Examples 1: singularities of surfaces

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In the following problems, you better use the following **Theorem** (see [AVGZ]): Let $f\colon \mathbb{C}^n, 0 \to \mathbb{C}, 0$ be an analytic function defining an isolated singularity $\{f = 0\} \subset \mathbb{C}^n$, let J be the ideal generated by the partial derivatives $\frac{\partial f}{\partial x_i}$ and let $\alpha = (a_1, \dots, a_n)$ be a weight.

Assume that

$$f = f_0 + h$$

where $f_0 = 0$ is an isolated singularity and $d = \deg_\alpha f_0 < \deg_\alpha h$. Out of a monomial basis of $\mathcal{O}/J_0$, let $x^{m_1}, \dots, x^{m_k}$ be those monomials with $\deg_\alpha x^{m_i} > d$. Then f is analytically equivalent to

$$f_0 + \sum c_i x^{m_i}$$

for some constants $c_i \in \mathbb{C}$.

(1) If $f = x^2 y + O(4)$ defines an isolated singularity $\{f = 0\} \subset \mathbb{C}^2$, then f is analytically equivalent to

$$x^2 y + y^k$$

for some $k \geq 4$.

(2) In this question, we classify *canonical surface singularities* $\{f(x, y, z) = 0\} \subset \mathbb{C}^3$:

(i) Show that for all weightings α of the variables x, y, z, $\deg_\alpha(xyz) \geq \deg_\alpha(f) + 1$.

(ii) Prove that f has a nonzero part f_2 of degree 2, so in suitable coordinates:

$$\begin{aligned} f_2 &= x^2 + y^2 + z^2, \text{ or} \\ &= xy, \text{ or} \\ &= x^2 \end{aligned}$$

In the first case we have an A_1-singularity; otherwise:

(iii) If $f_2 = x^2$, then is suitable coordinates $f = xy + z^k$.

(iv) If $f_2 = x^2$, then $x^2 + g(y,z)$ and $g_3 \neq 0$. Further cases are:

(v)

$$\begin{aligned} g_3 &= y^3 + z^3, \text{ then } x^2 + y^3 + z^3 \text{ or} \\ &= y^2 z, \text{ then } x^2 + y^2 z + z^k \ (k \geq 4), \text{ or} \\ &= y^3. \end{aligned}$$

In the last case we have the three possibilities

$$\begin{cases} x^2 + y^3 + yz^3 \\ x^2 + y^3 + z^4 \\ x^2 + y^3 + z^5. \end{cases}$$

(vi) Explicitly resolve all these singularities and check that they are canonical.

(3) Let ζ be a primitive $2n$th root of 1 and consider the matrices

$$A = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Check that A, B generate a finite group $G \subset SL_2(\mathbb{C})$ of order $4n$. Prove that $\mathbb{C}^2/G = \{x^2 + y^2 z + z^n = 0\}$.

(4) Resolve the elliptic Gorenstein singularity

$$T_{p,q,r} \quad \text{defined by} \quad xyz + x^p + y^q + z^r = 0 \text{ for } \frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1.$$

[Hint: do the case $p, q, r \geq 3$ first.]

(5) Resolve the elliptic Gorenstein singularities given by:

$$\begin{aligned} x^2 + y^3 + z^k &= 0 \quad \text{for } k = 6, 7, 8, 9, 10, 11, \text{ and} \\ x^2 + y^3 + yz^l &= 0 \quad \text{for } l = 5, 7. \end{aligned}$$

[Hint: start out with a blowup with weights $3, 2, 1$.]