K3 Surfaces Examples
Lent 2005

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Starred questions are optional.

Scrolls

I use the following notation for scrolls: $\mathbb{F}(a_1, \ldots, a_n)$ is the standard scroll over $\mathbb{P}^1$ with "fibre coordinates" $x_1, \ldots, x_n$ and coordinates t_1, t_2 coming from the base. L is the line bundle with sections t_1, t_2 and M is the line bundle with sections $x_i f_{a_i}(t_1, t_2)$ where f_{a_i} is a homogeneous polynomial of degree a_i in the variables t_1, t_2.

(1) The subscroll corresponding to the set $\{a_{i_1}, \ldots, a_{i_m}\} \subset \{a_1, \ldots, a_n\}$ is the subvariety $\mathbb{F}(a_{i_1}, \ldots, a_{i_m}) \subset \mathbb{F}(a_1, \ldots, a_n)$ defined by the equations $x_j = 0$ for $j \notin \{a_{i_1}, \ldots, a_{i_m}\}$. For any b let B_b be the subscroll corresponding to the subset $\{a_i \mid a_i \leq b\}$. Suppose that $a_1 \leq \cdots \leq a_n$. Show that

- The base locus of $|-(b+1)L + M|$ is B_b.
- If $b = a_m$, then B_b is contained with multiplicity $< \mu$ in the base locus of $|eL + dM|$ if and only if

$$e + bd + (a_n - b)(\mu - 1) \geq 0.$$

(2) Prove that

$$\mathbb{F}(a_1, \ldots, a_n) \cong \mathbb{F}(b_1, \ldots, b_n) \quad \text{if and only if} \quad \{a_1, \ldots, a_n\} = \{b_1 + c, \ldots, b_n + c\}$$

for some c.

[*Hint: Use the description of* $\operatorname{Pic} \mathbb{F}$ *and the previous question.*]

(3) Recall that a nonhyperelliptic curve C of genus g is *trigonal* if there is a 3-to-1 morphism $C \to \mathbb{P}^1$. Show that the canonical image of a trigonal curve is contained in a surface scroll $C \subset \mathbb{F}(a_1, a_2) \subset \mathbb{P}^{g-1}$ where $g = a_1 + a_2 + 2$ and the pencil L on $\mathbb{F}$ cuts out the g^1_3. Suppose that $a_1 \leq a_2$, set $a = a_2 - a_1$, and let $B \subset \mathbb{F}_a$ be the negative section. Verify that $\mathbb{F}(a_1, a_2)$ is $\mathbb{F}_a$ embedded by $a_2 L + B$ and show that

$$\mathbb{F}_a \supset C \in |(a + a_2 + 2)L + 3B|$$

Show that a general element of this linear system is nonsingular if and only if one of the following equivalent conditions holds

- $3a \le g + 2$, *or*
- $3a_2 \le 2g - 2$, *or*
- $3a_1 \ge g - 4$.

(4) Consider a 3-fold scroll $\mathbb{F} = \mathbb{F}(a_1, a_2, a_3) \to \mathbb{P}^1$ with $0 = a_1 \le a_2 \le a_3$ and a surface $X \subset \mathbb{F}$ meeting the general fibre of $\mathbb{F} \to \mathbb{P}^1$ in a nonsingular cubic curve. Then

$$X \in |(k + 2 - \sum a_i)L + 3M|$$

for some $k \in \mathbb{Z}$. Using the notation of Example 1, show that X is nonsingular at the generic fibre of $\mathbb{F} \to \mathbb{P}^1$ if and only if $B_{a_2} \not\subset X$ and $2B_{a_1} \not\subset X$.

Show that $B_{a_2} \not\subset X$ if and only if $a_2 + k + 2 \ge a_3 - a_2$.

Similarly, show that $2B_{a_1} \not\subset X$ if and only if $k + 2 \ge a_2$.

Show that X is nonsingular in the extreme case $a_2 = k + 2$, $a_3 = 3(k + 2)$.

(5) Let $X = X_{2,e} \subset \mathbb{F}(a_1, a_2, a_3)$ be a surface of bidegree $(2, e)$. The fibres of $X \to \mathbb{P}^1$ are plane conics. Prove that, if X is nonsingular, then every fibre is either a nonsingular conic or a pair of distinct lines. Find a formula for the number of line pairs.

(6) Suppose that $a_1 \le \cdots \le a_n$ and $b_1 \le \cdots \le b_m$; prove that there exists a surjective sheaf homomorphism

$$\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n) \to \mathcal{O}_{\mathbb{P}^1}(b_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(b_m)$$

if and only if $m \le n$ and for every i,

$$a_i \le b_i \quad \text{and:} \quad \text{If } (a_1, \ldots, a_i) \ne (b_1, \ldots, b_i), \text{ then also } b_{i+1} \le a_i.$$

If $0 < a_1$, deduce necessary and sufficient conditions for $\mathbb{F}(b_1, \ldots, b_{n-1})$ to be a hyperplane section of $\mathbb{F}(a_1, \ldots, a_n)$.

(7) If $a_1 \le a_2$ and $a'_1 \le a'_2$, prove that $\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2)$ has a small deformation isomorphic to $\mathcal{O}_{\mathbb{P}^1}(a'_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a'_2)$ if and only if $a_1 + a_2 = a'_1 + a'_2$ and $a_1 \le a'_1 \le a'_2 \le a_2$.

[*Hint: Find a small deformation of* $\mathbb{F}(a_1, a_2)$ *by taking a special hyperplane section of a 3-fold scroll* $\mathbb{F}(b_1, b_2, b_3)$ and then varying the hyperplane.]

(8) Find all values (a_1, a_2, a_3) and e such that the general hypersurface $X_{3,e} \in |eL + 3M|$ is a nonsingular $K3$.

[*Hint: Do Example 4 first. When $|eL + 3M|$ has a base locus, you have to check for isolated singularities along the base locus.*]

(9) Show that the ideal of a trigonal curve of genus g is generated by the $(g-2)(g-3)/2$ quadrics that contain it, and $g-3$ additional cubics.

Curves

(10) Let C be a curve of genus $g = 2$. Show that C can not be embedded as a curve of degree $d \leq 4$ in $\mathbb{P}^r$. Describe an embedding $\varphi\colon C \subset \mathbb{P}^3$ as a curve of degree 5. How many quadric and cubics contain the image $\varphi(C)$? Among the cubics, how many are not already in the ideal generated by the quadrics containing C?

(11) Prove carefully Max Noether's Theorem: If C is a nonhyperelliptic curve of genus g, then the canonical ring

$$R(C, K_C) = \oplus_{n\geq 0} H^0(C, nK_C)$$

is generated by its elements of degree 1.

[*You must state and use the base point free pencil trick.*]

(12$*$) In this example we prove in several steps the following generalisation of Noether's theorem due to Castelnuovo: Let $|D|$ be a complete base point free linear system of dimension $r \geq 3$ on a curve C, and assume that the map

$$\varphi_D\colon C \to \mathbb{P}^r$$

is birational on its image. Then the natural map

$$S^l H^0(C, D) \otimes H^0(C, K) \to H^0(C, K + lD)$$

is surjective for all $l \geq 1$.

(a) Let L be a line bundle on C such that $H^1(C, L(-D)) = (0)$, and let $V \subset H^0(C, D)$ give a base point free pencil. Then the natural map

$$V \otimes H^0(C, L) \to H^0(C, L + D)$$

is surjective.

[*This follows easily from the base point free pencil trick.*]

(b) With the notation of (a), show that the image of the natural map

$$S^l V \otimes H^0(C, K) \to H^0(C, K + lD)$$

is of codimension $lr - 1$.

[*Do $l = 1, 2$ by hand. For $l \geq 3$, by (a) applied to $L = K + lD$, deduce that every element of $H^0(C, K + lD)$ is of the form*

$$\sum P_\alpha \omega_\alpha + Q\eta$$

where $\omega_1, \dots, \omega_g$ is a basis of $H^0(C, K)$, $\eta \in H^0(C, 2D)$, $P_\alpha \in S^l V$, and $Q \in S^{l-2}V$. Proceed by induction.]

(c) Suppose that $P_1 + \cdots P_d$ is a general divisor in $|D|$, and set $E = P_1 + \cdots + P_{r-2}$. Show that $|D - E|$ is a base point free pencil and in the exact cohomology sequence of

$$0 \to \mathcal{O}(D - E) \to \mathcal{O}(D) \to \mathcal{O}_E(D) \to 0$$

the map $H^0(C, D) \to H^0(E, D_{|E})$ is surjective.

[*You may use the general position Theorem without proof: Let $C \subset \mathbb{P}^r$, $r \geq 2$, be an irreducible nondegenerate, possibly singular, curve of degree d. Then a general hyperplane meets C in d points, no r of which are linearly independent.*]

(d) By tensoring the exact cohomology sequence of (c) with $H^0(C, K + (l - 1)D)$ and arguing by induction on l, complete the proof of the statement.

Surfaces

(13) Prove the *algebraic index Theorem*: If X is a nonsingular surface, H is ample on X, and D is a divisor on X, then $HD = 0$ implies $D^2 \leq 0$; moreover, if $D^2 = 0$, then $D \overset{\text{num}}{\sim} 0$ is numerically equivalent to 0.

Equivalently, if D_1, D_2 are divisors on X, and $(\lambda D_1 + \mu D_2)^2 > 0$ for some $\lambda, \mu \in \mathbb{R}$, then

$$\det \begin{pmatrix} D_1^2 & D_1 D_2 \\ D_1 D_2 & D_2^2 \end{pmatrix} \leq 0,$$

with equality if and only if some nonzero rational linear combination is numerically equivalent to zero, that is $\alpha D_1 + \beta D_2 \overset{\text{num}}{\sim} 0$.

(14) Let D be an effective divisor on a surface X; the *Zariski decomposition* of D is an expression $D = P + N$ where P (the *p*ositive part) is nef and $N = \sum q_i \Gamma_i$ (the *n*egative part) with $q_i \in \mathbb{Q}_+$, the intersection matrix $\Gamma_i \Gamma_j$ is negative definite, and $P\Gamma_i = 0$ for all i. Show that this exists and is unique.

(15) Suppose that $D = \sum n_i \Gamma_i$ with all $\Gamma_i^2 = 0$. When does D fail to be numerically n-connected? For any n, give an example of such a divisor which is numerically n-connected but not numerically $n + 1$-connected.

(16) Let $E = f^{-1}(P)$ be a nonsingular reduced fibre of a morphism $f \colon X \to B$ of a surface to a base curve. Prove that $H^0(\mathcal{O}_X(aE)) = k[t]/t^a$.

[*Consider appropriate exact sequences.*]

Hodge Theory

(17) Let X be a Kähler manifold. Define carefully the natural maps involved in the diagram:

$$\begin{array}{ccc} \check{H}^m(X,\mathbb{Z}) & \longrightarrow & H^m_{dR}(X,\mathbb{C}) \\ \downarrow & & \downarrow \\ \check{H}^m(X,\mathcal{O}) & \longrightarrow & H^m_{\overline{\partial}}(\mathcal{A}^{0,\bullet}_X) \end{array}$$

Use the main statement of Hodge theory to show that the diagram is commutative.

[*Try to be more rigorous than I was in class and use the statement of Hodge theory in the form*

$$\mathcal{A}^{p,q} = \mathcal{H}^{p,q} \perp \Delta G = \mathcal{H}^{p,q} \perp \overline{\partial}\mathcal{A}^{p,q-1} \perp \overline{\partial}^* \mathcal{A}^{p,q+1}$$

where G is the resolvent of the Laplacian.]

(18) Consider a scroll $\mathbb{F} = \mathbb{F}(a_1, \ldots, a_n)$. I stated in class that $\operatorname{Pic}\mathbb{F} = \mathbb{Z}^2$ generated by L, M. Prove this statement.

[*Hint: show that* $h^1(\mathcal{O}) = h^2(\mathcal{O}) = 0$ *and hence, by the long exact cohomology sequence of the exponential sequence, that* $\operatorname{Pic}\mathbb{F} = H^2(\mathbb{F},\mathbb{Z})$*; then show that (cycle classes of sections of) L, M form a basis of $H^2(\mathbb{F},\mathbb{Z})$. For this you need to calculate $H^2(\mathbb{F},\mathbb{Z})$. This is an elementary calculation in topology and it can be done in various ways. In general, if E is a vector bundle of rank r over a manifold X, and $P = \mathbb{P}(E) \to X$,*

$$H^\bullet P = \frac{H^\bullet X[t]}{t^{r+1} + \sum(-1)^i c_i t^{r-i}}$$

where $c_i = c_i(E) \in H^{2i}X$ is the i-th Chern class of E.]

The Kodaira-Spencer map

(19∗) We check that two constructions of the Kodaira-Spencer map are compatible. Indeed, let $\pi\colon \mathcal{X} \to B$ be a deformation of a Kähler manifold X parameterised by a small disk $0 \in B \subset \mathbb{C}$. As we did in the lectures, we think of this as a power series:

$$B \ni t \to \varphi(t) = t\eta + O(t^2) \in \mathcal{A}^{0,1}(T^{1,0})$$

where η is harmonic. Then η determines a cohomology class in $H^1_{\overline{\partial}}(X,\Theta)$. Show that, under the Dolbeault isomorphism, this is the Kodaira-Spencer class $\rho(d/dt) \in \check{H}^1(X,\Theta)$ of the deformation.

[*Choose C^∞ charts $U_\alpha \times B$ with coordinates $(z_1^\alpha, \dots, z_n^\alpha; t)$. We can find holomorphic coordinates $\zeta_i^\alpha = z_i^\alpha + t w_j^\alpha(z) + O(t^2)$ extending the z_i^α. Then write*

$$\varphi(t) = t \sum_{i,j} \eta_i^{\alpha \bar{j}} d\bar{z}_j^\alpha \otimes \frac{\partial}{\partial z_i^\alpha} + O(t^2);$$

where ζ_i^α holomorphic means $(\overline{\partial} + \varphi)\zeta_i^\alpha = 0$, that is

$$\overline{\partial} w_i^\alpha + \sum_j \eta_i^{\alpha \bar{j}} d\bar{z}_j^\alpha = 0,$$

hence in local coordinates $\eta = -\sum_i \overline{\partial} w_i^\alpha \otimes (\partial/\partial z_i^\alpha)$. Now set

$$\Lambda^\alpha = -\sum_i w_i^\alpha \otimes (\partial/\partial z_i^\alpha);$$

you need to verify that $\Lambda^\alpha_{|t=0} = \Lambda^\beta_{|t=0}$. If you have difficulties, look into F. Catanese, "Theory of Moduli", pg. 17.]

Kuranishi family

(20∗) Let $B = \mathbb{C}$ with coordinate s and consider the family of hypersurfaces:

$$X_s = (s x_1 + t_1 x_2 + t_2 x_3 = 0) \subset \mathbb{F}(0,1,1).$$

Show that $X_0 = \mathbb{F}_2$ and $X_s = \mathbb{P}^1 \times \mathbb{P}^1$ for $s \neq 0$. Show that the Kodaira-Spencer map $\rho \colon T_{B,0} \to H^1(\mathbb{F}_2, \Theta)$ is an isomorphism. Conclude that $\{X_s \mid s \in B\}$ is the Kuranishi family. Show that $h^0(X_s, \Theta)$ jumps at $s = 0$ and that the Kuranishi family is not versal.

[*Good luck.*]

Gauss-Manin

(21) Show the following result that was used implicitly in Deligne's construction of the Gauss-Manin's connection. Let M be a C^∞ manifold and $X \in C^\infty(M, TM)$ a global C^∞ vector field on M. Show that the *Lie derivative*

$$\mathcal{L}_X \colon \mathcal{A}_M^\bullet \to \mathcal{A}_M^\bullet$$

acting on the de Rham complex of M is Chain homotopic to the zero map (it follows that infinitesimal diffeomorphisms act trivially on de Rham cohomology, which is geometrically obvious since $H^m_{dR}(M, \mathbb{R}) = H^m(M, \mathbb{Z})$, the action is derived from an action on $H^m(M, \mathbb{Z})$, and $H^m(M, \mathbb{Z})$ is discrete.

[*Hint: you must write down an explicit homotopy operator $i \colon \mathcal{A}^k \to \mathcal{A}^{k-1}$; I advise you to take $i = i_X$ the contraction with the vector field X. Then you need to prove the formula $\mathcal{L}_X = i_X d + d i_X$. If you have difficulty with this, look into Warner, "Foundations of differentiable manifolds and Lie groups".*]

Griffiths domain

(22) Fix a nondegenerate antisymmetric (symplectic) form

$$\psi = \begin{pmatrix} 0 & -I_r \\ I_r & 0 \end{pmatrix}$$

on $\mathbb{Z}^{2r}$; consider Hodge structures on $\mathbb{Z}^{2r}$ polarised by the form ψ. Show that $\check{D} = SpGr(r, 2r)$ is the Grassmannian of r-dimensional complex subspaces $H \subset \mathbb{C}^{2r}$ which are isotropic for ψ. Calculate the dimension of $\check{D}$. Describe an explicit identification:

$$D = \{Z \in M_r(\mathbb{C}) \mid {}^tZ = Z, \Im Z > 0\}$$

and describe explicitly the action of $Sp_{2r}(\mathbb{Z})$ on D. If you know about Abelian varieties, identify $D/Sp_{2r}(\mathbb{Z})$ with the moduli space of principally polarised r-dimensional Abelian varieties.