Groups Rings and Fields, Example Sheet 1

Dr Alessio Corti a.corti@dpmms.cam.ac.uk

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(1)

1. What are the orders of elements of the group S_4? How many elements are there of each order?
2. How many subgroups of order 2 has S_4? Of order 3? Of order 4?
3. Identify a non-cyclic subgroup $V \leq S_4$ of order 4. How many of these are there?
4. Identify a subgroup $D \leq S_4$ of order 8. How many of these are there?

(2)

1. Suppose that G is a non-abelian group of order p^3 where p is prime.
2. Show that the order of the centre $Z(G)$ is p.
3. Show that if $g \notin Z(G)$ then the order of the centralizer $C(g)$ is p^2.
4. Hence determine the sizes and numbers of the conjugacy classes.

(3) This question treats an important example of finite groups, that is, groups of matrices over finite fields.

1. Recall that the set $k = \mathbb{Z}/(p)$ of integers modulo p, where p is a prime, is a field. Put $V = k^n$. Find the order of $G = GL(V)$. (Hint: fix a basis of V and think what an element of G can do to the elements of this basis.)

2. Find a p-Sylow subgroup P of G (Hint: if $g \in G$ has p-power order, find the characteristic polynomial of g. What does this tell you about the shape of g as a matrix with respect to a suitable basis of V?)

3. Define the subgroup $SL_n(k)$ of $GL_n(k)$ by the condition $\det g = 1$. For V as above, what is the order of $SL_n(k)$?

4. How many 1-dimensional subspaces of k^2 are there, when $k = \mathbb{Z}/(p)$? Find an interesting homomorphism $\phi : SL_2(k) \to S_{p+1}$. What is its kernel?

5. (*) When $p = 5$, show that the image of ϕ is isomorphic to A_5.

(4) Calculate the size of the conjugacy class of (123) as an element of S_4, as an element of S_5 and as an element of S_6. Find in each case the centralizer. Hence calculate the size of the conjugacy class of (123) as an element of A_4, as an element of A_5 and as an element of A_6.

(5)

1. In question 1 we found the number of 2-Sylow subgroups and 3-Sylow subgroups of S_4. Check that your answer is consistent with Sylow's theorems. (Note that if you did not then quite complete proofs for subgroups of order 8, you can do so now.) Identify the normalizers of the 2-Sylow subgroups and 3-Sylow subgroups.

2. For $p = 2$, 3, 5 find a p-Sylow subgroup of A_5 and find the normalizer of the subgroup.

(6)

1. Show that A_5 has no subgroups of index 2, 3 or 4. Exhibit a subgroup of index 5.

2. Show that A_5 is generated by (12)(34) and (135). (Multiply the two elements, hence show that they generate a subgroup of order 30 or 60. Conclude using the simpicity of A_5.)