Basic Algebraic Geometry
Example Sheet 1

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(1) The "linear system" (=set, if you like) of (projective) plane conics passing through 4 points has dimension 1 unless the 4 points all lie on a line.

(2) Let C, D be plane cubics intersecting in 9 distinct points. Assume that 3 of these points lie on a line L. Conclude that the remaining 6 points of intersection lie on a conic [Hint: 1 curve in the linear system $\lambda C + \mu D$ contains the line L].

(3) Do problem 2.12 on pages 41–42 of Reid's UAG, following the hints given there.

(4) In this problem I help you to revise the (19th century) proof of Hilbert Nullstellensatz, using the resultant, as I discussed it in class. While not as "neat" as the modern algebraic approach, this proof is much easier to understand.

Let R be a UFD and $g_1, ..., g_r \in R[X]$ polynomials with coefficients in R. Let

$$\tilde{R} = R[\lambda_1, ..., \lambda_r; \mu_1, ..., \mu_r]$$

be the UFD obtained adjoining the variables λ_i, μ_i to R. Consider $f = \sum \lambda_i g_i$ and $g = \sum \mu_i g_i$; think of them as polynomials in $\tilde{R}[X]$. We can write the resultant $r(f, g) \in \tilde{R}[X]$ in the form

$$r(f, g) = \sum_{I,J} \alpha_{I,J} \lambda^I \mu^J$$

over multi-indices I, J, and $\alpha_{I,J} \in R$. We define the *resultant system* of the g_is to be the set $\{\alpha_{I,J}\}$.

(a) There is a $\varphi \in R[X]$ dividing all the g_is, if and only if all $\alpha_{I,J} = 0$.

(b) Show that $(\alpha_{I,J}) \subset (g_1, ..., g_r) \cap R$, that is, every $\alpha_{I,J}$ belongs to the ideal of $R[X]$ generated by the g_is.

(c) Show by example that the inclusion in (b) is strict.

(d) Let K be an infinite field and $f \in K[X_1, ..., X_n]$ be a polynomial of degree d. Show that, after a linear change in the variables $X_1, ..., X_n$, we may assume that $X_n^d \in f$, that is, the monomial X_n^d appears in f with nonzero coefficient. [Hint: this is in the proof of Noether normalization]

(e) Use the resultant system to prove the weak form of Hilbert Nullstellensatz, that is if $g_1, ..., g_r \in K[X_1, ..., X_n]$ have no common zeros, they generate the unit ideal. [Hint: proceed by induction on the number n of variables, use (d) to put your polynomials in a favorable shape—convince yourself that this really is necessary]

(5) Let A be a UFD, K its field of fractions (more generally you could assume that A is an integral domain which is integrally closed in its field of fractions). Let $K \subset L$ be an algebraic extension of fields. Then if $b \in L$ is integral over A, the norm $N_K^L(b)$ is an element of A. [Note: we proved this, and used it repeatedly, in the proof of Krull's theorem in dimension theory.]

(6) Do problem 4.11 on page 78 of Reid's UAG, following the hints given there.

(7) Prove that $\mathbb{A}^2 \setminus \{0, 0\}$ is not (isomorphic to) an affine variety [this is problem 4.12 on page 78 of Reid's UAG].

(8) Let X, Y be topological spaces and $f : X \to Y$ a continuous map.
If $\mathcal{F}$ is a sheaf on X, define the sheaf $f_\bullet \mathcal{F}$ on Y.
Similarly, for a sheaf $\mathcal{G}$ on Y, define the sheaf $f^\bullet \mathcal{G}$ on X.
Prove the formula:

$$\mathrm{Hom}_X(f^\bullet \mathcal{G}, \mathcal{F}) = \mathrm{Hom}_Y(\mathcal{G}, f_\bullet \mathcal{F})$$

(9) Let $(X, \mathcal{O}_X)$ be an algebraic prevariety. Recall that an open subprevariety of X is a (Zariski) open subset $U \subset X$ with the sheaf of functions:

$$\mathcal{O}_U := \mathcal{O}_X | U$$

[recall that, for any sheaf $\mathcal{F}$ on X, and denoting $j : U \hookrightarrow X$ the inclusion, we use the notation

$$\mathcal{F}|U := j^\bullet \mathcal{F}$$

We justify the change of notation on the grounds that $j^\bullet$, for the inclusion $j : U \hookrightarrow X$ of an open subset, is much easier than $f^\bullet$ for an arbitrary continuous map f.]

Given a finite collection X_α of algebraic prevarieties, and open subprevarieties $X_{\alpha\beta} \subset X_\alpha$ and *isomorphisms*:

$$\psi_{\alpha\beta} : X_{\alpha\beta} \to X_{\beta\alpha}$$

satisfying

$$\psi_{\alpha\gamma} = \psi_{\beta\gamma} \circ \psi_{\alpha\beta}$$

(whenever both sides are defined), construct an algebraic prevariety gluing the X_α. Check that the ensuing object is an algebraic prevariety as pedantically as you can at the same time using up no more than 5 handwritten pages.

(10) Let X be an algebraic variety, U and V open subvarieties. Assume that U and V are affine (i.e., isomorphic to affine varieties). Prove that $U \cap V$ is also affine [hint: if $i : U \subset X$, $j : V \subset X$ are the inclusions and $(i, j) : U \times V \to X \times X$ is their product, $U \cap V = (i, j)^{-1}\Delta$]. Show by example that the statement is wrong if X is a prevariety.

(11) Prove that the product of 2 projective varieties is again a projective variety [hint: it is enough to prove that $\mathbb{P}^n \times \mathbb{P}^m$ is a projective variety. Think of mapping $\mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^{nm+n+m}$ via $(x_i; y_j) \to (x_i y_j)$. This is problem 5.11 on page 92 of Reid's UAG; you can get some hints there as well]. Conclude that a projective "variety" is a variety (that is, show that it is separated).

(12) Do problem 5.12 on pages 92–93 of Reid's UAG, following the hints given there.

(13) Do problem 5.12 on pages 92–93 of Reid's UAG, following the hints given there. When you are done with it, remember that projective varieties are proper and appreciate.