1. (a) $y'' + 3y' - 4y = 7\,e^{3x}$.

General Solution(GS) = Complementary Function(CF) + Particular Integral(PI).

To find CF: put $y(x) = e^{\lambda x}$ in the homogeneous form of the ODE, so $(\lambda^2 + 3\lambda - 4)\,e^{\lambda x} = 0$. Then the auxiliary equation is $\lambda^2 + 3\lambda - 4 = 0$ i.e. $(\lambda - 1)(\lambda + 4) = 0$. Hence $\lambda = 1$ or $\lambda = -4$, so CF $= Ae^{x} + Be^{-4x}$, where A and B are arbitrary constants.

To find the PI: put $y(x) = C\,e^{3x}$ in the full form of the ODE, where C is a constant to be found. [e^{3x} does not appear in the CF, so one expects this substitution to work.]
Then the ODE becomes $C\,(9 + 9 - 4)\,e^{3x} = 7e^{3x}$, so $C = 7/14 = 1/2$.
Hence the General Solution = CF + PI = $Ae^{x} + Be^{-4x} + \frac{1}{2}e^{3x}$.

(b) $y(x) = e^{\lambda x}$ in homogeneous form of ODE gives $\lambda = 1$ or $\lambda = 2$, so CF $= Ae^{x} + Be^{2x}$.
To find PI: e^{2x} appears in the CF, so $C\,e^{2x}$ will not work - check this if necessary. So try $y(x) = C\,x\,e^{2x}$ in the full form of the ODE.
Then $y' = C(1 + 2x)e^{2x}$ and $y'' = C(2 + 2 + 4x)e^{2x} = 4C(1 + x)e^{2x}$, so substituting in full form of the ODE, $4C(1 + x)e^{2x} - 3C(1 + 2x)e^{2x} + 2Cxe^{2x} = e^{2x}$, giving $C = 1$.
Hence General Solution = CF + PI = $Ae^{x} + Be^{2x} + xe^{2x}$.

(c) CF: auxiliary equation is $\lambda^2 - 4\lambda + 4 = 0$, so $\lambda = 2$, repeated. So CF $= (A + Bx)\,e^{2x}$.
PI: Since e^{2x} and xe^{2x} also appear in the CF, try $y = C\,x^2\,e^{2x}$ in full ODE.

(d) CF: auxiliary equation is $2\lambda^2 - 3\lambda + 1 = 0$, so $\lambda = 1$ or $\frac{1}{2}$. So CF $= Ae^{x} + Be^{x/2}$.
PI: Since RHS is quadratic in x, try general quadratic $y = C + Dx + Ex^2$. Substituting in full ODE gives $2E2 - 3(D + 2Ex) + C + Dx + Ex^2 = x^2$. Since this identity must hold for all values of x, coefficients of x^0, x and x^2 on both sides must be equal. Hence $4E - 3D + C = 0$, $-6E + D = 0$ and $E = 1$, so $D = 6$, $C = 14$. Thus the general solution is $y = Ae^{x} + Be^{x/2} + 14 + 6x + x^2$.

(e) CF: $\lambda^2 - 2\lambda + 2 = 0$, so $\lambda = 1 \pm i$. So CF $= ae^{(1+i)x} + be^{(1-i)x} = Ae^{x}\cos x + Be^{x}\sin x$, where A, B are real constants (a, b are complex conjugate constants).
PI: Since $e^{x}\sin x$ appears in RHS of ODE, try $y = (C\cos x + D\sin x)\,x\,e^{x}$.

(f) CF: $\lambda^2 + \lambda + 1 = 0$, so $\lambda = \frac{-1 \pm i\sqrt{3}}{2}$. So CF $= e^{-x/2}\left\{A\cos\left(\frac{x\sqrt{3}}{2}\right) + B\sin\left(\frac{x\sqrt{3}}{2}\right)\right\}$.
PI: Try $y = (C + Dx)\,e^{2x}$.

(g) CF: $\lambda^2 - 2\lambda + 1 = 0$, so $\lambda = 1$, repeated. So CF $= (A + Bx)\,e^{x}$.
PI: Since RHS is cubic in x, try general cubic $y = C + Dx + Ex^2 + Fx^3$.

(h) CF: $\lambda^2 + 9 = 0$, so $\lambda = \pm 3i$. So CF $= A\cos 3x + B\sin 3x$.
PI: Replace $\cos^2 x$ by an expression linear in terms of trig functions, i.e.
put $\cos^2 x = (1 + \cos 2x)/2$. Then RHS of full ODE contains two terms - find their corresponding PIs separately and add. PI for 1/2: try $y = C$, giving $C = \frac{1}{18}$.
PI for $\frac{1}{2}\cos 2x$: try $y = D\cos 2x$. Then $D(-4 + 9) = \frac{1}{2}$, so $D = \frac{1}{10}$.
Hence general solution is $y = A\cos 3x + B\sin 3x + \frac{1}{18} + \frac{1}{10}\cos 2x$.

2. CF: $\lambda^2+6\lambda+8=0$, so $\lambda=-2,\ -4$. So CF $=Ae^{-2x}+Be^{-4x}$.
PI: **IF** one tries $y=C\cosh 2x+D\sinh 2x$. Then
$4C\cosh 2x+4D\sinh 2x+12C\sinh x+12D\cosh x+8C\cosh 2x+8D\sinh 2x=12\cosh 2x$.
Equating coefficients of $\cosh x$ and of $\sinh x$, $4C+12D+8C=12$ and $4D+12C+8D=0$.
But the last two equations are not consistent. This method failed since the term e^{-2x} appears both in the CF and in RHS of the ODE, i.e. $6e^{2x}+6e^{-2x}$. So try PI of form $Ee^{2x}+(F+Gx)e^{-2x}$. Then
$4Ee^{2x}+4Fe^{-2x}+4G(-1+x)e^{-2x}+12Ee^{2x}-12Fe^{-2x}+6G(1-2x)e^{-2x}+8Ee^{2x}+8(F+Gx)e^{-2x}=6e^{2x}+6e^{-2x}$.
Equate coefficients of e^{2x} and e^{-2x}, noting coefficient of xe^{2x} in LHS is 0.
$4E+12E+8E=6$ and $4F-4G-12F+6G+8F=6$, so $E=1/4$, $G=3$ and F is undetermined. So take $F=0$, noting a term in e^{-2x} already appears in CF.
Hence general solution is $y=Ae^{-2x}+Be^{-4x}+\frac{1}{4}e^{2x}+3xe^{-2x}$.
To satisfy $y(0)=0$ and $y'(0)=1$, requires $A+B+\frac{1}{4}=0$ and $-2A-4B+\frac{1}{2}+3=1$.
So $A=-\frac{7}{4}$, $B=\frac{3}{2}$ and solution is $y=-\frac{7}{4}e^{-2x}+\frac{3}{2}e^{-4x}+\frac{1}{4}e^{2x}+3xe^{-2x}$.

3. $x=e^t$. So $\frac{dx}{dt}=e^t=x$ and $\frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx}=\frac{1}{x}\frac{dy}{dt}$. Hence $x\frac{dy}{dx}=\frac{dy}{dt}$. Differentiate the last with respect to x. $x\frac{d^2y}{dx^2}+\frac{dy}{dx}=\frac{d}{dx}\frac{dy}{dt}=\left(\frac{d}{dt}\frac{dy}{dt}\right)\frac{dt}{dx}=\frac{1}{x}\frac{d^2y}{dt^2}$ so $x^2\frac{d^2y}{dx^2}=\frac{d^2y}{dt^2}-\frac{dy}{dt}$.
Hence the ODE becomes $a\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)+b\frac{dy}{dt}+cy=0$, i.e. $a\frac{d^2y}{dt^2}+(b-a)\frac{dy}{dt}+cy=0$.
For $a=b=c=1$, $\frac{d^2y}{dt^2}+y=0$, with general solution $y(t)=A\cos t+B\sin t$, i.e. $y(x)=A\cos(\ln x)+B\sin(\ln x)$.

4. Between launch and first coming to rest, the equation of motion is $mx''=-m\omega^2x-mk$.
When $t=0$, $x=0$ and $x'=V$. ODE is $x''+\omega^2x=-k$. So CF is $A\cos\omega t+B\sin\omega t$.
RHS is constant, so try $y=C$ for PI, giving $\omega^2C=-k$. Hence PI $=-k\omega^{-2}$.
Thus general solution is $x(t)=A\cos\omega t+B\sin\omega t-k\omega^{-2}$.
Applying the conditions $x=0$ and $x'=V$ when $t=0$ gives $0=A-k\omega^{-2}$ and $V=B\omega$, so that $A=k\omega^{-2}$ and $B=V\omega^{-1}$ and $x(t)=k\omega^{-2}(\cos\omega t-1)+V\omega^{-1}\sin\omega t$.
(Optional Part - Harder) The particle first comes to rest at time T when $x'(T)=0$ i.e.
$$-k\omega^{-1}\sin\omega T+V\cos\omega T=0,\quad \text{i.e. } \tan\omega T=V\omega/k, \text{ as required.}$$
Then $\cos\omega T=k\left(k^2+V^2\omega^2\right)^{-1/2}$ and $\sin\omega T=V\omega\left(k^2+V^2\omega^2\right)^{-1/2}$, so that $x(T)=(k^2\omega^{-2}+V^2)(k^2+V^2\omega^2)^{-1/2}-k\omega^{-2}=\omega^{-2}\left\{(k^2+V^2\omega^2)^{1/2}-k\right\}$.
For $t>T$, $x''+\omega^2x=k$, so the general solution then becomes
$$x(t)=D\cos\omega(t-T)+E\sin\omega(t-T)t+k\omega^{-2}.$$
But for $t=T$, $x(T)=D+k\omega^{-2}$ so that $D=\omega^{-2}\left\{(k^2+V^2\omega^2)^{1/2}-2k\right\}$.
Also $x'(T)=0=E\omega$ so that $E=0$. Hence $x'(t)=-D\sin\omega(t-T)$. So motion for $t>T$ can occur only if $D>0$ i.e. $(V^2\omega^2+k^2)^{1/2}-2k>0$ i.e. $V^2\omega^2>3k$.

5. Taking the positive x axis to be vertically downwards, the equation of motion is Mass $\times$ Acceleration $=$ Resultant Force vertically downwards, so that $mx''=mg-mkv^2$, where $v=x'$ ($'$ means $\frac{d}{dt}$) and the resistance to motion is mkv^2. Hence $v'=g-kv^2$.
But, using a Chain Rule, $v'=\frac{dv}{dt}=\frac{dv}{dx}\frac{dx}{dt}=\frac{dv}{dx}v=\frac{1}{2}\frac{d}{dx}v^2$, so that $\frac{1}{2}\frac{d}{dx}v^2=g-kv^2$.
Hence, by putting $y=v^2$, this gives $\frac{dy}{dx}=2(g-ky)$. Also at $t=0$, we have $x=0$ and $v=0$, so that $y(0)=0$. The ODE for $y(x)$ is both separable and linear.
Using separation gives $-2k(x+c)=\ln|g-ky|$. But $x=0$ and $y=0$ when $t=0$, so that $-2kc=\ln g$. Hence on substituting for c, $-2kx=\ln|(g-ky)/g|$. Solving for y, $y=v^2=gk^{-1}\left(1-e^{-2kx}\right)$. As $x\to\infty$, the velocity $v\to$ the terminal value $\left(gk^{-1}\right)^{1/2}$.