Introduction to mixed Hodge theory: a lecture to the LSGNT

AC

26^{th} February, 2016

Topics

(1) Motivation and definition of MHS

(2) There are four theories: get used to it

(3) Grothendieck's algebraic de Rham theorem

(4) Constructible sheaves and the six operations. Exact sequences

(5) Cauchy's theorem and the Tate Hodge structure

(6) Extensions of MHS

(7) Proper base change. Cohomology of singular varieties

(8) What next? Invariant cycles and mixed Hodge modules

In the last un-numbered section, I give a short guide to the literature including references for everything that I say.

1 Motivation and definition of MHS

1.1 Motivation

You learned that if X is a nonsingular and projective (proper is enough) algebraic variety over $\mathbb{C}$, then the singular cohomology groups $H^m(X;\mathbb{Z})$ carry a (functorial) Hodge structure of pure weight m.

In this lecture I explain that if X is a possibly singular and/or nonproper algebraic variety over $\mathbb{C}$, then $H^m(X;\mathbb{Z})$ carries a functorial *mixed* Hodge structure.

I also secretly attempt to give an introduction to the *motivic* point of view. There is something special about the topological spaces underlying algebraic varieties that results into various *improvements* of the functors of algebraic topology (homology, cohomology). The mixed Hodge structure is an example of an improvement. In turn, these improvements *motivate* the special properties of topological spaces underlying algebraic varieties.

To motivate the definition, consider a nonsingular and proper algebraic variety X and a Zariski closed subset $Y \subset X$. We are interested in the nonproper Zariski open complement $U = X \setminus Y$. We have an exact sequence (see also the discussion in section 4 below):

$$\cdots H^{m-1}(X;\mathbb{Z}) \to H^{m-1}(Y;\mathbb{Z}) \to H^m_c(U;\mathbb{Z}) \to H^m(X,\mathbb{Z}) \to H^m(Y,\mathbb{Z}) \cdots$$

For instance now if, to fix ideas, $Y \subset X$ were a nonsingular divisor, then, whatever Hodge-type structure $H^m_c(U;\mathbb{Z})$ may have, it has **mixed weight** $m-1$ and m.[1]

1.2 Mixed Hodge structures

Definition 1. A *mixed Hodge structure* is a triple $(H, W_\bullet, F^\bullet)$ where

1. H is a finitely generated $\mathbb{Z}$-module;

2. $$W_\bullet = \cdots W_l \subset W_{l+1} \cdots$$
   is an *increasing* filtration of H by $\mathbb{Z}$-submodules[2];

3. $$F^\bullet = \cdots F^p \supset F^{p+1} \cdots$$

[1] In terms of Algebraic Topology 101, this is the exact sequence for the *homology* of the pair (X, U): you need to persuade yourself that $H_m(X,U;\mathbb{Z}) = H_{m-2}(Y;\mathbb{Z})$ (use excision & Thom).

In the sequence $H^m_c(U;\mathbb{Z})$ is the *compactly supported cohomology* of U. Since we are assuming that U is nonsingular, you may as well let $d = \dim X$ be the algebraic dimension of X and think of this group as $H_{2d-m}(U;\mathbb{Z})$.

[2] I am not absolutely sure about this, sorry. Usually $W_\bullet$ is a filtration of $H_\mathbb{Q}$ by rational vector spaces. I don't see any good reason to make this compromise.

is a *decreasing* filtration of $H_{\mathbb{C}} = H \otimes \mathbb{C}$ by $\mathbb{C}$-vector spaces;

such that $F^\bullet$ induces a Hodge structure of pure weight l on $\mathrm{gr}_l^W H$. The two filtrations $W_\bullet$ and $F^\bullet$ are called the *weight* and the *Hodge* filtrations.

Definition 2. Let A, B be MHS; a *morphism* of MHS is a $\mathbb{Z}$-module homomorphism $f\colon A \to B$ that respects filtrations: for all $l \in \mathbb{Z}$, $f(W_l) \subset W_l$ and for all $p \in \mathbb{Z}$, $f_{\mathbb{C}}(F^p) \subset F^p$.

The category of MHS is an abelian category with tensor products and internal Hom. The only nontrivial point is to show that image is the same of co-image. Another way to say this is that morphisms of MHS are strict wrt the weight filtration.

Exercise 3. Let $f\colon A \to B$ be a morphism of MHS. Then f is *strict* with respect to both filtrations: for all l, if $b \in W_l(B)$ is in the image of f, then b comes from $W_l(A)$ and, for all $p \in \mathbb{Z}$, if $b \in F^p(B)$ is in the image of $f_{\mathbb{C}}$, then b comes from $F^p(A)$.

This is not a terribly easy exercise but I absolutely need you to do it so I give you several hints. It is crucial to understand that you must use *both* filtrations.

The first thing to understand is why a morphism of *pure* HS is strict wrt the F-filtration. So let $f\colon A \to B$ be a morphism of HS of weight m. Here it is best to use the Hodge decomposition:

$$A_{\mathbb{C}} = \bigoplus_p A^{p,m-p}, \quad \text{where} \quad F^p(A) = \bigoplus_{p' \geq p} A^{p',m-p'}$$

so that

$$A^{p,q} = F^p(A) \cap \overline{F^{m-q}(A)} \quad \text{and} \quad \forall p \; A_{\mathbb{C}} = F^p(A) \oplus \overline{F^{m-p-1}(A)}$$

Suppose now that $b \in F^{p_0}(B)$ and let $a \in A_{\mathbb{C}}$ such that $f(a) = b$. Writing $a = \sum_p a_{p,q}$ it is clear that for all $p > p_0$ $f(a_{p,q}) = 0$.

For morphisms of MHS, the key point is to construct something like a Hodge decomposition. So let H be a MHS, and write

$$I^{p,q} =$$
$$= (F^p \cap W_{p+q}(\mathbb{C})) \cap \big(\overline{F_q} \cap W_{p+q}(\mathbb{C}) + \overline{F^{q-1}} \cap W_{p+q-2}(\mathbb{C}) + \overline{F^{q-2}} \cap W_{p+q-3}(\mathbb{C}) + \cdots\big)$$

(note the subscripts: $p+q, p+q-2, p+q-3, \dots$). Prove that

$$W_m(\mathbb{C}) = \oplus_{p+q\le m} I^{p,q} \quad \text{and} \quad F^p = \bigoplus_{p'\ge p} \bigoplus_{q} I^{p',q}$$

Finally use this $I^{p,q}$-decomposition to finish the exercise.

For simplicity in this note I don't define what is a *polarized* MHS. Polarizations are very important, but excessive detail can distract from some of the more basic issues.

1.3 Tate twist

Definition 4. The Tate Hodge structure is the Hodge structure $\mathbb{Z}(1)$ on $\mathbb{Z}$ of pure weight -2 "with underlying $\mathbb{Z}$-module $2\pi\mathtt{i}\mathbb{Z} \subset \mathbb{C}$."

If V is a MHS, $V(n)$ denotes $V \otimes \mathbb{Z}(n)$: twisting by n has the effect of raising all the weights by $-2n$ (or, if you prefer, lowering all the weights by $2n$).

Note that the definition of Tate Hodge structure does not make sense: of course, all HS of weight $2n$ on $\mathbb{Z}$ are isomorphic, so what is this business of $2\pi\mathtt{i}$? The wording means that $\mathbb{Z}(1)\otimes\mathbb{C}$ has a *natural* $\mathbb{C}$-*basis* $\alpha = \frac{1}{2\pi\mathtt{i}}$. This is indeed true and I explain the point in § 5 below. For now, just think of $\mathbb{Z}(1)$ as the unique HS on $\mathbb{Z}$ of pure weight -2.

2 There are four theories: get used to it

For topological spaces X satisfying some mild assumptions, there are four theories: $H^m(X;\mathbb{Z})$, $H^m_c(X;\mathbb{Z})$, $H_m(X;\mathbb{Z})$, and $H^{BM}_m(X;\mathbb{Z})$ (Borel–Moore homology), each with their own functorialities. Here H_m and H^m_c are compactly supported theories, whereas H^m and H^{BM}_m have infinite support.

These theories arise from the category of constructible sheaves with six operations, as derived functors:

$$H^m(X;\mathbb{Z}) = R^m\Gamma(X, \mathbb{Z}_X), \quad H^m_c(X;\mathbb{Z}_X) = R^m\Gamma_c(X,\mathbb{Z}_X),$$
$$H^{BM}_m(X;\mathbb{Z}) = R^{-m}\Gamma(X, D_X), \quad H_m(X;\mathbb{Z}) = R^{-m}\Gamma_c(X, D_X)$$

where $\Gamma(X,\ )$ is global sections, $\Gamma_c(X,\ )$ is global sections with compact support, and D_X is the dualising sheaf of X.

From now on in this lecture, X is an algebraic variety. If X is singular and not proper, the four theories are (in general) different. You better get used to this fact. There are some obvious maps between them. The most important is the Poincaré map: if X is an algebraic variety of pure (algebraic) dimension d, then X it is a pseudomanifold: consequently there is a homomorphism:

$$P\colon \mathbb{Z}_X[2d](d) \to D_X$$

called the *Poincaré homomorphism* giving rise to a fundamental class $[X]$ in $H^0(X, D_X[-2d])(-d) = H^{BM}_{2d}(X;\mathbb{Z})(-d)$ and Poincaré maps

$$P\colon H^m(X;\mathbb{Z}) \to H^{BM}_{2d-m}(X;\mathbb{Z})(-d), \text{ and } P\colon H^m_c(X;\mathbb{Z}) \to H_{2d-m}(X;\mathbb{Z})(-d) \tag{1}$$

If X is nonsingular, the Poincaré homomorphism is an isomorphism and these last two homomorphisms are isomorphisms.

All these theories carry functorial mixed Hodge structures, all reasonable morphisms are morphisms of mixed Hodge structures, and all reasonable exact sequences are sequences of mixed Hodge structures.[3]

3 Grothendieck's algebraic de Rham theorem

Theorem 5. *If X is a nonsingular over $\mathbb{C}$, then*

$$H^m(X;\mathbb{C}) = H^m(X, \Omega^\bullet_{X/\mathbb{C}})$$

[3]In this discussion, $\mathbb{Z}_X[2d]$ denotes the complex $\mathbb{Z}_X$ shifted $2d$ places to the **left**.

I will not explain the Tate twist at the level of sheaves. All you really need to know is that on all X there is a "mixed sheaf" $\mathbb{Z}_X(1)$; if F is a "mixed sheaf" then $F(n) = F \otimes \mathbb{Z}_X(1)^{\otimes n}$ is a mixed sheaf with the property that $H^m(X, F(n)) = H^m(X, F)(n)$. You don't need to know what these things are to use the formalism.

A thing to remember is that a shift of $2n$ places to the left is almost always accompanied by a Tate twist by (n); here we have $\mathbb{Z}_X[2d](d)$. If you have an exact sequence with a term where this is not the case, then you probably made a mistake. Go back and chase this right now!

If X is nonsingular and proper, the theory works so that $H_m(X;\mathbb{Z})$ has pure weight $-m$ (this is natural and logical). So $H^m(X;\mathbb{Z})$ has weight m and, if d is the algebraic dimension of X, $H_{2d-m}(X;\mathbb{Z})$ has weight $m-2d$: so, for P to be a morphism of HS, we need to raise the weights of $H_{2d-m}(X;\mathbb{Z})$ by $2d$, i.e. Tate twist by $(-d)$. I am saying that, in complete generality, P is a morphism of MHS.

I **strongly advise you** to take the habit of inserting your Tate twists from the beginning of learning this theory. This additional investment is totally worth it. If you don't do this, you will write exact sequences where you never know what weights your HS actually have, thus suffering a total degradation of information.

In general, if X is nonsingular over a field k (for example $k = \mathbb{Q}$), then X has a de Rham complex $\Omega^\bullet_{X/k}$ (this is a complex of coherent sheaves although the differential is of course not $\mathcal{O}_X$-linear). The (hyper) cohomology of this complex is the *de Rham cohomology* of X, $H^m_{dR}(X)$. Note that this thing is a vector space over k. In characteristic p, differentiating can get very tricky so you always want to assume that k has characteristic 0! The de Rham complex has a natural decreasing "stupid" filtration

$$F^p = \bigoplus_{p' \geq p} \Omega^{p'}_X,$$

thus

(1) $H^m_{dR}(X)$ comes endowed with a decreasing filtration $F^p \supset F^{p+1} \supset \cdots$, and

(2) There is a spectral sequence $E_\bullet$ computing this filtration with first page:

$$E_1^{p.q} = H^q(\Omega^p_{X/k}) \Rightarrow H^{p+q}_{dR}(X)$$

When X is complex projective (or compact Kähler) you can interpret Hodge theory as saying that for all $r \geq 1$ $E_1 = E_r$—that is, the spectral sequence degenerates on the first page. The reason for this degeneration is deep and it is the most crucial way in which topological spaces underlying algebraic varieties are special.

The beauty of the theorem is that the group on the LHS is the singular cohomology of X, which is all about the (classical) topology of X, whereas you can compute the group on the RHS purely from the Zariski topology of X.

The theorem makes a deep statement that asks a tantalising question: if X is defined over a subfield $k \subset \mathbb{C}$, then the singular cohomology $H^m(X, \mathbb{C})$ has a natural k-structure:

$$H^m(X; \mathbb{C}) = H^m_{dR}(X) \otimes_k \mathbb{C}$$

The question is: what exactly is this k-structure?

4 Constructible sheaves and six operations. Exact sequences

In Algebraic Topology 101 you learn how to compute the homology of a space by standard exact sequences: sequence of the pair, Mayer–Vietoris sequence. When working with algebraic varieties, these tools are not very useful. It is much better to get used to working with the Zariski topology, and with operations that are natural in algebraic geometry such as Zariski closed subspaces and fibred products. This section is about algebraic-geometric analogs of the sequence of the pair; in § 7 below I discuss the algebraic-geometric analog of the Mayer–Vietoris sequence.

Suppose that X is a variety (or scheme) over $k \subset \mathbb{C}$, let $i \colon Y \hookrightarrow X$ be the closed embedding of a subvariety (or subscheme), and denote by $j \colon U \hookrightarrow X$ the (open) inclusion of the complement $U = X \setminus Y$. This situation gives rise to several exact sequences that I now discuss.

These exact sequences arise from two exact triangles that exist for all Zariski cohomologically constructible (complexes of) sheaves F on X that are locally constant for the *classical* topology:

$$i_\star i^! F \to F \to Rj_\star j^\star F \xrightarrow{+1} \tag{2}$$

and

$$Rj_! j^\star F \to F \to i_\star i^\star F \xrightarrow{+1} \tag{3}$$

4.1 An exact sequence

Plugging $F = \mathbb{Z}_X$ in the triangle 2 we get an exact triangle:

$$i_\star i^! \mathbb{Z}_X \to \mathbb{Z}_X \to Rj_\star \mathbb{Z}_U \xrightarrow{+1}$$

This is not much use unless we know what $i^!\mathbb{Z}_X$ is and for this we usually need some additional assumptions.

If there is **one thing** that you need to know about Verdier duality is that $i^! D_X = D_Y$.

If X is nonsingular of (algebraic) dimension $d = \dim X$, then $D_X = \mathbb{Z}_X[2d](d)$ so, in this case, $i^!\mathbb{Z}_X = D_Y[-2d](-d)$ and the triangle gives rise to an exact sequence of MHS:

$$\cdots H^{BM}_{2d-m}(Y;\mathbb{Z})(-d) \to H^m(X;\mathbb{Z}) \to H^m(U;\mathbb{Z}) \to H^{BM}_{2d-m-1}(Y;\mathbb{Z})(-d) \cdots \tag{4}$$

Note that this sequence is a little weird as it mixes (Borel–Moore) homology with cohomology. Note that $H^{BM}_{2d-m}(Y;\mathbb{Z})$ will tend to have weight $m-2d$ so we need to twist up by $(-d)$ for it to have a morphism to $H^m(X;\mathbb{Z})$.

A further specialization useful in applications is when $Y \subset X$ is a **simple normal crossing divisor**. In this case, we can pin down easily D_Y but, for lack of time, I am not discussing this—but see exercise 7.

An even further specialization is when $Y \subset X$ is a **nonsingular divisor**: in this case $i^!\mathbb{Z}_X = \mathbb{Z}_Y[-2](-1)$ and we get an exact sequence of MHS:

$$\cdots H^{m-2}(Y;\mathbb{Z})(-1) \to H^m(X;\mathbb{Z}) \to H^m(U;\mathbb{Z}) \to H^{m-1}(Y;\mathbb{Z})(-1)\cdots \quad (5)$$

which is very nice and reasonable since it is all about standard cohomology groups. (Note again that the weights of $H^{m-2}(Y;\mathbb{Z})$ have been raised by 2 so it can have a morphism to $H^m(X;\mathbb{Z})$.)

4.2 Another exact sequence

Plugging $F = \mathbb{Z}_X$ in the triangle 3 we get an exact triangle:

$$Rj_!\mathbb{Z}_U \to \mathbb{Z}_X \to i_\star\mathbb{Z}_Y \xrightarrow{+1}$$

giving rise to a long exact sequence of MHS:

$$\cdots H^m_c(U;\mathbb{Z}) \to H^m(X;\mathbb{Z}) \to H^m(Y;\mathbb{Z}) \to H^{m+1}_c(U;\mathbb{Z})\cdots \quad (6)$$

4.3 More exact sequences

Exercise 6. Plug $F = D_X$ (instead of $F = \mathbb{Z}_X$) in the triangles 2 and 3 and obtain two more exact sequences. Stare at them until they make sense to you. Interpret all four exact sequences as exact sequences of pairs from Algebraic Topology 101.

Exercise 7. Let $Y = \cup_j Y_j$ be a simple normal crossing variety. Write

$$Y^{[p]} = \coprod_{j_0<j_1\cdots<j_p} Y_{j_0}\cap Y_{j_1}\cdots\cap Y_{j_p}$$

so that $Y^{[\bullet]}$ is a (strict) simplicial resolution of Y. This situation gives rise to two "resolutions:"

$$\mathbb{Q}_Y \to \mathbb{Q}_{Y^{[0]}} \to \mathbb{Q}_{Y^{[1]}}\cdots$$

and

$$D_Y \leftarrow D_{Y^{[0]}} \leftarrow D_{Y^{[1]}} \cdots$$

Suppose now that U is a smooth but not proper algebraic variety. Then there exists a smooth and proper compactification $U \subset X$ such that the complement $Y = X \setminus U$ is a proper simple normal crossing variety. Using exact sequences 4 and 5 now convince yourselves that $H^m(U;\mathbb{Z})$ has weights $\geq m$ and $H^m_c(U;\mathbb{Z})$ has weights $\leq m$. Also, for example, the natural morphism $H^m(X;\mathbb{Z}) \to \operatorname{gr}^W_m H^m(U;\mathbb{Z})$ is surjective and $\operatorname{gr}^W_m H^m_c(U;\mathbb{Z}) \to H^m(Z;\mathbb{Z})$ injective.

5 Cauchy's theorem and the Tate Hodge structure

Theorem 8. *$H^1(\mathbb{G}_m, \mathbb{Z}) = \mathbb{Z}(-1)$ as Hodge structures.*

Proof. Write $i\colon \{0\} \hookrightarrow \mathbb{A}^1$ and $j\colon \mathbb{G}_m \hookrightarrow \mathbb{A}^1$, where $\mathbb{G}_m$ is the multiplicative group: for all rings R, $\mathbb{G}_m(R) = R^\times$.

Recall that if $i\colon Y \hookrightarrow X$ is a smooth divisor in a smooth space and $j\colon U \hookrightarrow X$ is the inclusion of the open complement, we have an exact sequence:

$$\cdots H^{m-2}(Y;\mathbb{Z})(-1) \to H^m(X;\mathbb{Z}) \to H^m(U;\mathbb{Z}) \to H^{m-1}(Y;\mathbb{Z})(-1) \cdots$$

If we apply this to $j\colon \mathbb{C}^\times \hookrightarrow \mathbb{C}$ we obtain at once an isomorphism:

$$\operatorname{Res}\colon H^1(\mathbb{C}^\times;\mathbb{Z}) \xrightarrow{\cong} H^0(\{0\};\mathbb{Z})(-1) = \mathbb{Z}(-1)$$

of HS of pure weight 2. □

Let's understand this better and, in the process, also understand the funny business with $2\pi\mathtt{i}$. Over a field $k \subset \mathbb{C}$, there is a corresponding exact sequence for algebraic de Rham cohomology **over** k, and it is induced by an exact sequence of de Rham complexes over k:

$$0 \to \Omega^\bullet_{X/k} \to \Omega^\bullet_{X/k}(\log Y) \xrightarrow{\operatorname{Res}} \Omega^{-1+\bullet}_{Y/k} \to 0$$

Simon tells me that he defined all these complexes and the residue homomorphism. Something to keep in mind if you want to understand this stuff is the *residue theorem* (which is a harmless generalization of Cauchy's theorem): (closed) differential forms can be integrated on homology classes and

$$\int_\gamma \operatorname{Res}\omega = \frac{1}{2\pi\mathtt{i}}\int_{\tau(\gamma)}\omega$$

where $\gamma \in H_m(Y;\mathbb{Z})$ and $\tau\colon H_m(Y;\mathbb{Z}) \to H_{m+1}(X;\mathbb{Z})$ is the Griffiths "tube map" and ω is a *closed* algebraic differential form of degree $m+1$ with a logarithmic pole along Y.

We want to understand all this for $j\colon \mathbb{G}_m \hookrightarrow \mathbb{A}^1$ over any field k of characteristic 0. Both these varieties are affine so no higher derived functors: computing hypercohomology is a breeze:

$$\begin{tikzcd} k[z]dz \arrow[r] & k[z]\frac{d}{dz} \arrow[r, "\mathrm{Res}"] & k \\ k[z] \arrow[u, "d"] \arrow[r] & k[z] \arrow[u, "d"] & \end{tikzcd}$$

Nothing much is going on here: if $f(z) \in k[z]$ is a polynomial, then $df(z) = \frac{df}{dz}dz$ is a polynomial 1-form. You don't need any calculus to know what $\frac{df}{dz}$ is! We deduce that $H^1_{dR}(\mathbb{G}_m/k) = k$ with canonical basis $\frac{dz}{z}$. If we are working over a field k, the choice of basis is a no-brainer: we must choose $\frac{dz}{z}$.

Because $\mathbb{G}_m$ is defined over $\mathbb{Q}$ (in fact over $\operatorname{Spec}\mathbb{Z}$, but never mind that), the de Rham cohomology group $H^1_{dR}(\mathbb{G}_m/\mathbb{Q})$ is $\mathbb{Q}$, the 1-dimensional vector space over $\mathbb{Q}$ with basis $\frac{dz}{z}$. Thus $H^1_{dR}(\mathbb{G}_m/\mathbb{C}) = H^1_{dR}(\mathbb{G}_m/\mathbb{Q}) \otimes_\mathbb{Q} \mathbb{C}$ (flat base change theorem for coherent cohomology) has a natural $\mathbb{Q}$-structure with basis $\frac{dz}{z}$. On the other hand, by Grothendieck, $H^1_{dR}(\mathbb{G}_m/\mathbb{C})$ is the singular cohomology group $H^1(\mathbb{C}^\times;\mathbb{C})$ and from topology it inherits the lattice $H^1(\mathbb{C}^\times,\mathbb{Z})$, and hence also a rational structure. These two rational structures are not the same! Indeed, Poincaré duality gives a perfect pairing (the cap product):

$$\cap\colon H^1(\mathbb{C}^\times;\mathbb{Z}) \times H_1(\mathbb{C}^\times;\mathbb{Z}) \to H_0(\mathbb{C}^\times;\mathbb{Z}) = \mathbb{Z}$$

The natural integral basis of $H^1(\mathbb{C}^\times;\mathbb{Z})$ is the cohomology class α such that $\alpha\cap\gamma = 1$ where $\gamma \in H_1(\mathbb{C}^\times;\mathbb{Z})$ is the class of the couterclockwise loop around the origin. We all knew as undergraduates that:

$$\frac{dz}{z}\cap\gamma = \int_\gamma \frac{dz}{z} = 2\pi\mathtt{i}$$

hence $\alpha = \frac{1}{2\pi\mathtt{i}} \frac{dz}{z}$: if we identify $H^1(\mathbb{C}^\times;\mathbb{C})$ with $\mathbb{C}$ via the basis $\frac{dz}{z}$, and we argued that this is the natural and logical thing to do, then it makes sense to say that "the underlying lattice"–the one given by topology as $H^1(\mathbb{C}^\times;\mathbb{Z})$, is $\frac{1}{2\pi\mathtt{i}}\mathbb{Z}$; that is, again $H^1(\mathbb{C}^\times;\mathbb{Z}) = \mathbb{Z}(-1)$ as a HS.

6 Extensions of MHS

There is absolutely no point in knowing about MHS if you don't know how to mix pure HS of different weights to form a MHS cocktail. Here I only sketch the simplest possible case.

6.1 Intermediate Jacobians

Definition 9. Let H be a MHS, and $p > 1/2$(highest weight of H); then the p^{th} intermediate of H is the group $J^p(H) = H_\mathbb{C}/F^p + H_\mathbb{Z}$.

It is important to practice a few relevant cases to get a grip on what this thing is.

Suppose that H is a pure HS of weight 1, so $H_\mathbb{C} = H^{0,1} \oplus H^{1,0}$ with $F^0 = H_\mathbb{C} \supset F^1 = H^{1,0}$; then $J^1H = H^{0,1}/H_\mathbb{Z}$. The dual $H^\vee$ of H is a HS of weight -1, and $H^\vee_\mathbb{C} = H^{-1,0} \oplus H^{0,-1}$ with $F^{-1} = H^\vee_\mathbb{C} \supset F^0 = H^{0,-1}$, and $H^{-1,0} = H^{1,0\,\vee}$, $H^{0,-1} = H^{0,1\,\vee}$, so $J^0H^\vee = H^{1,0\,\vee}/H^\vee_\mathbb{Z}$.

If X is a nonsingular and proper algebraic curve over $\mathbb{C}$, then $H^1(X;\mathbb{Z})$ is a HS of weight 1 and:

$$J^1H^1(X;\mathbb{Z}) = H^{0,1}/H_\mathbb{Z} = H^1(X,\mathcal{O}_X)/H^1(X;\mathbb{Z}) = \mathrm{Pic}^\tau(X)$$

is the *Picard variety* of X. On the other hand, $H_1(X;\mathbb{Z})$ is a HS of weight -1 and

$$J^0H_1(X;\mathbb{Z}) = H^{-1,0}/H_\mathbb{Z} = H^{1,0\,\vee}/H_\mathbb{Z} = H^0(X,\Omega^1_X)^\vee/H_1(X;\mathbb{Z}) = \mathrm{Alb}(X)$$

is the *Albanese variety* of X.[4]

[4] All nonsingular proper algebraic varieties X over any field k have a Picard variety and an Albanese variety. In higher dimensions these abelian varieties are dual to each other.

If X is one-dimensional, the **Abel theorem** states that the *Abel–Jacobi map* $a\colon \mathrm{Pic}^\tau(X) \to \mathrm{Alb}(X)$ is an isomorphism. The Abel theorem is a beautiful and deep result that is unfortunately rarely proved in Algebraic Geometry 101.

Here is another case to contemplate: if H is a HS of weight 3, $H_{\mathbb{C}} = H^{0,3} \oplus H^{1,2} \oplus H^{2,1} \oplus H^{3,0}$ then $J^2 H = (H^{0,3} \oplus H^{1,2})/H_{\mathbb{Z}}$.

I forgot the most basic case: the Tate Hodge structure $\mathbb{Z}(1)$ has weight -2 and $J^0\mathbb{Z}(1) = \mathbb{C}/2\pi\mathtt{i}\mathbb{Z} = \mathbb{C}^{\times}$.

6.2 Extensions

Theorem 10. *Let A and B be MHS. Denote by* $\operatorname{Ext}(B, A)$ *the group of MHS* extensions*:*

$$0 \to A \to H \to B \to 0$$

Suppose that A, B are separated*, that is, the highest weight of A is $<$ then the lowest weight of B. Then* $\operatorname{Ext}(B, A) = J^0 \operatorname{Hom}(B, A)$*.*

6.3 Wait, what?

Let X be a nonsingular proper algebraic curve over $\mathbb{C}$, $S \subset X$ a finite set, and $U = X \setminus S \hookrightarrow X$ the open complement. We have an extension of separated MHS:

$$(0) \to H^0(S;\mathbb{Z})/H^0(X;\mathbb{Z}) \to H^1_c(U;\mathbb{Z}) \to H^1(X;\mathbb{Z}) \to (0)$$

We are told that the class of this extension lies in the group:

$$\begin{aligned} J^0 \operatorname{Hom}(H^1(X;\mathbb{Z}), H^0(S;\mathbb{Z})/H^0(X;\mathbb{Z})) &= \\ = \operatorname{Alb}(X) \otimes \big(H^0(S;\mathbb{Z})/H^0(X;\mathbb{Z})\big) &= \\ = \operatorname{Hom}\big(V_S, \operatorname{Alb}(X)\big) \end{aligned}$$

where $V_S = \big(H^0(S;\mathbb{Z})/H^0(X;\mathbb{Z})\big)^{\vee}$, that is V_S is the group of divisors $D = \sum_{P\in S} n_P P$ supported on S and of degree $\deg D = \sum n_P = 0$. The class of the extension is the *Albanese homomorphism* (aka *Abel–Jacobi map*) $a\colon V_S \to \operatorname{Alb}(X)$; for example for $P, Q \in S$:

$$a(P - Q) = \int_Q^P \in H^0(X, \Omega^1_X)^{\vee}/H_1(X;\mathbb{Z}) :$$

a holomorphic 1-form ω on X is closed and, by Stokes, the value of the integral $\int_Q^P \omega$ is well-defined modulo a closed path in $H_1(X;\mathbb{Z})$.

6.4 Proof of theorem 10

Let's just do the simplest case; the general case is not much harder. Let us consider extensions:

$$(0) \to \mathbb{Z} \to H \to B \to (0)$$

where B is a (pure) HS of weight 1. Assume for simplicity that B is torsion-free so we can split the extension and the weight filtration of H over the integers, and in fact let us assume that we have already done that so $H = \mathbb{Z} \oplus B$ and $W_0(H) = \mathbb{Z}$, $W_1(H) = H$. It remains to endow H with an F-filtration. Let us write $B_{\mathbb{C}} = B^{0,1} \oplus B^{1,0}$ and $\mathbb{Z} \otimes \mathbb{C} = A^{0,0}$.

We are looking to endow H with an F-filtration

$$F = F^0 \supset F^1 \supset F^2$$

in such a way that

$$F^0 \cap H^{0,0} = H^{0,0},\ F^1 \cap H^{0,0} = (0); \quad \text{and} \quad F^0/H^{0,0} = B_{\mathbb{C}},\ F^1 = B^{1,0}\ F^2 = (0)$$

It follows that $F^0 = H$ and $F^2 = (0)$: we only need to decide F^1 and it follows from the conditions that F^1 is the graph of a linear homomorphism $f\colon B^{1,0} \to H^{0,0} = \mathbb{C}$:

$$F^1 = \{(a, b) \mid a = f(b)\}$$

We still need to mod out by integral automorphisms of $\mathbb{Z} \oplus B$ that are the identity on both $\mathbb{Z}$ and B: and these are in 1-to-1 correspondence with integral linear homomorphisms $B \to \mathbb{Z}$. In summary:

$$\operatorname{Ext}(B, \mathbb{Z}) = B^{1,0\vee}/B^{\vee}_{\mathbb{Z}} = J^0(B^{\vee})$$

7 Proper base change. Cohomology of singular varieties

Theorem 11 (Proper base change theorem). *Let X be a scheme over $\mathbb{C}$, $f\colon Y \to X$ a proper morphism, $g\colon S \to Y$ any morphism and $E = S \times_X Y$, so that we have a fibre square:*

$$\begin{tikzcd} E \arrow[r, "g'"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ S \arrow[r, "g"'] & X \end{tikzcd}$$

Then for all $F \in D^b_{cc}(X)$ the natural base change homomorphism:

$$g^\star Rf_\star F \to Rf'_\star(g'^\star F)$$

is an isomorphism.

Let X be a singular proper algebraic variety over $\mathbb{C}$, $f\colon Y \to X$ a resolution of singularities, $g\colon S \hookrightarrow X$ the inclusion of the singular set, E the exceptional set, and $j\colon U = X \setminus S \hookrightarrow X$ the inclusion of the complement. It is part of the definition of resolution of singularities that $f\colon f^{-1}U \to U$ is an isomorphism, thus we identify $f^{-1}U$ with U via f. Applying the proper base change theorem to $F = \mathbb{Q}_Y$, we get a morphism of exact triangles:

$$\begin{tikzcd}
Rj_!\mathbb{Z}_U \arrow[r] & Rf_\star\mathbb{Z}_Y \arrow[r] & i_\star Rf_\star\mathbb{Z}_E \arrow[r, "+1"] & {} \\
Rj_!\mathbb{Z}_U \arrow[r] \arrow[u, equal] & \mathbb{Z}_X \arrow[r] \arrow[u] & i_\star\mathbb{Z}_S \arrow[r, "+1"] \arrow[u] & {}
\end{tikzcd}$$

that induces an exact triangle

$$\mathbb{Z}_X \to Rf_\star\mathbb{Z}_y \oplus i_\star\mathbb{Z}_S \to Rf_\star\mathbb{Z}_E \stackrel{+1}{\to}$$

that, in turn, gives a Mayer–Vietoris type long exact sequence of MHS:[5]

$$\cdots H^{m-1}(E;\mathbb{Z}) \to H^m(X;\mathbb{Z}) \to H^m(Y;\mathbb{Z}) \oplus H^m(S;\mathbb{Z}) \to H^m(E;\mathbb{Z}) \cdots \quad (7)$$

A breeze! It slices the cohomology of X by putting a rational weight filtration on it the graded pieces of which are either pure HS or MHS of simpler (lower-dimensional) schemes.

This incredibly simple structure is the only non-formal content of Deligne's theory of cohomological descent.

Exercise 12. Let X be an algebraic curve over $\mathbb{C}$ with n punctures and k nodes. Compute the four mixed Hodge structures $H^1_c(X;\mathbb{Z})$, $H_1(X;\mathbb{Z})$, $H^1(X;\mathbb{Z})$, $H^{BM}_1(X;\mathbb{Z})$. Make sure that you have a geometric interpretation of all extension classes. Stare at the result and practice is in your head until it seems completely trivial.

[5]Suppose for simplicity that X has a unique singular point $x \in X$, so $S = \{x\}$. In this case, X is obtained as a topological space by collapsing $E \subset Y$ to a point. Then we recognise this sequence as the long exact cohomology sequence of the pair (Y, E) from Algebraic Topology 101.

If you produce a book or article with this exact sequence in it, I will open a good bottle and drink it with you.

8 What next? Invariant cycles, mixed Hodge modules

I want to state and prove a really amazing fact: the global invariant cycle theorem. Before I state it, I need some preparations. Let $\overline{X}$ be a nonsingular projective variety over $\mathbb{C}$, $\overline{Y}$ a nonsingular projective curve, and $\overline{f}\colon \overline{X} \to \overline{Y}$ a morphism. Denote by $S \subset \overline{Y}$ the set of singular values of $\overline{f}$, $Y = \overline{Y} \setminus S$ the set of regular values, $X = f^{-1}(Y)$ and $f\colon X \to Y$ the induced smooth projective morphism. Fix any $y \in Y$ and let $X_y = f^{-1}(y)$ be the fibre at y. For all m, the fundamental group $\pi = \pi_1(Y, y)$ acts on $H^m(X_y; \mathbb{Z})$: this action is called the *monodromy* action.

Theorem 13 (Deligne's global invariant cycle theorem)**.** *For all m, the group $H^m(X_y; \mathbb{Q})^\pi$ of monodromy invariant cohomology classes (cycles) on X_y is the image of the natural homomorphism $H^m(\overline{X}; \mathbb{Q}) \to H^m(X_y; \mathbb{Q})$.*

Proof. Deligne [Del68] shows that the Leray spectral sequence with first page

$$E_1^{p,q} = H^p(R^q f_\star \mathbb{Q}_X) \Rightarrow H^{p+q}(X; \mathbb{Q})$$

degenerates on the first page: for all $r \geq 1$, $E_1^{p,q} = E_r^{p,q}$. (In fact he proves more, namely that there is a (noncanonical) isomorphism in $D^b_{cc}(Y)$: $Rf_\star \mathbb{Q}_X = \oplus R^m f_\star \mathbb{Q}_X[-m]$.). Thus we have a surjection of MHS:

$$H^m(X; \mathbb{Q}) \to H^0(Y, R^m f_\star \mathbb{Q}_Y) = H^0(X_y; \mathbb{Q})^\pi \subset H^m(X_y, \mathbb{Q})$$

But then $H^m(\overline{X}; \mathbb{Q}) \to H^m(X_y; \mathbb{Q})^\pi$ is also surjective, by stricness and the fact—shown in exercise 7—that $H^m(\overline{X}; \mathbb{Q}) \to \operatorname{gr}^W_m H^m(X; \mathbb{Q})$ is surjective. □

Looking at what we have done, it is natural to wish for a theory "mixed sheaves" on schemes with six operations, "improving" the category of cohomologically constructible sheaves, such that, for example: if F is a mixed sheaf on X and $x \in X$ is a point, then the fibre F_x is a mixed Hodge complex; the cohomology groups $H^m(X; F)$ carry MHS; etcetera. In fact we had this theory for at least 30 years, and it is Morihiko Saito's category of *mixed Hodge modules*. If you think about it, I have been using it all the time in this lecture (without understanding what it is).

A short guide to the literature

I limit myself to a few suggestions to help you get started.

This theory is really tough to learn properly. For starters, the motivation for developing mixed Hodge theory comes from the Weil conjectures and étale cohomology. If you are a number theorist, you are probably better off leaving MHS alone and study étale cohomology first. You can come back to MHS later and they will make sense.

If you are a geometer, and you want to use Hodge theory to learn the motivic point of view, you should start from the nice-and-easy [Gro69] and Milne's note *Motives—Grothendieck's dream* (avaliable for download from his website). Then I recommend the paper of Deligne and Milne on Tannakian categories in [DMOS82] (also downloadable from Milne's website): this one is very nicely written and easy to follow, but you must get to the end of it, where they speak of motives for absolute Hodge cycles.

The least painful introduction to mixed Hodge theory for someone with a complex-analytic background and therefore used to C^∞ forms (such as yourselves after attending Simon's lectures) is [GS75]. I warn you, however, that this paper is long and hard; it is long-winded and dated and doesn't go very far; on the plus side, it gives an introduction to more advanced topics such as the SL_2-orbit theorem and nilpotent orbit theorem (degenerations of HS).

For Grothendieck's algebraic de Rham theorem, I totally recommend the original paper [Gro66]: it is short, it is clear, it is written in English. (I mention this because, sadly, it may make a difference to you. It makes no difference to me.) Reasons why spectral sequences of topological origin degenerate in algebraic geometry are given in [DI87] and [Del68]: both papers are nice and (relatively speaking) readable.

A very concise summary in the topological setting of the derived category of cohomologically constructible (complexes of) sheaves and the six operations is in [GM83, §1].

The material on extensions of mixed Hodge structures is explained really well in the paper of Jim Carlson [Car80].

If you are serious about studying MHS, then even today you have to go back to the papers by Deligne. Hodge II [Del71] is readable (but very tough) and I recommend it without much hesitation. The global invariant cycle theorem is Theorem 4.1.1 in Hodge II. The main problem with Hodge III [Del74] is that you need to know in advance Deligne's theory of cohomological descent. The original presentation in SGA4 is a prime example of how a pure thinker (alchemist) can trasmute something that is at bottom really simple (the alchemical equivalent of gold) into something extremely complicated (the alchemical equivalent of a much baser metal or substance). I believe that this single fact is the perverse reason why mixed Hodge theory still has a reputation for being a black art, and so few algebraic geometers can use it confidently. Fortunately Brian Conrad did a huge service to humanity by writing up this theory in (almost) human-readable from in a note titled *Cohomological Descent* available for download from his website.

At this point you will know the basics. From now on the road is even steeper. The theory of mixed Hodge modules is due to Morihiko Saito. A nice summary of how things work is in his unpublished RIMS preprint *On the formalism of mixed sheaves* (it's on the `arXiv`).

References

[Car80] James A. Carlson. Extensions of mixed Hodge structures. In *Journées de Géometrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979*, pages 107–127. Sijthoff & Noordhoff, Alphen aan den Rijn—Germantown, Md., 1980.

[Del68] P. Deligne. Théorème de Lefschetz et critères de dégénérescence de suites spectrales. *Inst. Hautes Études Sci. Publ. Math.*, (35):259–278, 1968.

[Del71] Pierre Deligne. Théorie de Hodge. II. *Inst. Hautes Études Sci. Publ. Math.*, (40):5–57, 1971.

[Del74] Pierre Deligne. Théorie de Hodge. III. *Inst. Hautes Études Sci. Publ. Math.*, (44):5–77, 1974.

[DI87] Pierre Deligne and Luc Illusie. Relèvements modulo p^2 et décomposition du complexe de de Rham. *Invent. Math.*, 89(2):247–270, 1987.

[DMOS82] Pierre Deligne, James S. Milne, Arthur Ogus, and Kuang-yen Shih. *Hodge cycles, motives, and Shimura varieties*, volume 900 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1982.

[GM83] Mark Goresky and Robert MacPherson. Intersection homology. II. *Invent. Math.*, 72(1):77–129, 1983.

[Gro66] A. Grothendieck. On the de Rham cohomology of algebraic varieties. *Inst. Hautes Études Sci. Publ. Math.*, (29):95–103, 1966.

[Gro69] A. Grothendieck. Standard conjectures on algebraic cycles. In *Algebraic Geometry (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968)*, pages 193–199. Oxford Univ. Press, London, 1969.

[GS75] Phillip Griffiths and Wilfried Schmid. Recent developments in Hodge theory: a discussion of techniques and results. In *Discrete subgroups of Lie groups and applicatons to moduli (Internat. Colloq., Bombay, 1973)*, pages 31–127. Oxford Univ. Press, Bombay, 1975.