Examples 2: singularities of 3-folds

A. C.

In these problems, you will construct examples of 3-fold *divisorial contractions* $(E \subset Y) \to (P \in X)$. These are the 3-dimensional analogue of contracting a -1-curve on a surface. A comprehensive theory and classification of 3-fold divisorial contractions can be found in the work of Kawakita.

(1) Let $X = \frac{1}{r}(a, -a, r)$ where $\operatorname{hcf}(a, r) = 1$, and $Y \to X$ the blow up with weights $\frac{1}{r}(r - a, a, 1)$. Calculate the discrepancy and the singularities on Y.

(2) Do the same for

$$X = \{xy + z^r + t^n = 0\} \subset A = \frac{1}{r}(a, -a, 1, 0),$$

and $Y \subset B$ the proper transform under the blow up $B \to A$ with weights $\frac{1}{r}(r - a, a, 1, r)$. Deduce an algorithm to resolve X.

(3) Do the same now with

$$X = \{xy + f(z^r, t) = 0\} \subset A = \frac{1}{r}(a, -a, 1, 0),$$

$f = f(U, t) = 0$ an isolated curve singularity of multiplicity $k = \operatorname{mult} f$, and $Y \subset B$ the proper transform under the blow up $B \to A$ with weights $\left(i + \frac{a}{r}, k - i - \frac{a}{r}, \frac{1}{r}, 1\right)$ where $0 \le i \le k$. (All these have discrepancy $1/r$.)

(4)

$$X = \{xy + z^k + t^{km}) = 0\} \subset A = \mathbb{C}^4,$$

$B \to A$ the blowup with weights $(a, b, m, 1)$ where $a + b = km$, $\operatorname{hcf}(a, b) = 1$. (The discrepancy here is $\frac{a+b}{k}$).

(5)

$$X = \{xy + z^{2m} + t^k) = 0\} \subset A = \frac{1}{2}(1,1,1,0),$$

with $k \geq 2m$, m even, $B \to A$ the blowup with weights $(m,m,1,1)$.

(6) For $X = \mathbb{C}^3$, show that the blow up with weights (a_1, a_2, a_3) has terminal singularities if and only if (up to permutation) $(a_1, a_2, a_3) = (1, a, b)$ and $\mathrm{hcf}(a,b) = 1$. (This uses the "terminal lemma" of [YPG].)

(7)

$$X = \{x^2 + f(y,z,t) = 0\} \subset A = \frac{1}{2}(1,1,0,1)$$

where $f = O(3)$ contains either y^2t or yzt (this is an "exotic" 3-fold terminal singularity), $B \to A$ is the blowup with weights $\frac{1}{2}(1,1,2,3)$. Verify that the exceptional divisor of $Y \to X$ is not normal.