Groups Rings and Fields, Example Sheet 3

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(1) Write $X^2+Y^2+Z^2$ and $X^2Y^2+Y^2Z^2+Z^2X^2$ in terms of the elementary symmetric functions in X, Y and Z. Suppose that $X^3-a_1X^2+a_2X-a_3 \in \mathbb{C}[X]$ has roots α, β and γ. Find the monic polynomial with roots α^2, β^2 and γ^2.

Suppose that $X^3-a_1X^2+a_2X-a_3 \in \mathbb{C}[X]$ has roots α, β and γ all non-zero. Find the monic polynomial with roots $1/\alpha$, $1/\beta$ and $1/\gamma$. [Hint. This is easy.]

(2) Use Newton's Identities relating the symmetric powers to the elementary symmetric functions to express the symmetric powers p_2, p_3 and p_4 in three variables X, Y and Z in terms of the corresponding elementary symmetric functions.

(3) Let $G \cong C_4$ be the subgroup of $GL_2(\mathbb{C})$ generated by the matrix $A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. When G acts on $\mathbb{C}[X,Y]$ in the obvious way, show that $\mathbb{C}[X,Y]^G = \mathbb{C}[XY, X^4, Y^4]$. (You can use the Noether method as explained in class, or try to do it by "brute force").

Repeat the process for $G \cong C_4$, the subgroup of $GL_2(\mathbb{C})$ generated by the matrix $\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$.

(4) Let G be the subgroup of $GL_2(\mathbb{C})$ generated by the matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. G acts on $\mathbb{C}[X,Y]$ in the obvious way. Compute the ring of invariants as follows.

1. Noether's proof gives just one polynomial. Find it and give its coefficients.
2. Take the monomials of depth less than 4 and symmetrize them with respect to the group action.
3. Deduce that

$$\mathbb{C}[X,Y]^G = \mathbb{C}[X^2+Y^2, X^2Y^2, XY(X^2-Y^2)]\,.$$

Repeat the last question replacing A by $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. What is the ring of invariants?

(5) Let D be the subgroup of $GL_2(\mathbb{C})$ generated by the matrices $A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Show that D is the dihedral group of order 8. D acts on $\mathbb{C}[X,Y]$ in the obvious way. Show that

$$\mathbb{C}[X,Y]^D = \mathbb{C}[XY, X^4 + Y^4]\,.$$

(6) Let Q be the subgroup of $GL_2(\mathbb{C})$ generated by the matrices $A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, and $B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$. Show that Q is the quaternionic group of order 8. Q acts on $\mathbb{C}[X,Y]$ in the obvious way. Show that

$$\mathbb{C}[X,Y]^Q = \mathbb{C}[X^4 + Y^4, X^2Y^2, XY(X^4 - Y^4)]\,.$$

(7) In three variables Newton's recurrence formulae $e_1 = p_1$, $2e_2 = p_1e_1 - p_2$, $3e_3 = p_1e_2 - p_2e_1 + p_3$ and so on give the following

- $p_2 = e_1^2 - 2e_2$, so we can eliminate e_1^2 in favour of p_2;
- $p_4 = p_3e_1 - p_2e_2 + p_1e_3 = (p_2e_1 - p_1e_2 + 3e_3)e_1 - p_2e_2 + e_1e_3 = (p_2 - e_2)e_1^2 - p_2e_2 + 4e_1e_3$ so we can eliminate e_1e_3 in favour of p_4;
- $p_6 = p_5e_1 - p_4e_2 + p_3e_3 = p_4e_1^2 - p_3e_2e_1 + p_2e_3e_1 - p_4e_2 + p_2e_1e_3 - p_1e_2e_3 + 3e_3e_3$ whence iductively we can eliminate e_3e_3 in favour of p_6.

(8) Let $R[[X]]$ be the ring of formal power series $\sum_{i=0}^{\infty} a_iX^i$ with coefficients in R.

(i) Show that an element of form $1 + a_1X + a_2X^2 + \cdots$ is a unit in $R[[X]]$. What are the units?

(ii) Show that if R is Noetherian then so is $R[[X]]$.