§5 Rules for calculating with convergent series

Propos 5.1

(i) If $\sum_{n=0}^{\infty} a_n = A$ and $\lambda \in \mathbb{C}$

then $\sum_{n=0}^{\infty} \lambda a_n = \lambda A$.

(ii) If $\sum_{n=0}^{\infty} a_n = A$, & $\sum_{n=0}^{\infty} b_n = B$

$\Longrightarrow \sum_{n=0}^{\infty} (a_n + b_n) = A + B$.

pf: this follows easily from corresponding statement about sequences.

e.g. for (i):

let $s_n = a_0 + a_1 + \cdots + a_n$

be the sequence of partial sums.

To say that $\sum_{n=0}^{\infty} a_n = A$ is, by definition, to say that the sequence

$$s_n \longrightarrow A$$

But then we know from sequence theory that:

$$\lambda s_n \longrightarrow \lambda A.$$

Note that:

$$\lambda s_n = \lambda a_0 + \lambda a_1 + \cdots + \lambda a_n$$

"is" the sequence of partial sums of the series $\sum_{n=0}^{\infty} \lambda a_n$.

By definition of convergence of a series then $\sum_{n=0}^{\infty} \lambda a_n$ converges to:

$$\sum_{n=0}^{\infty} \lambda a_n = \lim_{n\to\infty} \lambda s_n = \lambda A$$

qed (i)

(ii) Exercise (write a short essay).

PRODUCTS OF SERIES

To explain the idea, I start with power series.

Goal of the theory:

start with $\sum_{n=0}^{\infty} a_n z^n$ r.o.c. $R_1 > 0$

& $\sum_{n=0}^{\infty} b_n z^n$ r.o.c. $R_2 > 0$.

let $R = \min\{R_1, R_2\}$.

for $z \in D(0,R) = \{ z \in \mathbb{C} \mid |z| < R \}$

we have defined two functions:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

$$g(z) = \sum_{n=0}^{\infty} b_n z^n.$$

Goal of the theory:

to "set up" a power series

$$\sum_{n=0}^{\infty} c_n z^n \quad \text{with r.o.c.} \geq R$$

such that for $z \in D(0,R)$:

$$f(z)g(z) = \sum_{n=0}^{\infty} c_n z^n$$

Idea:

$$\begin{array}{c|ccccc}
\vdots & \vdots & \vdots & & \vdots & \\
+ & & & & & \\
b_3 z^3 & a_0 b_3 z^3 & a_1 b_3 z^4 & \cdots & & \\
+ & & & & & \\
b_2 z^2 & a_0 b_2 z^2 & a_1 b_2 z^3 & \cdots & & \\
+ & & & & & \\
b_1 z & a_0 b_1 z & a_1 b_1 z^2 & a_2 b_1 z^3 & \cdots & \\
+ & & & & & \\
b_0 & a_0 b_0 & a_1 b_0 z & a_2 b_0 z^2 & a_3 b_0 z^3 & \cdots \\
\hline
 & a_0 & + a_1 z & + a_2 z^2 & + a_3 z^3 & + \cdots
\end{array}$$

$$\sum c_n z^n$$

$$= a_0 b_0 + (a_1 b_0 + a_0 b_1) z + (a_2 b_0 + a_1 b_1 + a_0 b_2) z^2$$

$$+ (a_3 b_0 + a_2 b_1 + a_1 b_2 + a_0 b_3) z^3 + \text{hot}$$

In other words:

$$c_n = a_n b_0 + a_{n-1} b_1 + a_{n-2} b_2 + \dots + a_0 b_n$$

$$= \sum_{k=0}^{n} a_{n-k} b_k .$$

---

defn 5.2 (Cauchy product of series)

The <u>Cauchy product</u> of two series $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$

★ is★ the series $\sum_{n=0}^{\infty} c_n$ where:

$$c_n = \sum_{k=0}^{n} a_{n-k} b_k$$

Theorem 5.3

Let $\sum_{n=0}^{\infty} a_n = A$ be <u>absolutely convergent</u>,

and $\sum_{n=0}^{\infty} b_n = B$ be <u>absolutely convergent</u>.

Consider the Cauchy product:

$$\sum_{n=0}^{\infty} c_n \qquad c_n = \sum_{k=0}^{n} a_{n-k} b_k .$$

Then $\sum_{n=0}^{\infty} c_n = AB$ is also <u>absolutely convergent</u>

We immediately get t.f. Corollary:

Corollary 5.4. Consider power series:

$\sum_{n=0}^{\infty} a_n z^n$ with roc R_1

$\sum_{n=0}^{\infty} b_n z^n$ with roc R_2

The Cauchy product is the power series:

$$\sum_{n=0}^{\infty} c_n z^n \quad \text{with } c_n = \sum_{k=0}^{n} a_{n-k} b_k$$

and it has roc $\geq \min\{R_1, R_2\}$

//

the pf of 5.3 is a little "tough", I will do it later.

Application: the (complex) exponential function.

define:

$$E(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

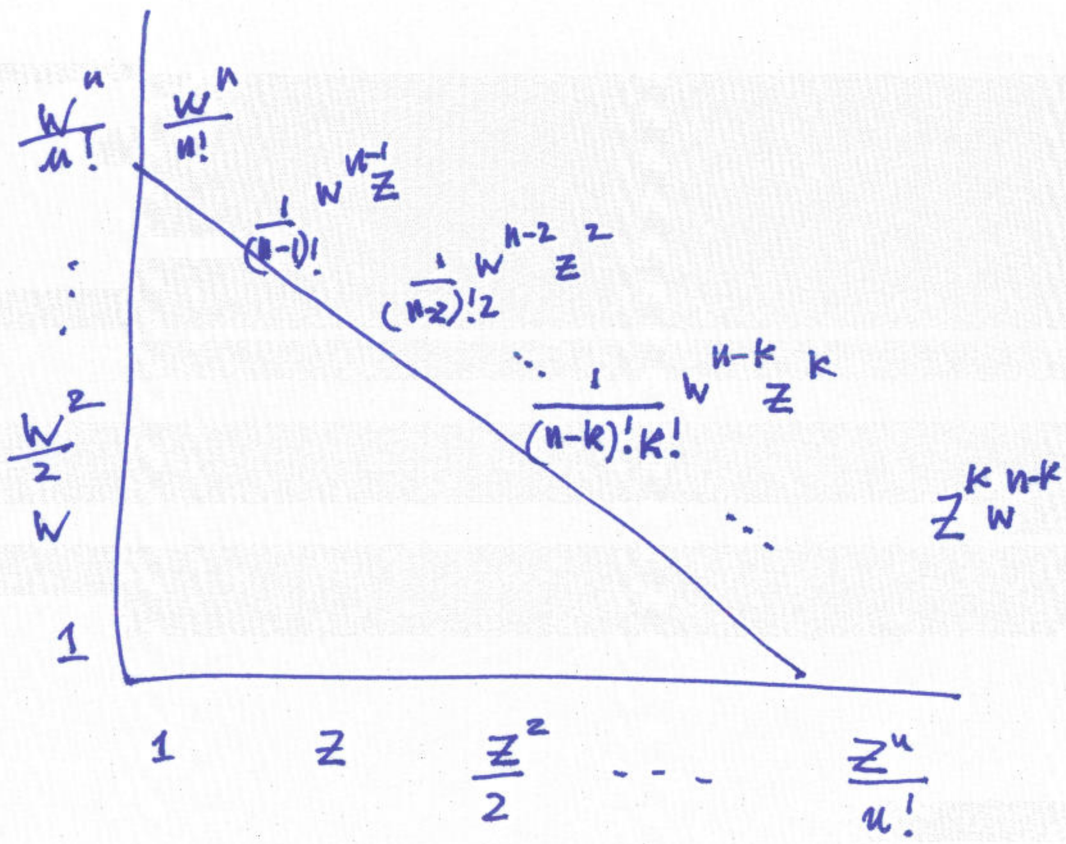
(we already decided long ago that roc $= \infty$)

(convention: $0! = 1$)

$$= 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \frac{z^5}{5!} + \text{hot}$$

Propos $E(z)\,E(w) = E(z+w)$

picture of Cauchy product:



Rmk since $E(0)=1$, then Propos implies that

$$E : \mathbb{C} \longrightarrow \mathbb{C}^\times$$

is a group homomorphism from

$\mathbb{C}, +$ to $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}, \times$

Pf of the proposition:

the Cauchy product of the series

$$E(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n ,$$

$$E(w) = \sum_{n=0}^{\infty} \frac{1}{n!} w^n \quad \text{is:}$$

$$E(z)E(w) = \sum_{n=0}^{\infty} c_n$$

where

$$c_n = \frac{w^n}{n!\,0!} + \frac{w^{n-1} z}{(n-1)!\,1!} + \cdots + \frac{w^{n-k} z^k}{(n-k)!\,k!} + \cdots + \frac{z^n}{n!}$$

$$= \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{(n-k)!\,k!} w^{n-k} z^k$$

$$= \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} w^{n-k} z^k$$

$$= \frac{1}{n!} (w+z)^n$$

So

$$E(z)E(w) = \sum_{n=0}^{\infty} \frac{(w+z)^n}{n!}$$

$$= E(z+w).$$

//

<u>define</u> $e := E(1)$

[END L: 19]

Last time we defined:

$$E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad \left[\begin{array}{l} \text{r.o.c} = \infty \\ E: \mathbb{C} \to \mathbb{C}^\times \end{array}\right.$$

we showed:

$$E(z+w) = E(z)E(w)$$

we <u>defined</u>:

$$e = E(1) = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{5!} + \cdots$$

<u>Proposition</u> For $r = \frac{p}{q} \in \mathbb{Q}$,

$$E(r) = e^r$$

(<u>note</u>: If $a > 0$, $r = \frac{p}{q} \in \mathbb{Q}$

$$a^r := \sqrt[q]{a^p}\ )$$

pf: $r = 0$ $\quad E(r) = E(0) = 1 = e^0$

$r > 0$ $\quad r = \frac{p}{q}$ $\quad p, q \in \mathbb{N}^\times$

$$E(p) = E(1)^p = e^p$$

$$E\left(\frac{p}{q}\right)^q = \underbrace{E\left(\frac{p}{q}\right) \times \cdots \times E\left(\frac{p}{q}\right)}_{q \text{ times}}$$

$$= E\Big(\underbrace{\frac{p}{q} + \frac{p}{q} + \cdots \frac{p}{q}}_{q \text{ times}}\Big) = E(p) = e^p$$

$$E\left(\frac{p}{q}\right) = \sqrt[q]{E\left(\frac{p}{q}\right)^q} = \sqrt[q]{E(p)} = \sqrt[q]{e^p} = e^r$$

$r<0$: note that

$$1 = E(0) = E(r-r)$$

$$= E(r) \cdot E(-r)$$

$$\begin{cases} E(r) = \dfrac{1}{E(-r)} \\ \text{similarly:} \\ e^r = \dfrac{1}{e^{-r}} \end{cases}$$

we showed: $E(-r) = e^{-r}$

$\Rightarrow E(r) = e^r$ //

Remark if $x \in \mathbb{R} \setminus \mathbb{Q}$ is irrational, we don't know (at this point) what to make of e^x.

def $e^x = E(x)$ (for $x \in \mathbb{R}$)

(for $x \in \mathbb{C}$)

we could do this

<u>Properties</u> of e^x, $x \in \mathbb{R}$:

E0 $\quad x \geq 0 \Rightarrow e^x \geq 1 \quad \left[x>0 \Rightarrow e^x > 1 \right]$

E1 $\quad x \in \mathbb{R} \Rightarrow e^x > 0$

E2 $\quad x \mapsto e^x$ is <u>strictly increasing</u>:

$$x<y \implies e^x < e^y$$

E3 If $|x|<1$, then

$$\left| e^x - 1 \right| < \frac{|x|}{1-|x|}$$

E4 $\quad x \mapsto e^x$ is <u>continuous</u>

(we will make sense of this shortly)

E5 $\quad e^{-} : \mathbb{R} \longrightarrow \mathbb{R}_{>0}$

is 1-to-1 & surjective.

<u>Note</u> I now prove E0 — E3.

I will prove E4, E5 later.

(when we know about continuous functions)

<u>pf of E0 — E3</u>

E0: easy:

$x \geq 0 \Rightarrow E(x)$

$$= \underset{\geq 0}{1} + \underset{\geq 0}{x} + \underset{\geq 0}{\frac{x^2}{2}} + \frac{x^3}{3!} + \cdots$$

$$\geq 1 + 0 + 0 + 0 + \cdots$$

$$= 1$$

pg 122/

E1

if $x \geq 0$ then $e^x \geq 1$

so ok.

if $x<0$ then $e^x = \frac{1}{e^{-x}} > 0$

(by E(0))

E2 say $x<y$:

$e^y = E(y) = E(x)\,E(y-x) > E(x) = e^x$

$\vee$

$y-x>0 \Rightarrow E(y-x)>1$

E3

$|e^x - 1| = \left| x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \right|$

$\leq |x| + \left|\frac{x^2}{2}\right| + \left|\frac{x^3}{3}\right| + \cdots$

$< |x| + |x|^2 + |x|^3 + \cdots$

since $|x|<1$ then this series converges and

$= \frac{|x|}{1-|x|}$

//

defn the log function:

$$\log : \mathbb{R}_{>0} \longrightarrow \mathbb{R}$$

is the <u>inverse</u> of e^{-}.

In other words:

$$y = \log x \iff e^y = x$$

<u>Properties</u> of $\log : \mathbb{R}_{>0} \to \mathbb{R}$:

L0 $e^{\log y} = y \qquad \log e^x = x$.

L1 $\log(xy) = \log x + \log y$

L2 log is <u>continuous</u>.

L3 $\cancel{}\, x < y \implies \ln x < \ln y$

these properties follow immediately from corresponding properties of e^x.

<u>The function $x \mapsto a^x$</u>

Here $a > 0$ is any real no.

<u>Note:</u> $r = \frac{p}{q} \in \mathbb{Q}$ then

$$a^r = e^{r \log a}$$

(you can do this as an exercise)

<u>definition</u> If $x \in \mathbb{R}$:

$$a^x = e^{x \log a}$$

Properties :

A1 : $a^{x+y} = a^x \, a^y$

A2 : $x \longmapsto a^x$ is continuous

$(\mathbb{R} \longmapsto \mathbb{R}_{>0})$

We could use the (complex) exponential to define all the trig functions & study their properties.

For instance:

for $\vartheta \in \mathbb{R}$ define

$$\left.\begin{aligned} \cos\vartheta &= \operatorname{Re} e^{i\vartheta} \\ \sin\vartheta &= \operatorname{Im} e^{i\vartheta} \end{aligned}\right\} \begin{array}{l} \text{where} \\ i \in \mathbb{C} \\ i^2 = -1 \end{array}$$

$$e^{i\vartheta} := \cos\vartheta + i\sin\vartheta.$$

e.g. $e^{i(\vartheta+\varphi)} = e^{i\vartheta} \cdot e^{i\varphi}$:

$$\cos(\vartheta+\varphi) + i\sin(\vartheta+\varphi)$$

$$(\cos\vartheta + i\sin\vartheta)(\cos\varphi + i\sin\varphi)$$

$$= (\cos\vartheta\cos\varphi - \sin\vartheta\sin\varphi) + i(\sin\vartheta\cos\varphi + \cos\vartheta\sin\varphi)$$

so we have :

$$\left.\begin{aligned} \cos(\vartheta+\varphi) &= \cos\vartheta\cos\varphi - \sin\vartheta\sin\varphi \\ \sin(\vartheta+\varphi) &= \sin\vartheta\cos\varphi + \cos\vartheta\sin\varphi \end{aligned}\right\}$$

the addition formulas of trig are here (from this perspective) proved as a consequence of :

$$E(z+w) = E(z)\cdot E(w).$$

[END L:20]

proof of Theorem 5.3

(Cauchy product of series)

I first prove $\sum_{n=0}^{\infty} c_n = AB$

(worry later about absolute convergence)

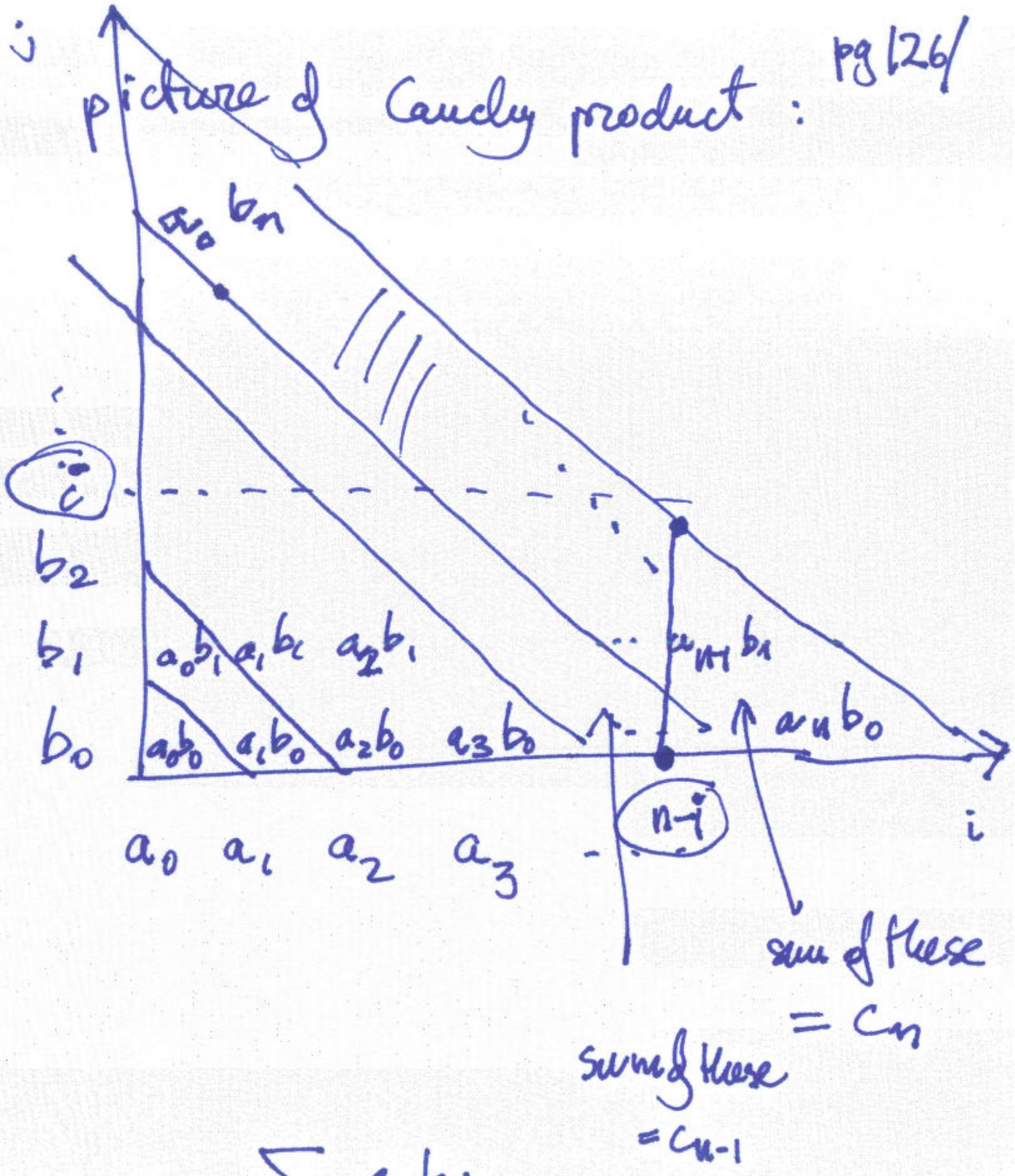
$s_n := a_0 + a_1 + \cdots + a_n$

$t_n := b_0 + b_1 + \cdots + b_n$

$u_n := c_0 + \cdots + c_n$

$$\overset{!}{=} \sum_{i=0}^{n} a_{n-i} \sum_{k=0}^{i} b_k$$

picture of Cauchy product:



$$u_n = \sum_{i,j \in \triangle} a_i b_j$$

Goal: to show $u_n \to AB$

$$u_n - AB = \sum_{i=0}^{n} a_{n-i} \underbrace{\sum_{k=0}^{i} b_k}_{t_i} - AB$$

$$= \sum_{i=0}^{n} a_{n-i} t_i - AB$$

$$- B\sum_{i=0}^{n} a_{n-i} + B\sum_{i=0}^{n} a_{n-i}$$

$$= \sum_{i=0}^{n} a_{n-i} (t_i - B) + \underbrace{B(s_n - A)}_{}$$

this term is problematic

this term is no prob.

BUT

$\begin{cases} i \text{ large} \Rightarrow t_i - B \text{ small.} \\ i \text{ small} \Rightarrow a_{n-i} \text{ is small.} \end{cases}$

Fix $\varepsilon > 0$.

Incantations

(I) $\exists N_1$ s.t. $n \geq N_1 \Rightarrow |s_n - A| < \frac{1}{3|B|}\varepsilon$

(II) $\exists N_2$ s.t. $i \geq N_2 \Rightarrow |t_i - B| < \dfrac{\varepsilon}{3\left(\sum_{n=0}^{\infty} |a_n|\right)}$

(III) $\exists N_3$ s.t. $k \geq N_3 \Rightarrow |a_k| < \dfrac{\varepsilon}{3\left(\sum_{i=0}^{N_2-1} |t_i - B|\right)}$

Claim : $n \geq N = \max\{N_1, N_2 + N_3\}$

$\implies |u_n - AB| < \varepsilon$

(this shows $u_n \to AB$, which is what we want).

Let us go back to $u_n - AB$

$$|u_n - AB| = \left|\sum_{i=0}^{n} a_{n-i}(t_i - B) + B(s_n - A)\right|$$

$$\leq |B|\,|s_n - A| + \quad (\mathrm{I})$$

$$+ \sum_{i=N_2}^{n} |t_i - B|\,|a_{n-i}| \quad (\mathrm{II})$$

$$+ \sum_{i=0}^{N_2-1} |t_i - B|\,|a_{n-i}| \quad (\mathrm{III})$$

Plan : to show :

$(\mathrm{I}) < \frac{\varepsilon}{3}$, $(\mathrm{II}) < \frac{\varepsilon}{3}$, $(\mathrm{III}) < \frac{\varepsilon}{3}$.

(I)

$$(\mathrm{I}) = |B|\,|s_n - A| \underset{(n \geq N_1)}{<} |B| \frac{\varepsilon}{3|B|} = \frac{\varepsilon}{3}$$

(II)

$$(\mathrm{II}) = \sum_{i=N_2}^{n} |t_i - B|\,|a_{n-i}|$$

$$\underset{(i \geq N_2)}{<} \frac{\varepsilon}{3\left(\sum_{n=0}^{\infty} |a_n|\right)} \sum_{i=N_2}^{n} |a_{n-i}|$$

$$\leq \frac{\varepsilon}{3\sum_{n=0}^{\infty} |a_n|} \sum_{n=0}^{\infty} |a_n| = \frac{\varepsilon}{3}$$

(III)

$$\sum_{i=0}^{N_2-1} |t_i - B|\,|a_{n-i}|$$

I can apply "incantation III" provided that $n-i \geq N_3$

We know $n \geq N_2 + N_3$

we are summing over $i \leq N_2 - 1$

$$\Rightarrow n-i \geq N_2 + N_3 - (N_2 - 1)$$
$$\geq N_3 + 1 \geq N_3 \quad !$$

I can indeed apply "incantation III":

$$\text{(III)} = \sum_{i=0}^{N_2-1} |t_i - B|\,|a_{n-i}|$$

$$\cancel{\leq \sum_{i=0}^{N_2-1} |t_i - B| \; \varepsilon}$$

$$< \frac{\varepsilon}{3\sum_{i=0}^{N_2-1} |t_i - B|} \sum_{i=0}^{N_2-1} |t_i - B|$$

$$= \frac{\varepsilon}{3}$$

We conclude:

$$|u_n - AB| = \text{(I)} + \text{(II)} + \text{(III)} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

//

What about absolute convergence?

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$$

$$|c_n| \leq |a_0||b_n| + |a_1||b_{n-1}| + \ldots |a_n||b_0|$$

$$= \gamma_n .$$

$\sum_{n=0}^{\infty} \gamma_n$ is the Cauchy product

of the series:

$$\sum_{n=0}^{\infty} |a_n| \quad \& \quad \sum_{n=0}^{\infty} |b_n|$$

We assume

$\sum |a_n|$, $\sum |b_n|$

converge. Therefore by what we've done:

$$\sum \gamma_n \quad \text{converges.}$$

but $0 \leq |c_n| \leq \gamma_n$

therefore, by the comparison test,

$$\sum |c_n| \quad \text{converges.}$$

That is, $\sum c_n$ absolutely converges

[END L.21]

§6 Limits & continuity of functions

Notation

$\Omega \subset \mathbb{R}^n$ a subset

we consider functions:

$f : \Omega \longrightarrow \mathbb{R}^m$

Most of the time & in examples:

$m = 1$, $\Omega = I \subset \mathbb{R}$, is an interval:

$I = (a,b)$, $[a,b]$, $(a,b] = \{t \in \mathbb{R} \mid a < t \leq b\}$

and $f : I \longrightarrow \mathbb{R}$.

$\boxed{\text{defn } 6.1}$ $\Omega \subset \mathbb{R}^n$ $f : \Omega \to \mathbb{R}^m$

Let $a \in \Omega$. We say that:

f is continuous at a if:

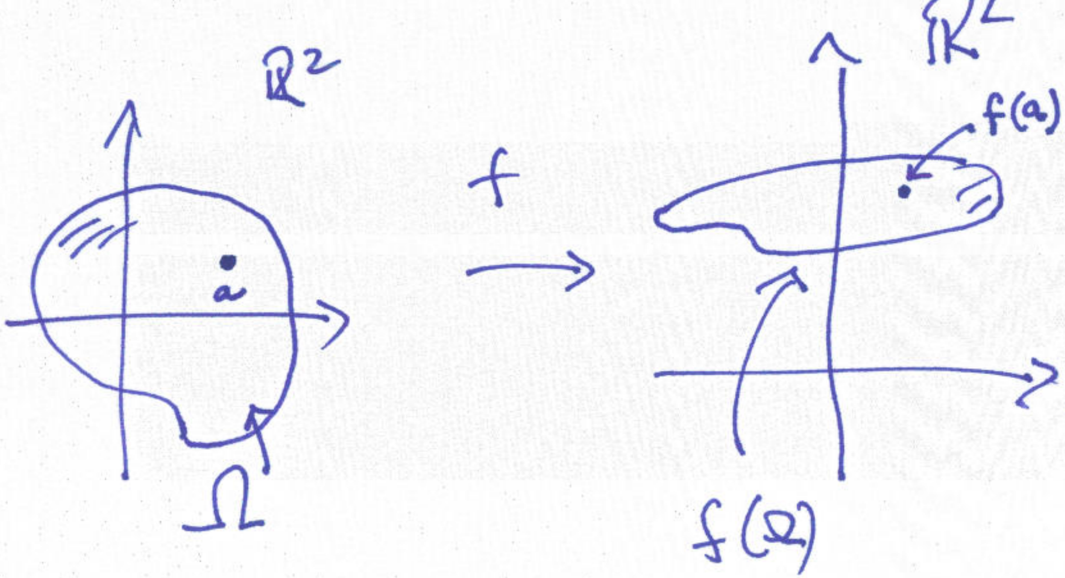
$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. if $x \in \Omega$:

$|x - a| < \delta \implies |f(x) - f(a)| < \varepsilon$

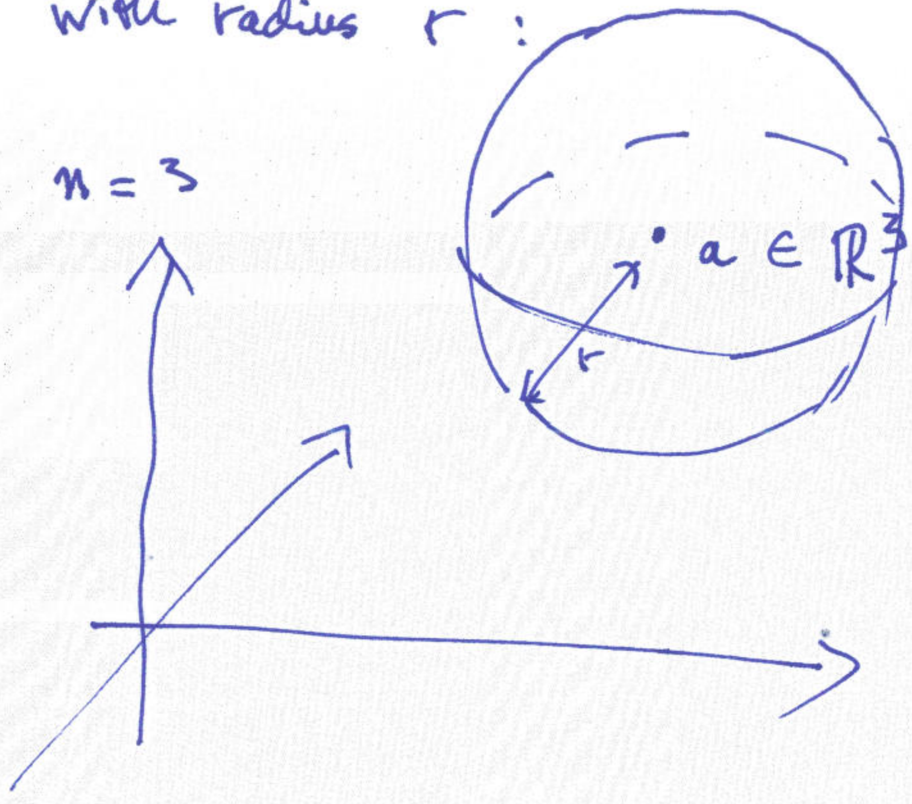
We say that f is continuous on Ω if f is continuous at a for all $a \in \Omega$.

A ramble tending to produce a mental picture of continuity in your minds.

A function from $\Omega \subset \mathbb{R}^2$ to $\mathbb{R}^2$

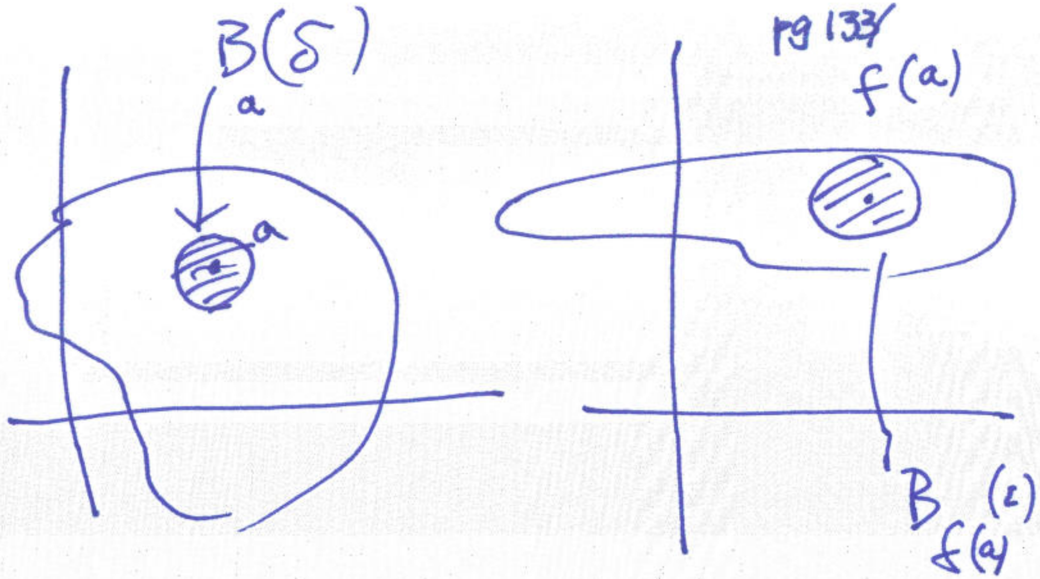


the open ball centred at $a \in \mathbb{R}^n$ with radius r:

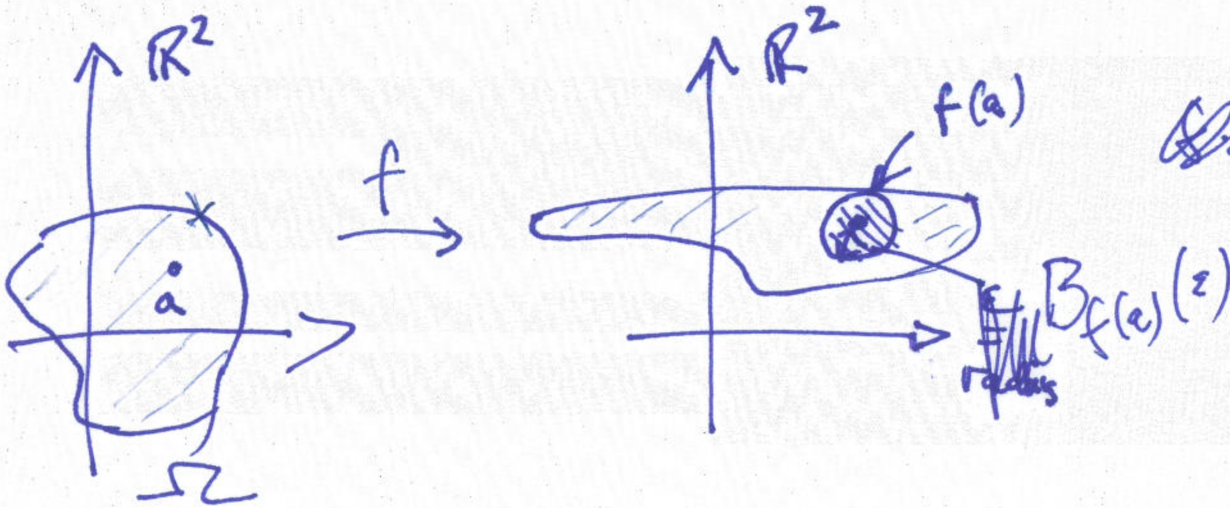


$$B_a(r) := \{ x \in \mathbb{R}^n \mid |x-a| < r \}$$

I re-state the definition of continuity.



$\forall \varepsilon > 0$ consider the ball $B_{f(a)}(\varepsilon)$:



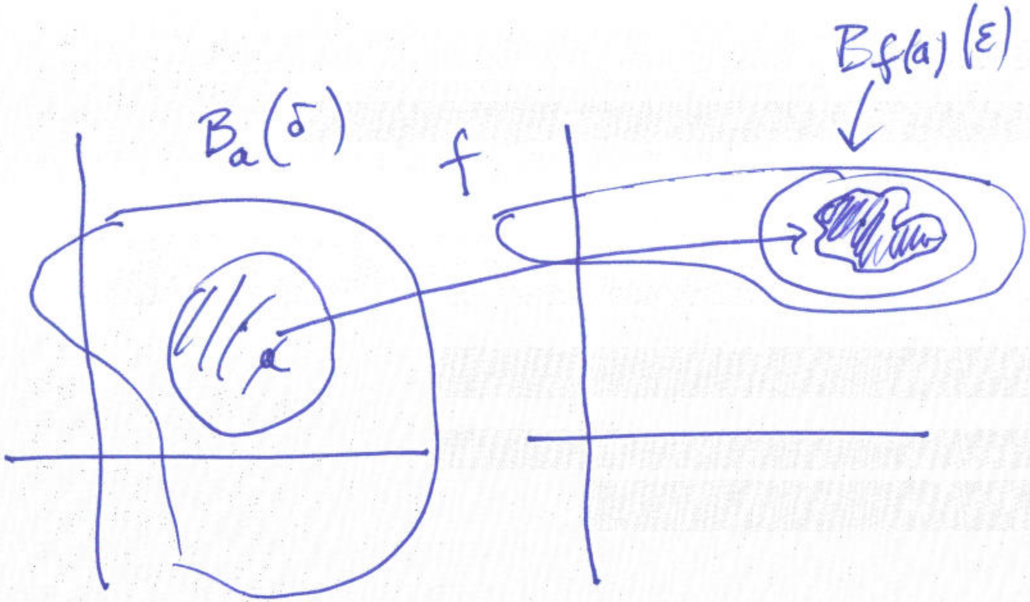
the definition says that there $\exists\, \delta$ s.that the ball $B_a(\delta)$:

such that : for $x \in \Omega$

$$x \in B_a(\delta) \Rightarrow f(x) \in B_{f(a)}(\varepsilon)$$

in other words:

$$f\left(B_a(\delta) \cap \Omega\right) \subset B_{f(a)}(\varepsilon)$$



<u>defn 6.2</u> $\Omega \subset \mathbb{R}^n$ $\quad f : \Omega \to \mathbb{R}^m$

Let $a \in \mathbb{R}^n$ (not necessarily in Ω)

We say that f tends to l

as x tends to a,

(write $l = \lim_{x \to a} f(x)$) if

$\forall\, \varepsilon > 0 \;\; \exists\, \delta > 0$ s.t. if $x \in \Omega$

$$0 < |x - a| < \delta \implies |f(x) - l| < \varepsilon$$

[<u>Note</u>: this is <u>very important</u>. We don't allow $x = a$ here. !!!]

<u>I re-state it again</u>:

$\forall$ balls B centred at $f(a)$

$\exists$ a ball B' centred at a

s.t. $f(B' \cap \Omega) \subset B$.

Translation in balls:

$\forall$ $B_l(\varepsilon)$ there $\exists$ $B_a(\delta)^* = B_a(\delta)\setminus\{a\}$

s.t. $f\left(B_a(\delta)^* \cap \Omega\right) \subset B_l(\varepsilon)$

Remark When $\lim_{x \to a} f(x) = l$, then

f is continuous at $a \in \Omega$ iff:

(1) $a \in \Omega$ i.e. f is defined at a, and

(2) $l = f(a)$.

Remark

In the definition of $\lim_{x\to a} f(x)$

(1) It makes no difference whether or not f is even defined at a

(2) Even if f were defined at a, the value $f(a)$ is utterly irrelevant to $\lim_{x\to a} f(x)$.

Examples 6.3

[END L:22]

(1) $f(x) = ax+b \quad f : \mathbb{R} \to \mathbb{R}$

(a, b constants)

(2) $\mathbb{R} \ni x \overset{f}{\mapsto} x^2 \in \mathbb{R}$

(3) $\mathbb{R} \setminus \{0\} \ni x \longrightarrow x \sin \frac{1}{x} \in \mathbb{R}$

(at $a = 0$)

(4) $\mathbb{R} \ni x \mapsto E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \in \mathbb{R}$

(5) $\mathbb{R} \ni x \mapsto a^x \quad (a > 0)$

(6) $f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$

(7) $f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

Example 6.3(1)

$f(x) = ax+b$ for $x \in \mathbb{R}$

(a, b are constants, $a \neq 0$.)

This f is continuous on $\mathbb{R}$

Let $x_0 \in \mathbb{R}$; I show f is continuous at x_0.

We study first:

$|f(x)-f(x_0)| < \varepsilon$, i.e.

$|ax+b-ax_0-b| = |a(x-x_0)|$

$= |a|\,|x-x_0| < \varepsilon$ OK if

$|x-x_0| < \dfrac{\varepsilon}{|a|}$.

Formal argument:

fix $\varepsilon > 0$.

Take $\delta = \dfrac{\varepsilon}{|a|}$; then:

$|x-x_0| < \delta$

$\Rightarrow |f(x)-f(x_0)| = |a|\,|x-x_0|$

$< |a|\,\delta = |a|\dfrac{\varepsilon}{|a|} = \varepsilon$

//

pg 138/

Example 6.3 (2)

$\mathbb{R} \ni x \xrightarrow{f} x^2 \in \mathbb{R}$

this function also is continuous at a for all $a \in \mathbb{R}$:

We study first:

$|f(x) - f(a)| < \varepsilon$ that is:

$$|x^2 - a^2| = |x-a||x+a| < \varepsilon$$

Note: If we make sure that $\boxed{\delta \leq 1}$ then:

$$|x+a| = |x-a+2a|$$
$$\leq |x-a| + 2|a| < \delta + 2|a|$$
$$\leq 2|a| + 1.$$

Then also:

$$|f(x) - f(a)| = |x-a||x+a|$$
$$< (2|a|+1)|x-a|$$
$$< \varepsilon$$

provided that $\boxed{\delta \leq \dfrac{\varepsilon}{2|a|+1}}$.

Formal argument:

fix $\varepsilon > 0$

Take $\delta = \min\left\{1, \dfrac{\varepsilon}{2|a|+1}\right\}$

Then:

$|x-a| < \delta \implies$

$$\begin{aligned}|f(x) - f(a)| &= |x+a|\,|x-a| \\ &\leq \big(|x-a| + |2a|\big)\big(|x-a|\big) \\ &< (\delta + 2|a|)\,\delta \\ &\leq (2|a|+1)\frac{\varepsilon}{2|a|+1} = \varepsilon\end{aligned}$$

Example 6.3(3)

$$\mathbb{R} \setminus \{0\} \ni x \xmapsto{f} x \sin\frac{1}{x} \in \mathbb{R}$$

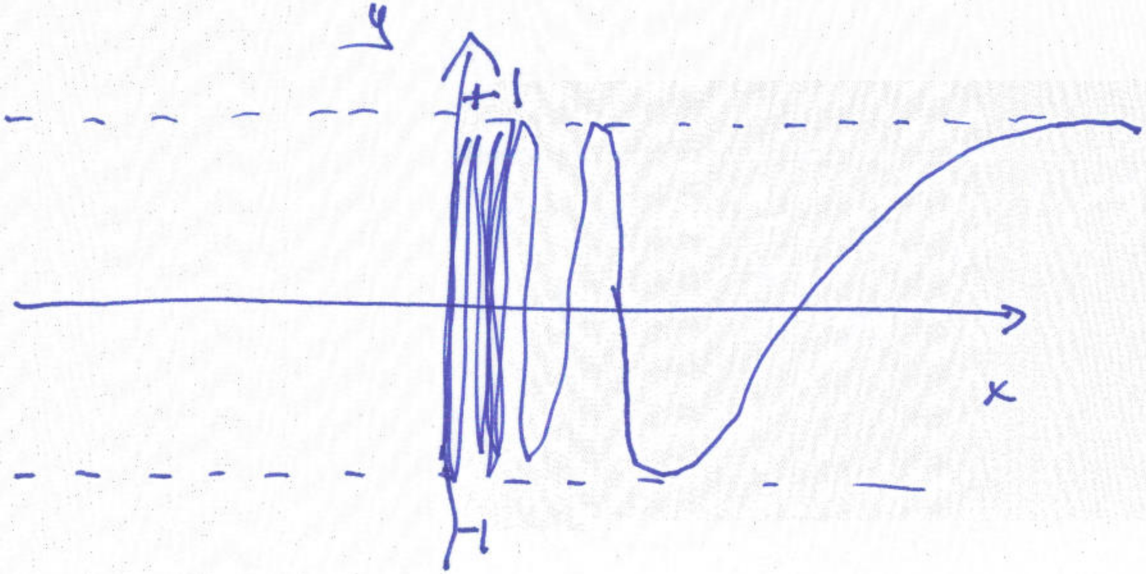
I will show $\lim\limits_{x \to 0} f(x) = 0$.

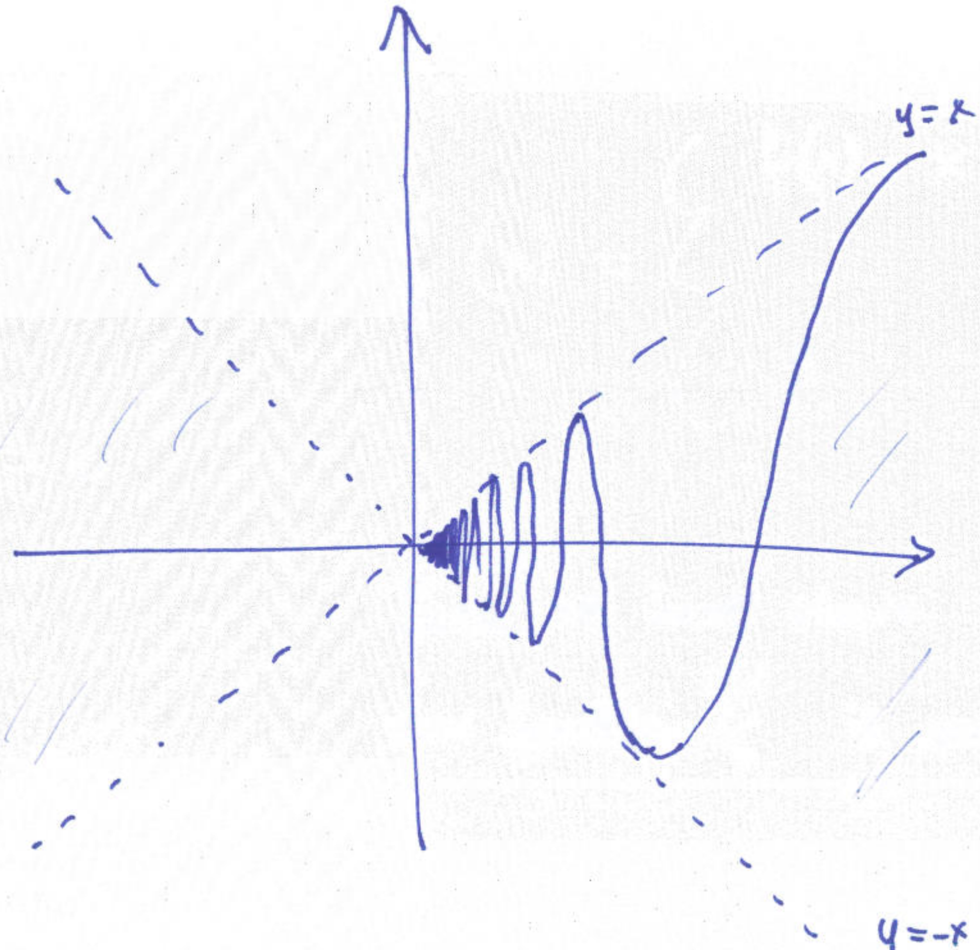
(It is true that f is continuous everywhere on $\Omega = \mathbb{R} - \{0\}$ but I won't discuss this now)

Pictures:

I am interested in $f(x) = x \sin\frac{1}{x}$

$y = \sin\frac{1}{x}$ for $x \in \mathbb{R} - \{0\}$





this function has <u>no limit</u>

as $x \longrightarrow 0$.

the picture suggests $\lim_{x \to 0} f(x) = 0$.

Formal argument :

fix $\varepsilon > 0$.

Take $\delta = \varepsilon$: if $0 < |x| < \delta$:

$$|f(x) - 0| = \left| x \sin\frac{1}{x} \right| \leq |x| < \delta = \varepsilon$$

'cause $|\sin \circledast| \leq 1$

//

At this point we are free to define a new function :

$$\tilde{f}(x) = \begin{cases} f(x) = x \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Then we have shown that this new function $\tilde{f}$ is continuous at 0.

Example 6.3 (4)

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for } x \in \mathbb{R}$$

This function is continuous on $\mathbb{R}$.

(this is E4 from last week)

step 1 : $E(x)$ is continuous at $x=0$.

Recall (E3) from last week

$|x| < 1 \implies$

$$\left| E(x) - E(0) \right| \leq \frac{|x|}{1-|x|} < \varepsilon$$

this means:

$$|x| < \varepsilon - |x|\varepsilon$$

$\iff$

$$|x|(1+\varepsilon) < \varepsilon$$

$\iff$

$$|x| < \frac{\varepsilon}{1+\varepsilon} \, ;$$

so:

fix $\varepsilon > 0$

take $\delta = \dfrac{\varepsilon}{1+\varepsilon}$ then

$|x| < \delta \implies$

$$\left| E(x) - E(0) \right| \leq \frac{|x|}{1-|x|} < \varepsilon$$

//

[END L:23]

Step 2 $E(x)$ is continuous at $x=a$ (for all $a \in \mathbb{R}$)

$$|E(x)-E(a)| = E(a)\,|(E(x-a)-E(0)|$$

fix $\varepsilon>0$. Since $E(x)$ is continuous at $x=0$, then $\exists\,\delta>0$:

$$|x-a|<\delta \Rightarrow |E(x-a)-E(0)| < \frac{\varepsilon}{E(a)}$$

But then:

$$|x-a|<\delta \Rightarrow |E(x)-E(a)| = E(a)\,|E(x-a)-E(0)| < E(a)\frac{\varepsilon}{E(a)} = \varepsilon \quad //$$

Theorem 6.4 $\Omega \subset \mathbb{R}^m$

$f: \Omega \to \mathbb{R}^n$

The following are equivalent (TFAE)

(1) $\lim_{x\to a} f(x) = l$

(2) For all sequences $x_n \in \Omega$ s.t.

(a) $\lim_{n\to\infty} x_n = a$,

(b) $\exists N$ s.t $n \geq N \Rightarrow x_n \neq a$,

we have

$$\lim_{n\to\infty} f(x_n) = l$$

Why do we do this ?

Because we can funnel (feed, pipe) all the theory of limits of sequences into the theory of limits of functions.

pf of Thm 6.4

$(1) \Rightarrow (2)$ this is easy.

Let x_n be a sequence with property (a) & (b).

I want to show $f(x_n) \to l$.

fix $\varepsilon > 0$. By (1), there $\exists\, \delta > 0$ s.t. : $x \in \Omega$

(*) $0 < |x-a| < \delta \Rightarrow |f(x) - l| < \varepsilon$

We know (a) that $x_n \to a$

so $\exists\, M_1 : n \geq M_1 \Rightarrow |x_n - a| < \delta$

let $M = \max\{M_1, N\}$

where N is as in (b). Now

$n \geq M \Rightarrow 0 < |x_n - a| < \delta$.

By (*) $\Rightarrow |f(x_n) - l| < \varepsilon$.

qed $(1) \Rightarrow (2)$

(2) $\Rightarrow$ (1)

It is best to prove the contrapositive :

not (1) $\Longrightarrow$ not (2)

That is, I prove:

Suppose $\lim_{x \to a} f(x) \neq l$,

then $\exists$ sequence $x_n \in \Omega$ (satisfying (a) & (b)) s.t. $\lim_{n \to \infty} f(x_n) \neq l$.

I first state explicitly assumption

$\lim_{x \to a} f(x) \neq l$.

Recall: $\lim_{x \to a} f(x) = l$ means:

$\forall \varepsilon > 0 \ \exists \delta$ s.t $0 < |x-a| < \delta \Rightarrow |f(x) - l| < \varepsilon$.

So our assumption $\lim_{x \to a} f(x) \neq l$ means:

$\exists \underset{\varepsilon_0}{\overset{\shortparallel}{\varepsilon}} > 0 \ \forall \delta \ \exists x_\delta$ $0 < |x_\delta - a| < \delta$,

and $|f(x_\delta) - l| > \varepsilon_0$

Fix ε_0 with this property, then

for $\delta = \frac{1}{n}$ $\exists x_n$ with $0 < |x_n - a| < \frac{1}{n}$

and $|f(x_n) - l| > \varepsilon_0$.

x_n is a sequence:

☆ It satisfies property (a), that is

$x_n \longrightarrow a$

(if $n \geq N \Rightarrow |x_n - a| < \frac{1}{n}$)

☆ It satisfies (b): indeed

for all n, $0 \neq |x_n - a|$ or (which is the same) for all n, $x_n \neq a$.

In addition:

$\exists \varepsilon = \varepsilon_0$ s.t. $|f(x_n) - l| > \varepsilon_0$.

This clearly implies $\lim_{n \to 0} f(x_n) \neq l$.

//

<u>Corollary 6.5</u> TFAE:

(1) f is continuous at a.

(2) for every sequence $x_n \in \Omega$

$x_n \to a \Rightarrow f(x_n) \to f(a)$.

(actually, to be precise, this is a Corollary of the proof of 6.4 not directly the statement of 6.4.

I'm saying that the proof of 6.4 can very easily be adapted to a proof of 6.5)

Example 6·3 (b)

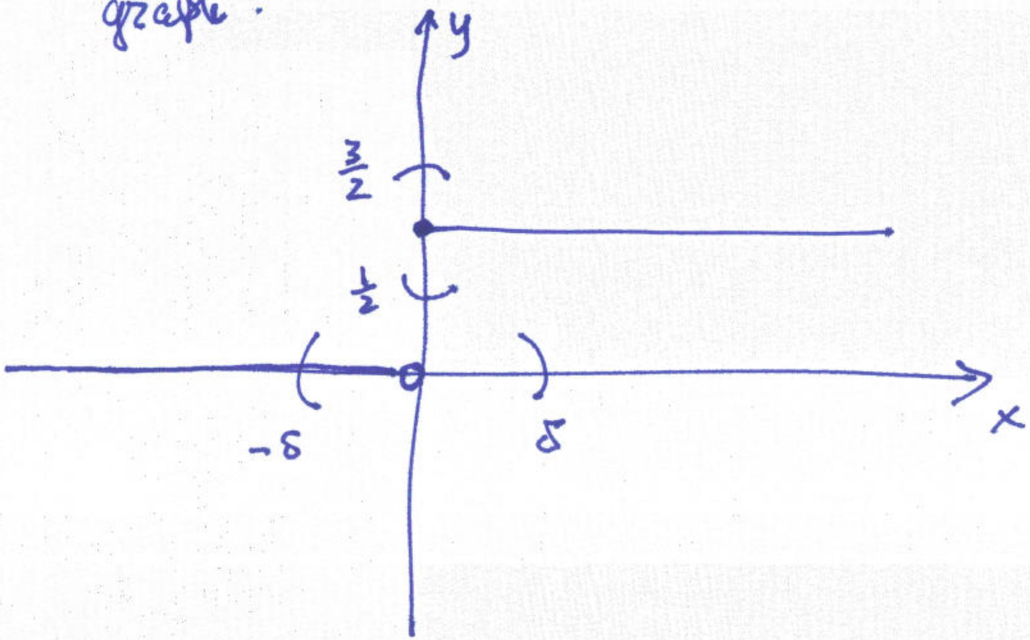
$$f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

(the Heaviside function)

[There is a "break" in the graph at $x=0$. This is not a proof.

In an exam situation this will get you 0 marks.]

graph:



this function is not continuous at $x=0$.

We could go back to the defn in terms of balls:

every $B_\delta(0)$ will contain some x with $x<0$

so $f(B_\delta(0)) \ni 0$.

it can never be contained in $B_{\frac{1}{2}}(f(0))$

This <u>is</u> a proof.

(Only slightly informally expressed).

Exercise re-state this more formally.

Here is another proof based on

Corollary 6.5:

I show: $\exists$ sequence $x_n \in \Omega$

s.t.

$x_n \to 0$ but $f(x_n) \not\to f(0)$.

just take $x_n = -\frac{1}{n} \to 0$

$f(x_n) = f\left(-\frac{1}{n}\right) = 0 \to 0 \neq 1 = f(0)$.

So $f(x_n)$ does not tend to $f(0)$.

[END L. 24]

<u>Example 6·3 (7)</u> The function:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

is <u>nowhere</u> <u>continuous</u>.

Indeed, if $a \in \mathbb{Q}$ consider

$$x_n = a + \frac{\sqrt{2}}{n} \notin \mathbb{Q}.$$

<u>Clearly</u> $x_n \to a$ and

$$f(x_n) = 0 \to 0 \neq 1 = f(a)$$

therefore, <u>by</u> Cor. 6·5, f is not continuous at a.

Consider now $a \in \mathbb{R} - \mathbb{Q}$.

<u>Recall:</u> If $\alpha \in \mathbb{R}$, then the <u>round down</u> $\lfloor \alpha \rfloor \in \mathbb{Z}$ is the unique integer s.t.

$$\lfloor \alpha \rfloor \leq \alpha < \lfloor \alpha \rfloor + 1$$

Define:

$$x_n = \frac{\lfloor a n \rfloor}{n} \in \mathbb{Q}$$

clearly: $x_n \leq a < x_n + \frac{1}{n}$

So $x_n \to a$, but:

$$f(x_n) = 1 \to 1 \neq 0 = f(a)$$

so again by 6·5 f is <u>not</u> continuous at a.

Thm 6.6 $\Omega \subset \mathbb{R}^n$

$f, g : \Omega \to \mathbb{R}^m$. Assume:

$$\lim_{x \to a} f(x) = \ell$$

$$\lim_{x \to a} g(x) = m$$

then

(1) If $\lambda, \mu \in \mathbb{R}$ then

$$\lim_{x \to a} (\lambda f + \mu g)(x) = \lambda \ell + \mu m$$

(2) $$\lim_{x \to a} fg \; (x) = \ell m$$

(3) assume $g(x) \neq 0$ for $x \in \Omega$ and $m \neq 0$. Then

$$\lim_{x \to a} \frac{f}{g}(x) = \frac{\ell}{m}$$

(4) $$\lim_{x \to a} |f(x)| = |\ell|$$

Proof: nothing to do: use 6.4 and corresponding properties of limits of sequences.

Cor. 6.7 $\Omega \subset \mathbb{R}^n$ $f, g : \Omega \to \mathbb{R}^m$

Suppose f, g are continuous at a. Then

(1) $\lambda, \mu \in \mathbb{R} \Rightarrow \lambda f + \mu g$ is continuous at a

(2) $f \cdot g$ is continuous at a

(3) $g \neq 0$ on $\Omega \Rightarrow f/g$ continuous at a

(4) $|f|$ is continuous at a.

Remark: All polynomial functions

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n$$

are continuous everywhere on $\mathbb{R}$.

All rational functions:

$$f(x) = \frac{P(x)}{Q(x)} \qquad (P, Q \text{ are polynomials})$$

are continuous on $\mathbb{R} - \{x \mid Q(x) = 0\}$

Theorem 6.8 Assume:

$$\lim_{x \to a} g(x) = l \quad \text{AND}$$

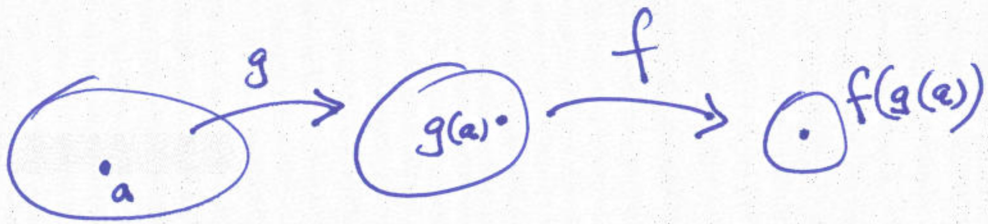
$f(x)$ continuous at l.

Then:

$$\lim_{x \to a} f(g(x)) = f(l)$$

Corollary 6.9

g continuous at a, f continuous at $g(a)$ $\Rightarrow$ $f \circ g$ cont. at a.



Pf of Thm 6.8 Use 6.4:

pick a sequence x_n such that:

(a) $\exists N : n \geq N \Rightarrow x_n \neq a$

(b) $x_n \longrightarrow a$.

then we want to show that

$$\lim_{n\to\infty} f(g(x_n)) = f(\ell).$$

Fix $\varepsilon > 0$.

We know f continuous at ℓ:

$\exists \delta$ s.t. $|y-\ell| < \delta$
$\Rightarrow |f(y) - f(\ell)| < \varepsilon$

We also assume that

$\lim_{x\to a} g(x) = \ell$. Hence by 6.4

again:

$$\lim_{n\to\infty} g(x_n) = \ell.$$

So $\exists N$ s.t.

$$n \geq N \Rightarrow |g(x_n) - \ell| < \delta.$$

$$\Rightarrow |f(g(x_n)) - f(\ell)| < \varepsilon.$$

That is, what we need to show.

[End of L: 25]

If $a>0$:

Example $\forall\ x \mapsto a^x$ is continuous on $\mathbb{R}$.

Indeed assume (see end of the course) that $\ln a$ makes sense. Then:

$$a^x = E\left[x \ln a\right]$$

we know:

$x \mapsto x \ln a$ continuous on $\mathbb{R}$

$y \mapsto E(y)$ continuous on $\mathbb{R}$

hence by 6·8:

$\left[x \mapsto a^x\right] = E(x \ln a)$ continuous on $\mathbb{R}$.

Everything we did works for $\Omega \subset \mathbb{C}$ and complex-valued $f : \Omega \to \mathbb{C}$.

<u>Example</u> $f, g : \Omega \to \mathbb{C}$.

f, g continuous at $a \in \Omega$

$\Rightarrow$ $f \cdot g$ also continuous at a.

pf. suppose $z_n \in \Omega$, $z_n \to a$

we want to show:

$$\lim_{n \to \infty} f(z_n) g(z_n) = f(a) g(a)$$

write
$$f(z) = f_1(z) + i f_2(z)$$
$$g(z) = g_1(z) + i g_2(z)$$

with f_1, f_2 $g_1, g_2 : \Omega \to \mathbb{R}$.

Then:

$$f(z) g(z) = f_1(z) g_1(z) - f_2(z) g_2(z) + \left(f_1(z) g_2(z) + f_2(z) g_1(z) \right) i$$

We already know that to show:

$$f(z_n) g(z_n) \longrightarrow f(a) g(a)$$

is the <u>same</u> as to show:

(I)

$$f_1(z_n)g_1(z_n) - f_2(z_n)g_2(z_n) \longrightarrow f_1(a)g_1(a) - f_2(a)g_2(a)$$

AND:

(II)

$$f_1(z_n)g_2(z_n) + f_2(z_n)g_1(z_n) \longrightarrow f_1(a)g_2(a) + f_2(a)g_1(a)$$

We now use them about real-valued sequences. We know:

$f_1(z_n) \to f_1(a)$ $\quad g_1(z_n) \to g_1(a)$

$f_2(z_n) \to f_2(a)$ $\quad g_2(z_n) \to g_2(a)$

this by assumption that f, g are continuous at a.

So (I) & (II) now follow from results on sequences of real numbers //

Corollary If $f: \mathbb{C} \to \mathbb{C}$ is a complex polynomial:

$$f(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$$

$$(a_0, a_1, \dots, a_n \in \mathbb{C}).$$

Then f is continuous everywhere on $\mathbb{C}$.

A lot of what we said also works for functions with values in $\mathbb{R}^m$.

<u>Exercise</u> $\Omega \subset \mathbb{R}^n$ $f : \Omega \to \mathbb{R}^m$

so I can write:

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{pmatrix} \in \mathbb{R}^m$$

where $f_i : \Omega \to \mathbb{R}$ is <u>real valued</u>

Show that f is continuous at a

$\iff$ f_i is continuous at a

$\forall\ i = 1, 2, \dots, m$.

I do the exercise.

The key point:

$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} \in \mathbb{R}^m$$

$$\sup\{|y_1|, \dots, |y_m|\}$$

$$\leq \sqrt{y_1^2 + y_2^2 + \dots y_m^2} = |y|$$

$$\leq \sqrt{m}\ \sup\{|y_1|, \dots |y_m|\}.$$

Suppose f is continuous at a

I show f_i is also cont. at a.

fix $\varepsilon > 0$.

$\exists \delta$ s.t. $|x-a|<\delta \Rightarrow |f(x)-f(a)| < \varepsilon.$

But then also:

$$|f_i(x)-f_i(a)| \leq \sup\{|f_1(x)-f_1(a)|, \ldots |f_m(x)-f_m(a)|\} \leq |f(x)-f(a)| < \varepsilon$$

I'm done.

S'pose $f_1, \ldots, f_m$ are all continuous at a.

I show f is continuous at a.

fix $\varepsilon > 0$.

$\exists \delta_1$:
$|x-a|<\delta_1 \Rightarrow |f_1(x)-f_1(a)| < \frac{\varepsilon}{\sqrt{m}}$

$\exists \delta_2$:
$|x-a|<\delta_2 \Rightarrow |f_2(x)-f_2(a)| < \frac{\varepsilon}{\sqrt{m}}$

$\vdots$

$\exists \delta_m$:
$|x-a|<\delta_m \Rightarrow |f_m(x)-f_m(a)| < \frac{\varepsilon}{\sqrt{m}}$

If $\delta = \min\{\delta_1, \delta_2, \dots, \delta_m\}$

$|x-a|<\delta$, then

$$\sup\{|f_1(x) - f_1(a)|, \dots, |f_m(x)-f_m(a)|\} < \frac{\varepsilon}{\sqrt{m}}.$$

But then:

$$|f(x)-f(a)|$$

$$\le \sqrt{m} \sup \left(\text{stuff above} \right)$$

$$< \sqrt{m} \frac{\varepsilon}{\sqrt{m}} = \varepsilon$$

and we are done.

Example consider

$$f(x_1, x_2) = \left(x_1 x_2, \frac{1}{x_2^4+1}, \frac{x_2}{x_1} \right)$$

$$f : \Omega = \mathbb{R}^2 - \{x_1 = 0\} \longrightarrow \mathbb{R}^3.$$

This function everywhere on Ω.

because the functions:

$$f_1(x_1, x_2) = x_1 x_2$$

$$f_2(x_1, x_2) = \frac{1}{x_2^4+1}$$

$$f_3(x_1, x_2) = \frac{x_2}{x}$$

are all continuous everywhere on Ω.

One-sided continuity

defn 6.10 $\Omega \subset \mathbb{R}$

$f : \Omega \to \mathbb{R}^m$

f is right-continuous at $a \in \Omega$ if:

$\forall \varepsilon > 0 \quad \exists \delta > 0$ s.t:

$a \leq x < a + \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

f is left continuous at $a \in \Omega$ if:

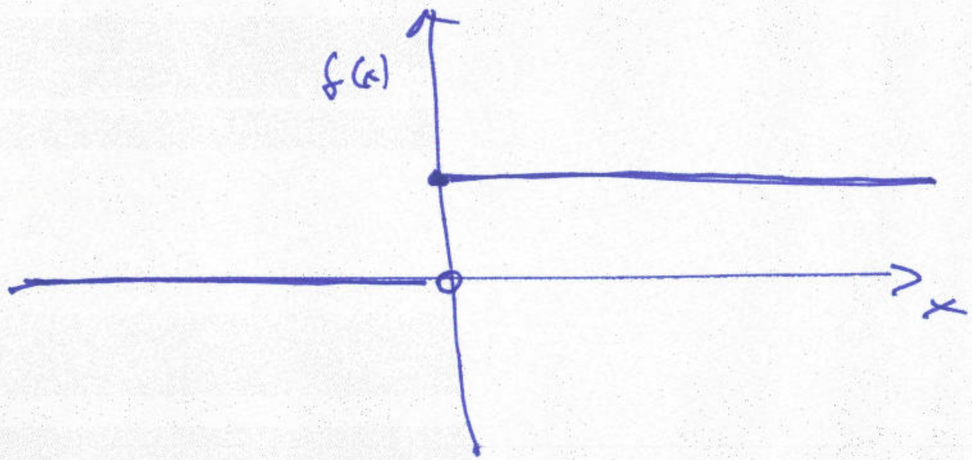
$\forall \varepsilon > 0 \quad \exists \delta > 0$ s.t.

$a - \delta < x \leq a \Rightarrow |f(x) - f(a)| < \varepsilon$.

Remark f continuous at a $\Leftrightarrow$ f r-continuous & l-continuous.

Example (Heaviside function)

$$f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$



f is right continuous at $a = 0$

f is not left continuous at $a = 0$

def 6.11

I say that the right-limit of f as x tends to a from the right is l; written:

$$\lim_{x \to a^+} f(x) = l$$

if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t $x \in \Omega$:

$$a < x < a + \delta \Rightarrow |f(x) - l| < \varepsilon$$

Similarly $\lim_{x \to a^-} f(x) = l$ if

$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t $x \in \Omega$:

$$a - \delta < x < a \Rightarrow |f(x) - l| < \varepsilon$$

Note

$$\lim_{x \to a} f(x) = l \iff$$

$$\lim_{x \to a^+} f(x) = l \ \& \ \lim_{x \to a^-} f(x) = l.$$

Example (Heaviside)

$$\lim_{x \to 0^+} f(x) = 1$$

$$\lim_{x \to 0^-} f(x) = 0$$

this is another way to see f has no limit –

[End of L 26]