diverges

(for if it converged, then R would not be an u. bound.)

//

Example:

$$\sum_{n=0}^{\infty} (-1)^n z^{2n}$$

$$= 1 - z^2 + z^4 - z^6 + z^8 - \ldots$$

(1) series converges absolutely if $|z| < 1$

(by comparison with $\sum r^n$ where $r = |z^2| < 1$)

(2) series diverges if $|z| \geq 1$

(e.g. 'cause $z^{2n} \not\to 0$
'cause $|z^{2n}| = 1 \not\to 0$)

In particular $R = 1$.

for $\{|z| < 1\}$:

$$\sum_{n=0}^{\infty} (-1)^n z^{2n} = \sum_{n=0}^{\infty} (-z^2)^n$$

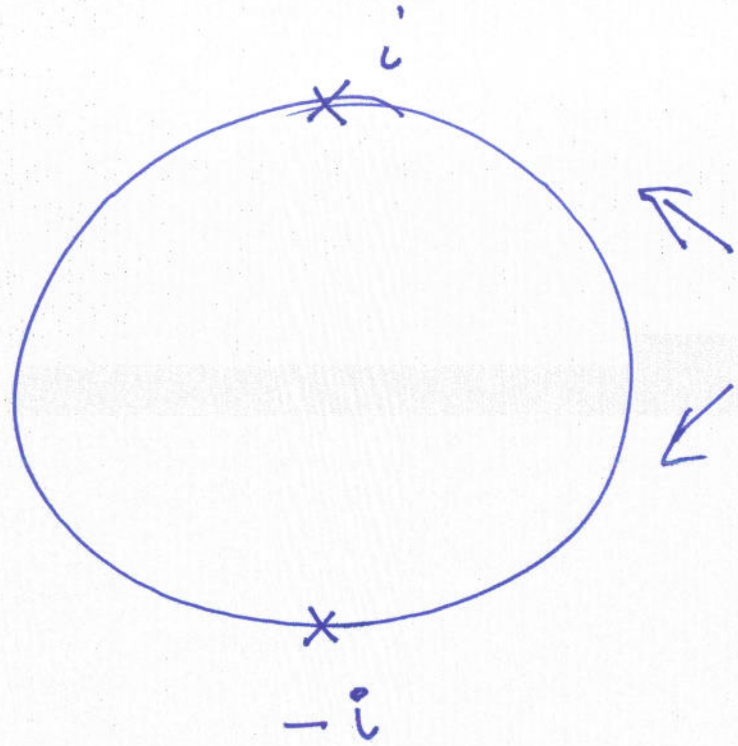
$$= \frac{1}{1+z^2}$$

The function $f(z) = \frac{1}{1+z^2}$

The domain is

$$\mathbb{C} \setminus \{\pm i\}$$

The fact also explains why I, following many other teachers before me, decided to do power series of complex numbers not just real numbers



$f(z) = \frac{1}{1+z^2}$

explodes here.

The fact that f explodes there completely explains that

$$R \leq 1$$

eg 106/

If I did <u>real numbers</u>

then

$$f(x) = \frac{1}{1+x^2}$$

is perfectly cool for $x \in \mathbb{R}$.

From the perspective of real nos alone we can't understand <u>why</u>, <u>why</u>

r.o.c = "only" $\underline{1}$.

[END L.17]

LAST WEEK'S TEST

Q1

(i) A sequence a_n converges to l

if: $\forall \varepsilon > 0$, $\exists N$ such that

$n \geq N \Rightarrow |a_n - l| < \varepsilon$.

(ii) $a_n = \dfrac{1 - 3n^3}{4n^3 - 1}$

$$= \frac{-3 + \frac{1}{n^3}}{4 - \frac{1}{n^3}}$$

By theorem on products of limits:

$$\frac{1}{n^3} = \frac{1}{n} \times \frac{1}{n} \times \frac{1}{n} \to 0$$

'cause $\frac{1}{n} \to 0$.

By the theorem on sums of limits:

the numerator $-3 + \frac{1}{n^3} \longrightarrow -3$,

and the denominator $4 - \frac{1}{n^3} \longrightarrow 4$.

The denominator $4 - \frac{1}{n^3}$ is always $\neq 0$, and tends to $4 \neq 0$, hence the assumptions of the theorem on quotients of limits are satisfied and:

$$a_n = \frac{-3+\frac{1}{n^3}}{4-\frac{1}{n^3}} \longrightarrow \frac{-3}{4}.$$

(iii) $l = -\frac{3}{4}$

$$|a_n - l| = \left|\frac{1-3n^3}{4n^3-1} + \frac{3}{4}\right|$$

$$= \left|\frac{4-12n^3+12n^3-3}{16n^3-4}\right|$$

$$= \frac{1}{16n^3-4}$$

Note: $\frac{1}{4(4n^3-1)} < \varepsilon$

if $4n^3 - 1 > \frac{1}{4\varepsilon}$

or $n^3 > \left(\frac{1}{4\varepsilon}+1\right)\frac{1}{4}$.

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Fix $\varepsilon > 0$.

Let $N = \sqrt[3]{\frac{1}{4}\left(\frac{1}{4\varepsilon}+1\right)} + 1$

then, if $n \geq N$:

$$\left|a_n + \frac{3}{4}\right| = \frac{1}{16n^3-4} < \varepsilon$$

//

Q2

$$a_n = \frac{1 + (-1)^n n^2}{n^2 - 2}$$

$$\cancel{= \frac{(-1)^n + \frac{1}{n^2}}{-2 + \frac{1}{n^2}}}$$

$$= \frac{(-1)^n + \frac{1}{n^2}}{1 - \frac{2}{n^2}}$$

Consider the subsequence

$$b_n = a_{2n} = \frac{1 + \frac{1}{4n^2}}{1 - \frac{2}{4n^2}} \to 1.$$

$b_n \to 1 \Rightarrow a_n \not\to -1$ by the theorem on limits of subsequences of a convergent sequence –

Q3

(i) $a_n = (-1)^n$ divergent

$b_n = \frac{1}{n}$ bounded

$a_n b_n = \frac{(-1)^n}{n} \to 0$ convergent

~~(ii)~~ Alternative even easier:

$a_n = (-1)^n$ divergent

$b_n = (-1)^n$ bounded

$a_n b_n = 1$ convergent.

(ii) $a_n = \begin{cases} n & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$

(iii) $a_n = \frac{1}{n} \to 0$

but $\sum \frac{1}{n}$ diverges.

Q4 $s_n = (1)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2 + \cdots + \left(\frac{1}{n}\right)^2$

(i) use: $\frac{1}{n^2} < \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$

to estimate $(m > n > 0)$:

$$s_m - s_n = \left(\frac{1}{n+1}\right)^2 + \left(\frac{1}{n+2}\right)^2 + \left(\frac{1}{n+3}\right)^2 + \cdots \left(\frac{1}{m}\right)^2$$

$$< \frac{1}{n} - \cancel{\frac{1}{n+1}} + \cancel{\frac{1}{n+1}} - \cancel{\frac{1}{n+2}} + \cancel{\frac{1}{n+2}} - \cancel{\frac{1}{n+3}} + \cdots + \cancel{\frac{1}{m-1}} - \frac{1}{m}$$

$$= \frac{1}{n} - \frac{1}{m}$$

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(ii) fix $\varepsilon > 0$.

[END L: 18]

Choose $N > \frac{1}{\varepsilon}$.

Then if $m \geq n \geq N$:

$$|s_m - s_n| = \cancel{(} s_m - s_n < \frac{1}{n} - \frac{1}{m}$$

$$\leq \frac{1}{n} < \varepsilon$$

so the sequence s_n is Cauchy.

(iii) By the theorem of convergence of Cauchy sequences, we know that s_n converges.

By definition of convergence of a series, then $\sum \frac{1}{n^2}$ converges.