Examples 3: base point freeness

A. C.

The first series of examples show how Francia's lemma from the lectures can be used to deduce effective base point freeness on surfaces of general type.

Recall that an effectice divisor D on a nonsingular surface X is *numerically m-connected* if $D = D_1 + D_2$ with D_1, D_2 effective and nonzero, implies $D_1 \cdot D_2 \geq m$.

Also recall *Francia's Lemma:* Let X be a nonsingular surface, D an effective divisor on X, and $x \in D$ a singular point of D. Denote by $f\colon Y \to X$ the blow up of the maximal ideal of $x \in X$ and $D' = f^*D - 2E$. Then $x \in \operatorname{Bs}|K + D|$ if and only if D' is disconnected.

(1) Let $D = \sum n_i D_i$ be a numerically 1-connected divisor, and $\mathcal{L}$ a line bundle on D with $\deg(\mathcal{L}_{|D_i}) = 0$ for all D_i. Then $h^0(D, \mathcal{L}) \leq 1$ and $= 1$ if and only if $\mathcal{L} = \mathcal{O}_D$ (in particular, 1-connected divisors are connected). [Hint: If $D = D_1 + D_2$, start by writing down the exact sequence $0 \to \mathcal{O}_{D_2}(-D_1) \to \mathcal{O}_D \to \mathcal{O}_{D_1} \to 0$.]

(2) Let D be a nef and big divisor on a nonsingular surface. Then D is numerically 1-connected. [Hint: You must use the *algebraic index theorem* of Hodge: if A is ample and $A \cdot D = 0$, then $D^2 \leq 0$, and if $D^2 = 0$ then D is numerically 0. Equivalently if D_1, D_2 are numerically independent divisors and $D = \lambda D_1 + \mu D_2$ has $D^2 > 0$ for some real λ, μ, then

$$\det \begin{pmatrix} D_1^2 & D_1 \cdot D_2 \\ D_1 \cdot D_2 & D_2^2 \end{pmatrix} < 0.$$

Now assume $D = D_1 + D_2$ with $D_1, D_2 \geq 0$; write

$$D_1^2 + D_1 D_2 = D_1 D \geq 0$$
$$D_2^2 + D_2 D_1 = D_2 D \geq 0$$

If $D_1 D_2 \leq 0$, then $D_1^2 D_2^2 \geq (D_1 D_2)^2$ contradicts the Hodge index theorem.]

(3) Let X be a *minimal* surface of *general type* (that is, K_X is nef and big). Unless $n = 2$ and $K_X^2 = 1$, every $D \in |nK_X|$ is 2-connected.

(4) Let X be a minimal surface of general type. Use Francia's lemma to show that for example $|4K_X|$ is free from base points (in fact if $K_X^2 \geq 2$, $|4K_X|$ defines an embedding away from -2-curves). [Hint: Riemann-Roch

$$p_n(X) = h^0(X, nK_X) = \binom{n}{2} K^2 + \chi(X)$$

and $\chi(X) > 0$ give $p_3 > 3$, hence for all $x \in X$, there is a divisor $D \in |3K_X|$ containing x with multiplicity $\operatorname{mult}_x D \geq 2$. By (3) D is 2-connected hence (exercise) if $f\colon Y \to X$ is the blow up of $x \in X$, then $D' = -2E + f^*D$ is 1-connected. Finally by (1) $h^0(D', \mathcal{O}_{D'}) = 1$, and by Francia x is not a base point of $|K_X + D|$.]

Recall that a K3 surface is a (nonsingular) surface X with $\mathcal{O}_X(K_X) = \mathcal{O}_X$ and $h^1(X, \mathcal{O}_X) = (0)$. The next few examples are about linear systems on K3 surfaces. The cool thing about K3s is of course that, by the methods taught, you can relate the singularities of D to the base locus of $|K_X + D|$, that is $|D|$ itself. In addition K3s have nice vanishing ($H^1(\mathcal{O}) = (0)$) and Riemann-Roch, Serre duality and adjunction $2p_a(D) - 2 = D^2$ etc have an especially nice form.

(5) Let D be a nef and big linear system on a K3 surface X. Then *either*:

1. $|D|$ has no *fixed part* (i.e. no base divisor), *or*
2. $D = \Gamma + kE$ where $\Gamma \subset X$ is a -2-curve (a $\mathbb{P}^1$ with selfintersection -2), $|E|$ is a free (elliptic) pencil, and $\Gamma \cdot E = 1$ (i.e., Γ is a section). In this case D is called "monogonal".

(6) If D as in (5) is nef and big and not monogonal, then $|D|$ is base point free. (More is true: the resulting morphism to $\mathbb{P}|D|^\vee$ is either 2-to-1 on its image (hyperelliptic K3) or an embedding away from -2-curves.)

(7) Extend the previous results to the case of a K3 X with DuVal singularities. In particular, show that if D is nef and big, then one of the following is true

- D is monogonal,
- D is free,
- $X = X_{2,6} \subset \mathbb{P}(1^3, 2, 3)$ is a special complete intersection of the form:

$$\begin{cases} q_2(x_0, x_1, x_2) & = 0 \\ z^2 + y^3 + a_6(x_0, x_1, x_2) & = 0 \end{cases}$$

  and $D \in |\mathcal{O}_X(1)|$.

(The new case corresponds to a monogonal system $D' = \Gamma + 2E$ on a nonsingular K3 Y and $Y \to X$ is the contraction of Γ.)

In the next question I ask you to show a statement that I made without proof in the lectures.

(8) Assumptions and notation as in the statement and proof of the theorem of Ein and Lazarsfeld. Show that if $L \cdot C \geq 3$ for all curves $C \ni x$, then $b > 2b_i$ for all i (we are thus spared half of the proof).

Recall that X is *Fano* if $-K_X$ is ample. In the final sequence of examples I ask you to study the anticanonical linear system $|-K_X|$ of a nonsingular Fano 3-fold.

(9) Show that the anticanonical linear system $|-K_X|$ of a nonsingular Fano 3-fold X is nonempty. [Hint: by Riemann-Roch $h^0(-K) = g + 2$.]

(10) Let X be a nonsingular Fano 3-fold, and $S \in |-K_X|$. Assume that S has nonrational singularities. This is equivalent to say (persuade yourselves) that S has a singularity which is not a rational double point, or that the pair (X, S) is not log terminal. Write as usual

$$\begin{aligned} K_Y &= f^*K_X + \sum a_i E_i \\ D' &= f^*D - \sum r_i E_i \end{aligned}$$

where $f\colon Y \to X$ is a nice resolution etc. By assumption some $r_i - a_i \geq 1$. Use the standard machinery of the X-method (with tie-braking) to write an expression

$$L = f^*(-K_X) = K_Y + \sum b_i B_i + M$$

where the b_i are rational numbers and

- all $b_i \le 1$ and exactly one of them, say b_0, is $= 1$,
- if $b_i < 0$, B_i is f-exceptional,
- M is an ample $\mathbb{Q}$-divisor

The restriction lemma says that the restriction map $H^0(Y, L+G) \to H^0(B_0, L+G_{|B_0})$, where G is a harmless effective divisor with support not containing B_0. Assuming that the centre $x = C(B_0, X)$ of B_0 on X is a point, conclude that $x \notin \operatorname{Bs}|-K_X|$.

(11) Notation and setting as in the previous question, assume now that the centre $C(B_0, X)$ of B_0 on X is a curve Γ in X. Show that Γ is a curve with arithmetic genus $p_a\Gamma \le 1$. [Hint: S is nonnormal along Γ. Now use the *subadjunction formula*.] Conclude again, using the setup of the previous question, that $\Gamma \not\subset \operatorname{Bs}|-K_X|$.

(12) Finally assume that B_0 is not exceptional, that is, the centre $S' = C(B_0, X)$ is a surface, necessarily a component of S of *multiplicity* > 1. Show that as a $\mathbb{Q}$-divisor $-K_{S'} =$ (nef& big) + (effective). Use Riemann-Roch on S' and show that $h^0(S', -K_{X|S'}) \neq 0$. Conclude as before that $S' \not\subset \operatorname{Bs}|-K_X|$. Put everything together and show the *general elephant conjecture* for X, that is, a general member of $|-K_X|$ is a K3 surface with Du Val singularities.

(13) Using question (7), show that a general member of $|-K|$ is nonsingular, *unless* X is a special complete intersection $X_{2,6} \subset \mathbb{P}(1^4, 2, 3)$.