# Basic Algebraic Geometry
# Example Sheet 2

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**(1)** A projective variety $X \subset \mathbb{P}^n$ is said to be a *local complete intersection* if, for every point $x \in X$, there is an affine neighbourhood $x \in U \subset \mathbb{P}^n$ such that the ideal of $X \cap U$ in $k[U]$ is generated by $n - \dim X$ equations. Prove that nonsingular implies local complete intersection. [Hint: use the exact sequence

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega^1_{\mathbb{P}^n|X} \to \Omega^1_X \to 0]$$

**(2)** Let $C \subset \mathbb{P}^3$ be the twisted cubic curve, that is, the image of $\mathbb{P}^1$ under the morphism

$$(x_0, x_1, x_2, x_3) = (t_0^3, t_0^2 t_1, t_0 t_1^2, t_1^3)$$

The purpose of the next three questions is to outline three different proofs that the homogeneous ideal of $C$ is generated by the tree obvious quadratic equations

$$(x_0 x_2 - x_1^2, x_0 x_3 - x_1 x_2, x_1 x_3 - x_2^2)$$

(a) $C$ lies on the quadric

$$Q = \{x_0 x_3 - x_1 x_2 = 0\}$$

This is $\mathbb{P}^1 \times \mathbb{P}^1$ with coordinates $(u_0, u_1; v_0, v_1)$; in terms of these coordinates the embedding can be given as

$$(x_0, x_1, x_2, x_3) = (u_0 v_0, u_0 v_1, u_1 v_0, u_1 v_1)$$

Show that, in general, algebraic curves on $Q$ are given by a single bi-homogeneous equation $F = 0$, that is $F = F(u_0, u_1, v_0, v_1)$ is homogeneous of degrees $d_1, d_2$ in each of the two sets of variables $(u_0, u_1)$ and $(v_0, v_1)$.

(b) The rational normal curve $C \subset Q$ is given by

$$(u_0, u_1, v_0, v_1) = (t_0^2, t_1^2, t_0, t_1)$$

and the bi-homogeneoous equation of $C$ is

$$C = \{F = u_0 v_1^2 - u_1 v_0^2 = 0\}$$

By what we said in the previous paragraph, the ideal $I(C, R)$ of $C$ in $R$ is generated by $F$. Now the image of the restriction

$$S \to R$$

is the subring $R^{\#} = \oplus R_{(d,d)}$. It is almost immediate that $I(C, R) \cap R^{\#}$ is generated by $u_0 F$ and $u_1 F$, the restrictions of $x_1^2 - x_0 x_2$ and $x_1 x_3 - x_2^2$.

**(3)** Same notation as the previous question. Order the monomials in $S = k[x_0, ..., x_3]$ lexicographically, that is, for example

$$x_0^{a_0} x_1^{a_1} \cdots > x_0^{b_0} x_1^{b_1} \cdots$$

if $a_0 > b_0$, or $a_0 = b_0$ and $a_1 > b_1$, etc. If $f \in S$, denote $\operatorname{lm} f$ the leading monomial of $f$.

(a) Given polynomials $g$ and $f_1, ..., f_k$, there are polynomials $h_i$ and $r$ (not necessarily unique) such that

$$g = r + \sum h_i f_i$$

and if $m \in r$ is a monomial of $r$, no leading monomial $\operatorname{lm} f_i$ divides $r$ (the "division" algorithm for polynomials of several variables).

(b) From now on we work with $f_1 = x_0 x_2 - x_1^2$, $f_2 = x_0 x_3 - x_1 x_2$, $f_3 = x_1 x_3 - x_2^2$; denote $m_1 = x_0 x_2$, $m_2 = x_0 x_3$, $m_3 = x_1 x_3$ the leading monomials. A degree $d$ monomial $m \in S_d$ is not divisible by any of the $m_i$ if and only if $m$ is one of the following monomials

(1) $m = x_0^a x_1^{d-a}$ with $a > 0$, *or*

(2) $m = x_1^a x_2^{d-a}$ with $a > 0$, *or*

(3) $m = x_2^a x_3^{d-a}$.

Note that there are $3d+1$ such monomials.

(c) Let $g \in S_d$ be a polynomial of degree $d$. By the previous two paragraphs, $g$ is congruent modulo $I = (f_1, f_2, f_3)$ to a polynomial $g_0$ which is the sum of monomials of type (1–3) in paragraph (2). This implies almost immediately that the ideal $I$ is a prime ideal but let me go through in more detail. Denote $R = R(C, \mathcal{O}(1))$ the homogeneous coordinate ring of $C$. We know that $S/I$ surjects to $R$, and $R_d = k[t_0, t_1]_{3d}$. We have just shown that $\dim_k (S/I)_d \leq 3d+1 = \dim R_d$, hence these dimensions must be equal and $S/I = R$.

To make all this even clearer, notice that, upon plugging $x_0 = t_0^3$, $x_1 = t_0^2 t_1$, etc., the monomials in paragraph (2) evaluate to $t_0^{3d}, t_0^{3d-1}t_1, ..., t_1^{3d}$

**(4)** Notation again as in the previous two questions.

(a) Show that the complex

$$0 \to 2\mathcal{O}_{\mathbb{P}^3}(-3) \xrightarrow{A} \mathcal{O}_{\mathbb{P}^3}(-2) \to \mathcal{I}_C \to 0$$

is exact, where

$$A = \begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}$$

Indeed this is a local statement. Consider for example the restriction to the affine open subset $\mathbb{A}^3 = \{x_0 = 1\}$, with affine coordinates $x = x_1/x_0, y = x_2/x_0, z = x_3/x_0$, set $C^0 = C \cap \mathbb{A}^3$, and $I^0$ the ideal of $C^0$. Then

$$I^0 = (x_2 - x_1^2, x_3 - x_1 x_2)$$

is a complete intersection. The complex in question splits the Koszul complex of the complete intersection: indeed the matrix

$$A^0 = \begin{pmatrix} 1 & x \\ x & y \\ y & z \end{pmatrix}$$

is equivalent, under column and row operations over $k[x, y, z]$, to

$$\begin{pmatrix} 1 & 0 \\ 0 & y - x^2 \\ 0 & z - xy \end{pmatrix}$$

Exactness of the complex then follows from exactness of the Koszul complex. Prove carefully that the Koszul complex is exact in this situation.

(b) From (1) we have a resolution

$$0 \to 2\mathcal{O}_{\mathbb{P}^3}(-3) \xrightarrow{A} 3\mathcal{O}_{\mathbb{P}^3}(-2) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_C \to 0$$

Breaking into two short exact sequences, and using $H^1(\mathbb{P}^3, \mathcal{O}(n)) = H^2(\mathbb{P}^3, \mathcal{O}(n)) = (0)$ for all $n$ (basic result of Serre on the cohomology of projective space), and long exact sequences of cohomology groups, show that the sequence stays exact upon taking global sections (and summing over $n$):

$$0 \to 2S(-3) \xrightarrow{A} 3S(-2) \to S \to R(C, \mathcal{O}(1)) \to 0$$

Here, of course, $R(C, \mathcal{O}(1)) = \oplus H^0(C, \mathcal{O}(1))$ denotes the homogeneous coordinate ring of $C$.

**(5)** Let $C \subset \mathbb{P}^4$ be the rational normal curve in $\mathbb{P}^4$, that is, the image of $\mathbb{P}^1$ under the morphism

$$(x_0, x_1, ...) = (t_0^4, t_0^3 t_1, ...)$$

Show that there are four quadratic equations $(q_1, ..., q_4)$ such that the ideal $I$ of $C$ is everywhere locally generated by $q_1, ..., q_4$ but $I$ needs six generators.

**(6)** Find generators of the homogeneous ideal (and prove carefully that they generate) of the Grassmannian $G(2,5)$ in its Plücker embedding in $\mathbb{P}^9$.

**(7)** To do this question, it maybe useful to know the Hurwitz formula for algebraic curves.

Let $C \in \mathbb{P}^2$ be a plane curve of degree $d$ and $C^* \subset \mathbb{P}^{2*}$ the *dual* curve (by definition this is the locus of tangent lines to $C$. Prove that $C^*$ is an algebraic curve and

$$C \ni p \to T_p C \in C^*$$

a morphism.

(a) Let $L \in \mathbb{P}^2$ be a line, not tangent to $C$. Define $\varphi : C \to L$ mapping $p \to T_p C \cap L$. Show that $\varphi$ is ramified at $p$ iff $p \in L$ or $p \in C$ is a flex.

(b) If $L$ is tangent to $C$ at $p_1, ..., p_r$ and none of the $p_i$ is a flex, then $L \in C^*$ is an ordinary r-fold point (i.e., by definition, a point of multiplicity $r$ with $r$ distinct tangents, and in particular $r$ distinct smooth branches).

(c) Let $p \in \mathbb{P}^2$ be a point not lying on $C$ nor on any inflectional or multiple tangent to $C$, $L$ a line not containing $p$, $\varphi : C \to L$ the projection from $p$. Use Hurwitz's formula to compute the degree of $C^*$ (you should get $d(d-1)$).

(d) A sufficiently general point $p \in C$ lies on $(d+1)(d-2)$ tangents of $C$ (not counting the tangent at $p$.

(e) Calculate the degree of the morphism $\varphi$ in (a) and use Hurwitz to count the flexes of $C$.

(f) Assume that $C^*$ has only ordinary nodes and cusps as singularities (this is true for sufficiently general $C$). Show that $C$ has

$$\frac{1}{2}d(d-2)(d-3)(d+3)$$

bitangents [this may be quite hard, but should be fun to try]. In particular a plane quartic has 28 bitangents. Do these have anything to do with the 27 lines on a cubic surface? [hint: 28=27+1. A more constructive hint would be to choose a point $p \in S$ on the cubic surface and project down to $\mathbb{P}^2$. This is a finite morphism of degree 2, branched along a 4-ic in $\mathbb{P}^2$. Try and see where do the 27 lines go...]

**(8)** Let $X$ be a proper and smooth algebraic surface. Define a suitable *intersection product*:

$$\operatorname{Pic} X \times \operatorname{Pic} X \to \mathbb{Z}$$

on the group of Line bundles on $X$, by generalising intersections of curves in $\mathbb{P}^2$. Then prove the Riemann-Roch theorem on $X$:

$$\chi\mathcal{L} = \frac{1}{2}\mathcal{L} \cdot (\mathcal{L} \otimes \omega_X^*) + 1 + p_a$$

where $\omega_X := \wedge^2\Omega^1$ is the canonical line bundle and $p_a := h^2\mathcal{O} - h^1\mathcal{O}$ [hint: first you should either study or make your own proof of Riemann-Roch for curves].